Control
and Cybernetics

VOL. 20 (1991) No. 3

Identification of a generalized dynamic system using the modulating element method

by

Hubert Wysocki

Marek Zellma

Department of Mathematics
Navy Academy
81 – 919 Gdynia, Poland

The paper presents identification technique for a generalized linear dynamic differential system by the modulating element method. The dynamic system is described in the Bittner operational calculus by means of an abstract linear diffrential equation with constant coefficients. Employing various models of the operational calculus we give the examples of applications of presented general method.

1. Introduction

In paper [2], which is a certain attempt at generalization of some problems from the theory of the linear dynamic systems S. Bellert has noticed that by application of the operational calculus "we obtain a more general theory of the linear dynamic systems, by what we avoid the necessity of elaborating the separate theories for various system types".

The idea of the operational calculus of Bellert has been largely developed and it has been formed as a compact mathematical theory by R. Bittner [3–5].

In this work an application of the Bittner operational calculus to identification of dynamic systems is presented. The notion of a generalized linear dynamic differential stationary system with compensating parameters is introduced. It may be any real system whose dynamics, after taking the proper model of the operational calculus, is described by the linear equation with constant coefficients

$$a_n S^n y + a_{n-1} S^{n-1} y + \ldots + a_1 S y + a_0 y = u, \tag{0}$$

where the linear operation S is the so called abstract derivative, with u and y denoting the input signal and the output signal of the system, respectively.

The notion of a modulating element is also introduced. An adequate functional, called identification quality index, is formed by means of the modulating element. The problem of choice of the best model of the form (0) describing the dynamics of the considered system is also solved. So, identification of the system (0) by determination of its optimal coefficients a_0^0, $a_1^0, \ldots, a_n^0$, i.e. the coefficients minimizing the identification quality index, is performed.

The general method presented stipulates that by defining different models of the operational calculus we obtain the possibility of identification of the systems whose dynamics is described by means of various types of differential equations.

In a particular case, when we consider the model with the ordinary derivative $S = \frac{d}{dt}$, we get the method of identification of the continuous systems by means of the modulating function. However, in the classical modulating function method [24, 16] the problem of choice of the best system model in the sense of finding of the best equation order by minimizing the identification quality index is not considered. This problem is treated in this article with the aid of the statistical test based on the decrease of the value of the identification quality index.

Using proper representations of the operational calculus identification of systems described, respectively, by an ordinary linear differential equation, an Euler differential equation and a quasi–linear partial differential equation is considered.

2. The operational calculus

The Bittner operational calculus, according to notation used e.g. in [5, p. 22], is the system

$$CO(L^0, L^1, S, T_q, s_q, q, Q),$$

where L^0 and L^1 are linear spaces over the field R^1 of real (or $\mathbb{C}$ of complex) numbers. The linear operation $S: L^1 \to L^0$ (denoted as $S \in L(L^1, L^0)$), called the (abstract) derivative, is a surjection. Moreover, Q is a nonempty arbitrary set of indexes q of the operations $T_q \in L(L^0, L^1)$ such that $ST_q w = w$, $w \in L^0$, called integrals, and of the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q Sx$, $x \in L^1$, called limit conditions.

By induction we define a sequence of spaces L^n, $n \in N$ such that

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}.$$

Then

$$\ldots \subset L^n \subset L^{n-1} \subset \ldots \subset L^1 \subset L^0$$

and

$$S^n(L^{m+n}) = L^m,$$

where

$$L(L^n, L^0) \ni S^n := \underbrace{S \circ S \circ \ldots S}_{n \text{ times}}, \quad n \in N, m \in N_0 := N \cup \{0\}.$$

The kernel of S, i.e. the set $\mathrm{Ker} S := \{c \in L^1 : Sc = 0\}$, is called the space of constants for the derivative S.

Assume that Q has at least two elements. The mapping $I_{q_1}^{q_2}: L^0 \to Ker\, S$ described by the formula

$$I_{q_1}^{q_2} w := (T_{q_1} - T_{q_2})w = s_{q_2} T_{q_1} w, \quad q_1, q_2 \in Q, w \in L^0 \tag{1}$$

is called the operation of definite integration.

