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Optimizing a quadratic function with fuzzy linear coefficients [1]

by

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The objective of this paper is to solve a particular fuzzy quadratic optimization problem, as a first approach to non linear fuzzy optimization. To this purpose we first introduce some properties concerning fuzzy numbers and the fuzzy extension of non-fuzzy functions. Moreover, we suggest a possible way for solving some constrained optimization problems by penalty function.

1. Introduction

In the recent past the quantity of new research in fuzzy linear programming has been large. None the less non linear fuzzy optimization seems to have been considered only by a small number of authors.

As a first step in this direction we intend to examine a specific case of quadratic unconstrained fuzzy optimization problem (section 3), after introducing some preliminary notions (section 2).

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To this purpose, let us consider a function $f(x,p)$, with $x \in \mathbb{R}^n$, $p \in \mathbb{R}^m$ and $f : \mathbb{R}^{n+m} \to \mathbb{R}$. Let us assume that the following problem:

$$\min_x f(x,p)$$

may have the vector $x_0 = g(p)$ as the unique minimizing solution, where $g : \mathbb{R}^m \to \mathbb{R}^n$.

We can easily get the extension to the fuzzy case of the non-fuzzy functions f and g ([4], p. 98).

The fuzzy number (= f.n.) $(\tilde{x}_0)_i$, the i-th component of $\tilde{x}_0 = g(\tilde{p})$, has the following membership function (= m.f.):

$$\mu_{(\tilde{x}_0)_i}(y) = \max_{p \in \mathbb{R}^m} \{\min[\mu_{\tilde{p}_1}(p_1), \ldots, \mu_{\tilde{p}_m}(p_m)] : y = g_i(p)\} \tag{1}$$

where $p = (p_1, p_2, \ldots, p_m)$ and $g(p) = (g_1(p), g_2(p), \ldots, g_n(p))$.

Let us assume, further, that the f.n.'s $\tilde{p}_i$ $(i = 1, \ldots, m)$ are noninteractive ([4], p. 70).

The purpose of the present paper is to determine

$$\tilde{z}_0 = \min_{\tilde{x}} f(\tilde{x}, \tilde{p})$$

via the following two steps:

i) $\tilde{x}_0 = g(\tilde{p})$,

ii) $\tilde{z}_0 = f(\tilde{x}_0, \tilde{p})$,

under suitable conditions concerning the functions f and g, as well as the m.f. of $\tilde{p}$.

In computing the f.n. $\tilde{z}_0$ through ii), we have to take into account the relation between $\tilde{x}_0$ and $\tilde{p}$, according to i).

In order to simplify matters, we restrict to study according to the following two conditions:

a) the components of $\tilde{p}$ are f.n.'s (convex normalized fuzzy sets) of L-R type;

b) the objective function is quadratic-linear with respect to vector $\tilde{x}$, i.e. $f(\tilde{x}, \tilde{p}) = \frac{1}{2}\tilde{x}' A \tilde{x} + \tilde{p}' \tilde{x}$.

In section 4, a possible way for solving some constrained optimization problems through the use of penalty functions is suggested.

In the following we shall represent the L-R type fuzzy number, as the triple (c, a_s, a_d), where c is the mean value, while a_s and a_d are the left- and right- no-negative spreads.

2. Some characteristics of the fuzzy numbers

DEFINITION 1 *Given a f.n. $\tilde{x}$, we define:*

$$Supp(\tilde{x}) = \{a \in \mathbb{R} : \mu_{\tilde{x}}(a) > 0\}$$

Let us consider the fuzzy extension, $\tilde{z} = h(\tilde{p})$, of a non-fuzzy function h, between a f.n. $\tilde{z}$ and a vector of f.n. $\tilde{p}$.

Some computational techniques have been developed for specific operations (for example, the sum and the product of two f.n.'s as increasing binary operation, see [8] p. 14-42), but generally their applications are limited to the case in which the fuzzy operands are noninteractive.