Let L^0 be an algebra and L^1 its subalgebra. We say that the derivative S satisfies the Leibniz condition if

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy, \quad x, y \in L^1. \tag{2}$$

We say that the limit condition s_q, $q \in Q$ is multiplicative if

$$s_q(x \cdot y) = s_q x \cdot s_q y, \quad x, y \in L^1. \tag{3}$$

3. The system identification

For further discussion we shall assume that

- Q has more than one element
- L^0 is a real algebra (with commutative multiplication of an element by a number) and L^1 is its subalgebra
- the derivative S satisfies the Leibniz condition (2).

Let us define an operation $R_{q_1}^{q_2}: L^1 \to \operatorname{Ker} S$ by means of the form

$$R_{q_1}^{q_2} x := (s_{q_2} - s_{q_1})x, \quad q_1, q_2 \in Q, x \in L^1. \tag{4}$$

Later we shall use the following lemma:

LEMMA 1 *The following formula for integration by parts holds:*

$$I_{q_1}^{q_2}(x \cdot S^n y) = \sum_{i=0}^{n-1} (-1)^i R_{q_1}^{q_2}(S^i x \cdot S^{n-1-i} y) + (-1)^n I_{q_1}^{q_2}(S^n x \cdot y), \tag{5}$$

where $S^0 x := x$, $q_1, q_2 \in Q$, $x, y \in L^n$, $n \in N$.

PROOF. We will prove the formula (5) by the induction for $n \in N$. From the Leibniz formula (2), for $x, y \in L^1$ we obtain

$$T_{q_i}(x \cdot Sy) = T_{q_i} S(x \cdot y) - T_{q_i}(Sx \cdot y), \quad i = 1, 2.$$

Herefrom and from the axiom $T_q Sx = x - s_q x$ we have

$$T_{q_i}(x \cdot Sy) = x \cdot y - s_{q_i}(x \cdot y) - T_{q_i}(Sx \cdot y), \quad i = 1, 2. \tag{6}$$

From (6) we obtain

$$(T_{q_1} - T_{q_2})(x \cdot Sy) = (s_{q_2} - s_{q_1})(x \cdot y) - (T_{q_1} - T_{q_2})(Sx \cdot y).$$

Using (1) and (4) in the last equality we receive the formula (5) for $n = 1$. Assuming the validity of the formula (5) for a fixed $n \in N$, for $x, y \in L^{n+1} \subset L^n$ we have

$$\begin{aligned} I_{q_1}^{q_2}(x \cdot S^{n+1} y) &= I_{q_1}^{q_2}[x \cdot S^n(Sy)] = \\ &= \sum_{i=0}^{n-1} (-1)^i R_{q_1}^{q_2}(S^i x \cdot S^{n-i} y) + (-1)^n I_{q_1}^{q_2}(S^n x \cdot Sy) = \\ &= \sum_{i=0}^{n-1} (-1)^i R_{q_1}^{q_2}(S^i x \cdot S^{n-i} y) + (-1)^n [R_{q_1}^{q_2}(S^n x \cdot y) - I_{q_1}^{q_2}(S^{n+1} x \cdot y)] = \\ &= \sum_{i=0}^{n} (-1)^i R_{q_1}^{q_2}(S^i x \cdot S^{n-i} y) + (-1)^{n+1} I_{q_1}^{q_2}(S^{n+1} x \cdot y). \end{aligned}$$

An application of the induction principle finishes the proof for any $n \in N$. ■

Let us consider all real systems whose dynamics after taking the suitable models of the operational calculus, is described by

$$a_n S^n y + a_{n-1} S^{n-1} y + \ldots + a_1 S y + a_0 y = u, \tag{7}$$

where $a_i \in R^1$, $i \in \overline{0,n} := \{0,1,\ldots,n\}$, $u \in L^0$, $y \in L^n$.

Model (7) of these systems will be called the generalized linear dynamic differential stationary system with compensating constants. The given element u and the unknown element y will be called input signal (control) and output signal (response) of the system (7), respectively. The set Q (of indexes for integrals and limit conditions) will be called the set of moments.