Therefore, it is not correct to calculate $\tilde{z} = h(\tilde{a}, \tilde{b}) \equiv \tilde{a}^2 + \tilde{a}\tilde{b}$ using the rules of summing up of noninteractive f.n.'s (for example, though the procedures indicated in [4] p. 42 and following ones), via the steps

$$\tilde{c} = \tilde{a}^2; \quad \tilde{d} = \tilde{a}\tilde{b}; \quad \tilde{z} = \tilde{c} + \tilde{d},$$

because $\tilde{c}$ and $\tilde{d}$ are interactive f.n.'s (see also [5] p. 56).

Given two f.n.'s $\tilde{x}$ and $\tilde{y}$, let us suppose now that there exists a bidimensional m.f. $\mu_{(\tilde{x},\tilde{y})}(a,b)$ (see [8] p. 146 and [6]). Likewise dependency and independency concepts in probability theory, we say that

DEFINITION 2 *The two f.n.'s $\tilde{x}$ and $\tilde{y}$ are* noninteractive *iff:*

$$\mu_{(\tilde{x},\tilde{y})}(a,b) = \min[\mu_{\tilde{x}}(a), \mu_{\tilde{y}}(b)] \quad \text{for every } a \in \mathbb{R} \text{ and } b \in \mathbb{R}. \tag{2}$$

Otherwise, we say that $\tilde{x}$ and $\tilde{y}$ are *interactive*, and in general we cannot obtain $\mu_{(\tilde{x},\tilde{y})}(a,b)$ only trough $\mu_{\tilde{x}}(a)$ and $\mu_{\tilde{y}}(b)$ ([8] p. 172).

In particular, we suppose that $\tilde{y} = g(\tilde{x})$, where g is a fuzzy extension of non-fuzzy function defined in $Supp(\tilde{x})$.

PROPOSITION 1 *For any $a \in Supp(\tilde{x})$, we have to assign to the ordered pair $(a, g(a))$ the same possibility degree as a, i.e. $\mu_{\tilde{x}}(a)$. Furthermore, the possibility degree of ordered pair (a,b) is zero, if $b \neq g(a)$. Hence,*

$$\mu_{(\tilde{x},\tilde{y})}(a,b) = \begin{cases} \mu_{\tilde{x}}(a) & \text{if } b = g(a) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

We notice that the relation introduced by $\tilde{y} = g(\tilde{x})$ is a more restrictive concept than *fuzzy interactivity* between $\tilde{y}$ and $\tilde{x}$.

In the following, if $\tilde{y} = g(\tilde{x})$ we'll call *linked by g* the two f.n.'s $\tilde{y}$ and $\tilde{x}$.

DEFINITION 3 *The f.n.'s* $\tilde{p}_1$ *and* $\tilde{p}_2$ *are said to be* strictly equal *if:*

i) $\mu_{\tilde{p}_1}(p) = \mu_{\tilde{p}_2}(p)$ *for any* $p \in \mathbb{R}$*;*

ii) $\mu_{(\tilde{p}_1,\tilde{p}_2)}(p_1,p_2) = \begin{cases} \mu_{\tilde{p}_1}(p_1) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 \neq p_2. \end{cases}$

We remark that $\tilde{y} = g(\tilde{x})$ means that the two f.n.'s $\tilde{y}$ and $g(\tilde{x})$ are *strictly equal.*

PROPOSITION 2 *Fuzzy extensions of no-fuzzy equivalent algebraic expressions remain equivalent.*

Therefore, the usual algebraic properties of non-f.n.'s keep their validity even with f.n.'s. In particular, the two f.n.'s $\tilde{z}$ and $\tilde{w}$, defined by:

$$\tilde{z} = \tilde{a}(\tilde{b} + \tilde{c}) \quad \text{and} \quad \tilde{w} = \tilde{a}\tilde{b} + \tilde{a}\tilde{c},$$

are strictly equal (distributive property of product over addition).

Likewise, the f.n.'s $\tilde{z}$ and $\tilde{w}$ defined by $\tilde{z} = (\tilde{a} + \tilde{b})^2$ and $\tilde{w} = \tilde{a}^2 + \tilde{b}^2 + 2\tilde{a}\tilde{b}$ are strictly equal (the square of a binomial).

In particular, we want to stress that the difference between two strictly equal f.n.'s is always 0 (fuzzy zero with zero spread). Such a consequence is coherent with the intuitive meaning of a f.n., considered as an approximate measure of a quantity: if a is the length of an object, whatever its value may be, it is obvious that the difference $a - a$ is always zero (crisp zero).

The case is different when we take into account the difference between two f.n.'s $\tilde{a}$ and $\tilde{b}$ characterized by the same m.f., but associated with measures of different quantities ($\tilde{a}$ and $\tilde{b}$ are noninteractive). In such case, the difference between $\tilde{a}$ and $\tilde{b}$ is $\tilde{0}$ (fuzzy), because all the possible values of $\tilde{a}$ and $\tilde{b}$ must be considered, according to the extension principle.

In fact, in the first case the extension principle reads:

$$\mu_{\tilde{a}-\tilde{b}}(z) = \sup_{(x,y):x-y=0} \{\mu_{(\tilde{a},\tilde{b})}(x,y) : z = x - y\} = \sup[\mu_{\tilde{a}}(x) : z = 0], \quad (4)$$

and so:

$$\mu_{\tilde{a}-\tilde{b}}(z) = \begin{cases} 0 & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases} \quad (5)$$

On the contrary, in the second case we obtain:

$$\mu_{\tilde{a}-\tilde{b}}(z) = \sup\{\min[\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)] : z = x - y\}. \quad (6)$$

EXAMPLE. *It is possible to calculate* $\tilde{z} = h(\tilde{a}, \tilde{b}) \equiv \tilde{a}^2 + \tilde{b} + \tilde{a}\tilde{b} + \tilde{a}$ *as* $\tilde{z} = g(\tilde{a}, \tilde{b}) \equiv (\tilde{a} + \tilde{b})(\tilde{a} + 1)$.

It is not correct, however, to calculate separately $\tilde{w} = \tilde{a} + \tilde{b}$ and $\tilde{y} = \tilde{a} + 1$, and successively the product $\tilde{w}\tilde{y}$ using the rule for the multiplication between noninteractive f.n.'s, because $\tilde{y}$ and $\tilde{w}$ are *linked* by $\tilde{y} = h(\tilde{w}) \equiv \tilde{w} - \tilde{b} + 1$.

PROPOSITION 3 *Given the f.n.* $\tilde{a}$, *there exists an unique f.n.* $\tilde{b}$, *linked with* $\tilde{a}$ *by* $\tilde{b} = g(\tilde{a}) \equiv -\tilde{a}$. *Its m.f. is given by:*

$$\mu_{\tilde{b}}(y) = \sup_x [\mu_{\tilde{a}}(x) : y = -x] = \mu_{\tilde{a}}(-y) \quad \textit{for any } y \in \mathbb{R}, \tag{7}$$

and the sum of $\tilde{a} = (c, a_s, a_d)$ *and* $\tilde{b} = (-c, a_d, a_s)$ *is the crisp number* $\tilde{0} = (0, 0, 0)$ *(i.e. fuzzy zero with zero spread).*

Let us recall first a simple order relation among f.n.'s.

DEFINITION 4 *Given two f.n.'s* $\tilde{a}$ *and* $\tilde{b}$ *we say that* $\tilde{a} \leq \tilde{b}$ *iff* $\tilde{a} = \min(\tilde{a}, \tilde{b})$ *([10] p. 188 and [4] p. 52). Likewise* $\tilde{a} \geq \tilde{b}$ *iff* $\tilde{a} = \max(\tilde{a}, \tilde{b})$.

The relation $\tilde{a} \leq \tilde{b}$ ($\tilde{a} \geq \tilde{b}$) among f.n.'s is a partial order.

PROPOSITION 4 *Given the f.n.* $\tilde{a}$, *there results* $\tilde{a} \leq \tilde{a} + \tilde{x}$ *for any f.n.* $\tilde{x} \geq 0$ *(fuzzy zero with zero spread).*

Such a characteristic, which holds for $\tilde{x}$ both non- and interactive with $\tilde{a}$, means that two f.n.'s $\tilde{a}$ and $\tilde{b} = \tilde{a} + \tilde{x}$, with $\tilde{x} \geq 0$ (fuzzy zero with zero spread), are always comparable in the relation "$\leq$" defined in def. 4.

EXAMPLE. *It results* $\tilde{a}^2 \geq 0$ *(fuzzy zero with zero spread) for any f.n.* $\tilde{a}$ *(Fig. 1). In other words the product* $\tilde{a}\tilde{b}$, *with* $\tilde{a}$ *and* $\tilde{b}$ *strictly equal, is a f.n. greater or equal to crisp number zero.*

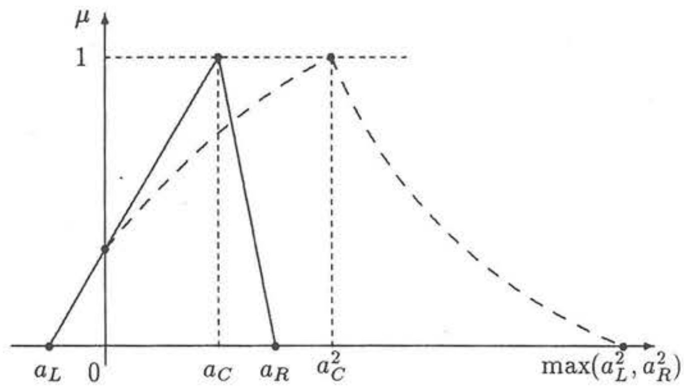

*Fig.*1. $\tilde{a}$ —— $\tilde{a}^2$ - -

PROPOSITION 5 *Let us assume $\tilde{z} = f(\tilde{x}, \tilde{p})$ as the fuzzy extension of the non-fuzzy function $f : \mathbb{R}^{n+m} \to \mathbb{R}$. Let us suppose again $\tilde{x} = g(\tilde{p})$, where g is a vector of functions, fuzzy extension of the non-fuzzy functions $g_i : \mathbb{R}^m \to \mathbb{R}$.*

Then the m.f. of $\tilde{z} = f(\tilde{x}, \tilde{p})$ is given by

$$\mu_{\tilde{z}}(y) = \begin{cases} \max_p\{\min[\mu_{\tilde{p}_1}(p_1), \ldots, \mu_{\tilde{p}_m}(p_m)] : y = f(g(p); p)\} \\ 0 \qquad\qquad\qquad\qquad \text{if } \{p : f(g(p), p) = y\} = \emptyset. \end{cases} \tag{8}$$

PROPOSITION 6 *Given a non fuzzy function $f : \mathbb{R}^n \to \mathbb{R}_+$ and its fuzzy extension $\tilde{z} = f(\tilde{x})$, where $\tilde{x}$ is a vector of f.n.'s, it turns out $\tilde{z} \geq 0$ (fuzzy zero with zero spread).*

In fact

$$\mu_{\tilde{z}}(y) = \begin{cases} \sup\{\min[\mu_{\tilde{x}_1}(x_1), \ldots, \mu_{\tilde{x}_n}(x_n)] : y = f(x)\} \\ 0 \qquad\qquad\qquad\qquad \text{if } f^{-1}(y) = \emptyset. \end{cases} \tag{9}$$

By assumption, it results $y \geq 0$ for any $x \in \mathbb{R}^n$, and $\mu_{\tilde{z}}(y) = 0$ if $y < 0$. Let us note, furthermore, that what stated in the example at the end of prop. 4, my be easily proved by prop. 6 setting $n = 1$ and $f(x) \equiv x^2$.

We note that the considerations developed above about calculus with fuzzy functions $\tilde{y} = g(\tilde{x})$, can be extended to other operators different from max and min. For example we can choose a T-norm (-conorm) [3, 6, 9] instead of min (max) operator, so characterizing a more general description of uncertain phenomena.

3. Fuzzy optimizing a quadratic-linear objective function

Let us set the following non-fuzzy unconstrained optimization problem:

$$\min_x z = f(x,b) \equiv \frac{1}{2}x'Ax + b'x, \tag{11}$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $f : \mathbb{R}^{n+m} \to \mathbb{R}$ and A is a symmetric and positive definite $n \times n$ matrix.

The minimization of (11) provides the minimizing vector

$$x_0 = -A^{-1}b \tag{12}$$

to which in objective function corresponds the minimum

$$z_0 = f(x_0, b) \equiv -\frac{1}{2}b'A^{-1}b. \tag{13}$$

Let us consider then

$$\min_{\tilde{x}} \tilde{z} = f(\tilde{x}, \tilde{b}) \equiv \frac{1}{2}\tilde{x}'A\tilde{x} + \tilde{b}'\tilde{x} \tag{14}$$

as a problem similar to (11) in a particular fuzzy case, i.e. assuming that $\tilde{b}$ and consequently $\tilde{x}$ are vectors of f.n.'s and $\tilde{z}$ is a f.n. (only A is a matrix of crisp number).

The purpose is to prove that the problem (14), containing the vector of f.n.'s $\tilde{b}$, has the same formal solution given by (12) and (13). Moreover, in such case, the computation of the minimizing vector $\tilde{x}_0$ is very easy (immediate if the components of $\tilde{b}$ are triangular f.n.'s), because it is a linear combination of the f.n.'s of vector $\tilde{b}$ with crisp coefficients. In fact we can prove the following

THEOREM. *Given any vector $\tilde{b}$ of noninteractive f.n.'s, setting $\tilde{z}_0 = -\frac{1}{2}\tilde{b}'A^{-1}\tilde{b}$ and $\tilde{z} = \frac{1}{2}\tilde{x}'A\tilde{x} + \tilde{b}'\tilde{x}$, it results $\tilde{z}_0 \leq \tilde{z}$, with respect to def. 4, for any vector of f.n.'s $\tilde{x}$. Therefore $\tilde{x} = -A^{-1}\tilde{b}$ is the minimizing vector solving (14).*

PROOF: Given the problem (14) it follows from prop. 2 that

$$\tilde{z} = f(\tilde{x}, \tilde{b}) \equiv \frac{1}{2}(\tilde{x} + A^{-1}\tilde{b})'A(\tilde{x} + A^{-1}\tilde{b}) - \frac{1}{2}\tilde{b}'A^{-1}\tilde{b}. \tag{15}$$

the first term of the sum in (15) is fuzzy quadratic form with symmetric and positive definite matrix A of non-f.n.'s. Consequently, from prop. 6, it results

$$\frac{1}{2}(\tilde{x} + A^{-1}\tilde{b})'A(\tilde{x} + A^{-1}\tilde{b}) \geq 0 \quad \text{(fuzzy zero with zero spread).}$$

Therefore, from prop. 4, the minimum value of $\tilde{z}$ is obtained at the vector of f.n.'s $\tilde{x}_0$ (if there exists one) that:

$$\tilde{x}_0 + A^{-1}\tilde{b} = 0 \quad \text{(vector of crisp numbers zero).} \tag{16}$$

From what we have explained in prop. 3, the minimizing vector $\tilde{x}_0$ exists and it is given by

$$\tilde{x}_0 = -A^{-1}\tilde{b}. \tag{17}$$

The f.n. $\tilde{z}_0$, corresponding to $\tilde{x}_0$ by (15), is

$$\tilde{z}_0 = -\frac{1}{2}\tilde{b}' A^{-1}\tilde{b}. \tag{18}$$

We note that:

- the computation of $\tilde{x}_0$ from (17) is not particularly hard, because it requires a linear combination of noninteractive f.n.'s with non-fuzzy coefficients;

- the computation of $\tilde{z}_0$ from (18) rises many problems, because we have to sum and multiply *linked* f.n.'s;

- if we read (12) as a function of the type $\tilde{x}_0 = g(\tilde{b})$, from prop. 5 we can directly compute the m.f. of $\tilde{z}_0$ by (8), obtaining

$$\mu_{\tilde{z}_0}(y) = \max_v\{\min[\mu_{\tilde{b}_1}(v_1), \ldots, \mu_{\tilde{b}_n}(v_n)] : y = -\frac{1}{2}v' A^{-1} v\}. \tag{19}$$

Considering that b is a vector of stochastic independent variables instead of a noninteractive f.n.'s, we obtain the probabilistic approach to problem (14). It is easy to verify that in latter case the vector x_0 is characterized by the following probability density:

$$f_{x_0}(a) = \int_C f_1(b_1) f_2(b_2) \ldots f_n(b_n) db,$$

where $C = [b \in \mathbb{R}^2 : a = -A^{-1}b]$, $a = [a_1, a_2, \ldots, a_n]$, $b = [b_1, b_2, \ldots, b_n]$ and f_i $(i = 1, 2, \ldots, n)$ is the probability density of i-th component of vector b.

4. Constrained optimization using penalty functions

In the study of fuzzy constrained optimization the approach generally used is due to Bellman and Zadeh [1], according to which the objective function is transformed into a constraint to be satisfied together with others.

In the present paper we follow a different approach introducing the non-satisfied constraints into the objective function. It is obvious that the latter method can be followed when the consequent unconstrained fuzzy optimization problem can be solved.

According to the previous section, the extension to the fuzzy case of the penalty function method, used for solving constrained optimization problems [7], becomes possible in presence of particular linear equality constraints.

Therefore, we consider the problem (14) subject to constraints

$$G\tilde{x} = \tilde{q} \tag{20}$$

where G is a $k \times n$ dimensional matrix of crisp numbers and $\tilde{q}$ is a vector of k noninteractive (neither between each others nor with $\tilde{b}$) f.n.'s.

Using the penalty function method we can reduce the problem (14)-(20) to the unconstrained case and so apply the results described in the previous section.

In this paper one of the simplest approaches to the penalty functions has been taken into account, as a possible suggestion for a resolution method. The use of more efficient methods – some of which are described in [7] chap. 12 – is highly possible and advisable.

Let us consider the following fuzzy penalty function:

$$J(\tilde{x}) = \frac{1}{2}\tilde{x}'A\tilde{x} + \tilde{b}'\tilde{x} + \frac{1}{2}(G\tilde{x} - \tilde{q})'S(G\tilde{x} - \tilde{q}) \tag{21}$$

where $S = diag(\sigma_i)$ is a $n \times n$ diagonal matrix of crisp numbers containing the penalties σ_i to be applied to the i-th constraint ($\sigma_i = 0$ if the constraint is satisfied).

According to the previous section we can write:

$$J(\tilde{x}) = \frac{1}{2}\tilde{x}'(A + G'SG)\tilde{x} + (\tilde{b} - G'S\tilde{q})'\tilde{x} + \frac{1}{2}\tilde{q}'S\tilde{q}. \tag{22}$$

The method, here suggested, for the solution of problem (14)-(20) consists in the minimization of (22) for sufficiently high values of σ_i. Some advices to overcome computational problems in the non-fuzzy case may be found in [2] and [7].

Let us note that the unconstrained fuzzy function $J(\tilde{x})$ in (22) is quadratic-linear as described in the section 3.

Therefore, the minimizing vector $\tilde{x}_0$ is given by

$$\tilde{x}_0 = -(A + G'SG)^{-1}(\tilde{b} - G'S\tilde{q}), \tag{23}$$

and it is computed as a linear combination of the vectors of f.n.'s $\tilde{b}$ and $\tilde{q}$ with non-fuzzy coefficients. Whereas the minimum value is

$$\tilde{z}_0 = -\frac{1}{2}(\tilde{b} - G'S\tilde{q})'(A + G'SG)^{-1}(\tilde{b} - G'S\tilde{q}) + \frac{1}{2}\tilde{q}'S\tilde{q}. \tag{24}$$

From (23) and (24) we can see that, if $S = 0$, we get (17) and (18), corresponding to unconstrained optimization case.

In order to explain further the above arguments, we present here a numeric and willingly simple example having $n = k = 1$ and symmetric triangular f.n.'s.

5. A numerical example

For homogeneity with section 2, a symmetric triangular f.n. is indicated by (c, a_s, a_d), where c is the mean value and a_s and a_d are left- and right-spreads (in this case equal).

Let us consider the problem (14) subject to

$$g\tilde{x} = \tilde{q} \tag{25}$$

with: $A = a = 3 = (3,0,0)$ $\tilde{b} = (1,2,2)$ $g = 4 = (4,0,0)$ $\tilde{q} = (5,1,1)$.

The (unconstrained) minimizing vector of problem (14), $\tilde{x}_0^0$, results:

$$\tilde{x}_0^0 = -(1/a)\tilde{b} = (-1/3, 2/3, 2/3) \tag{26}$$

which does not satisfy the constraint (25).

We introduce then the penalty function (22):

$$J(\tilde{x}) = 1/[2(3+16\sigma)]\tilde{x}^2 + (\tilde{b} - 4\sigma\tilde{q})\tilde{x} + \frac{1}{2}\sigma\tilde{q}^2, \tag{27}$$

and we compute the corresponding minimizing value (23)

$$\tilde{x}_0 = -1/(3+16\sigma)(\tilde{b} - 4\sigma\tilde{q}) \tag{28}$$

and the minimum value (24):

$$\tilde{z}_0 = -1/[2(3+16\sigma)](\tilde{b} - 4\sigma\tilde{q})^2 + \frac{1}{2}\sigma\tilde{q}^2. \tag{29}$$

Setting σ at the values $0, 1, 10, 100, 1000, 10000$ successively, we get the following values of the minimum $\tilde{z}_0^k$ and of the minimizing $\tilde{x}_0^k$:

$$\begin{array}{ll}
\tilde{z}_0^0 = (-0.167, 1.333, 0.167) & \tilde{x}_0^0 = (-0.333, 0.667, 0.667) \\
\tilde{z}_0^1 = (3.000, 2.605, 3.395) & \tilde{x}_0^1 = (1.000, 0.316, 0.316) \\
\tilde{z}_0^2 = (3.525, 3.037, 4.177) & \tilde{x}_0^2 = (1.221, 0.258, 0.258) \\
\tilde{z}_0^3 = (3.587, 3.088, 4.270) & \tilde{x}_0^3 = (1.247, 0.251, 0.251) \\
\tilde{z}_0^4 = (3.593, 3.093, 4.280) & \tilde{x}_0^4 = (1.250, 0.250, 0.250) \\
\tilde{z}_0^5 = (3.594, 3.094, 4.281) & \tilde{x}_0^5 = (1.250, 0.250, 0.250)
\end{array}$$

As $\sigma \to +\infty$, $\tilde{x}_0$ converges towards the symmetric triangular f.n. $(5/4, 1/4, 1/4)$, while $\tilde{z}_0$ goes to the non-symmetric and non-triangular f.n. $(3.594, 3.094, 4.281)$.

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Optymalizacja funkcji kwadratowej z rozmytymi współczynnikami liniowymi

Celem artykułu jest rozwiązanie szczególnego zadania rozmytego programowania kwadratowego, jako wstępnego kroku w rozmytej optymalizacji nieliniowej. Najpierw wprowadza się pewne własności dotyczące liczb rozmytych. Poza tym proponujemy możliwy sposób rozwiązania pewnych zadań optymalizacji z ograniczeniami przy pomocy funkcji kary.

Оптимизация квадратной фукции с размытыми линейными коэффициентами

Целью статьи является решение особой задачи размытой квадратной оптимизации, как предварительного шага к размытой нелинейной оптимизации. В первую очередь вводятся некоторые свойства, касающиеся размытых чисел и размытое расширение неразмытых функций. Кроме этого предлагается возможный способ решения некоторых задач оптимизации с ограничениями с помощью функции штрафа.