Assume that the following pairs

$$(u_\nu, y_\nu) \in L^0 \times L^n, \quad \nu \in \overline{1,m}, \ m \geq n+1$$

satisfy the abstract differential equation (7) with given coefficients $a_0, a_1, \ldots, a_n$. Hence for any $f \in L^n$ we have

$$a_n f S^n y_\nu + a_{n-1} f S^{n-1} y_\nu + \ldots + a_1 f S y_\nu + a_0 f y_\nu = f u_\nu, \quad \nu \in \overline{1,m}. \tag{8}$$

Operating on both sides of every equation of system (8) by an operation of definite integration $I_{q_1}^{q_2}$ we obtain

$$\sum_{i=0}^{n} a_i I_{q_1}^{q_2}(f \cdot S^i y_\nu) = I_{q_1}^{q_2}(f \cdot u_\nu), \quad \nu \in \overline{1,m}. \tag{9}$$

Using formula (5) of integration by parts for every component of the system of equations (9), including derivative S, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} a_i \left[\sum_{j=0}^{i-1} (-1)^j R_{q_1}^{q_2}(S^j f \cdot S^{i-j-1} y_\nu) \right] + \\ &\quad + \sum_{i=0}^{n} (-1)^i a_i I_{q_1}^{q_2}(S^i f \cdot y_\nu) = I_{q_1}^{q_2}(f \cdot u_\nu), \quad \nu \in \overline{1,m}. \end{aligned} \tag{10}$$

Assume that the element $f \in L^n$ satisfies the following conditions

$$f \notin \operatorname{Ker} S^n, \quad s_{q_1} S^i f = 0, \quad s_{q_2} S^i f = 0, \quad i \in \overline{0, n-1} \tag{11}$$

and that the operations s_{q_1}, s_{q_2} satisfy the multiplication condition

$$s_{q_i}(x \cdot y) = s_{q_i} x \cdot s_{q_i} y, \quad i = 1,2, \quad x, y \in L^1.$$

The element $f \in L^n$ satisfying conditions (11) will be called the modulating element of the system (7) at the moments q_1, $q_2 \in Q$. With the above assumptions, we obtain from equations (10)

$$\sum_{i=0}^{n}(-1)^i a_i I_{q_1}^{q_2}(S^i f \cdot y_\nu) = I_{q_1}^{q_2}(f \cdot u_\nu), \quad \nu \in \overline{1,m}. \tag{12}$$

The system of equations (12) may be written in the form

$$\sum_{i=0}^{n} a_i \bar{v}_i = \bar{w},$$

where

$$\bar{v}_i := \begin{bmatrix} (-1)^i I_{q_1}^{q_2}(S^i f \cdot y_1) \\ \vdots \\ (-1)^i I_{q_1}^{q_2}(S^i f \cdot y_m) \end{bmatrix}, \quad \bar{w} := \begin{bmatrix} I_{q_1}^{q_2}(f \cdot u_1) \\ \vdots \\ I_{q_1}^{q_2}(f \cdot u_m) \end{bmatrix}, \quad i \in \overline{0,n}. \tag{13}$$

From definition of the operation $I_{q_1}^{q_2}$ it follows that

$$\bar{v}_i, \bar{w} \in (\operatorname{Ker} S)_m := \bigoplus_{\nu=1}^{m} \operatorname{Ker} S, \quad i \in \overline{0,n}. \tag{14}$$

where "$\oplus$" denotes the direct sum.

In this work by identification of a dynamic system (7) we shall understand the problem of choice of coefficients of equation (7) with given elements $(u_\nu^*, y_\nu^*) \in L^0 \times L^n$, $\nu \in \overline{1,m}$, $m \geq n+1$ so that for a certain modulating element $f \in L^n$ the functional (the so called identification quality index)

$$J_f(a_0, a_1, \ldots, a_n) := \|\sum_{i=0}^{n} a_i \bar{v}_\nu^* - \bar{w}^*\| \tag{15}$$

attains its minimum, where $\|\cdot\|$ is the norm induced by the scalar product $(\cdot|\cdot)$ in a fixed Hilbert space H and $\bar{w}^*, \bar{v}_i^* \in (\operatorname{Ker} S)_m$, $i \in \overline{0,n}$ are vectors of the form (13) determined for u_ν^*, y_ν^*, $\nu \in \overline{1,m}$.

In other reports of the present authors [33, 34] other methods for obtaining the system of the form (12) are considered. In [33] for identification of the dynamic system (7) the knowledge of at least $n+1$ modulating elements, one input signal and one output signal corresponding to it is required. In [34] we have to know only one modulating element and only one pair of input and output signals corresponding to each other. This has a special importance in cases when application of multiple measurements is expensive, troublesome or unfeasible.

Assume that the set $B := \{\bar{v}_0^*, \bar{v}_1^*, \dots, \bar{v}_n^*\}$ is a system of linearly independent vectors in H.

Let

$$\mathrm{Lin}\, B := \{\bar{w} = \sum_{i=0}^{n} a_i \bar{v}_i^* : a_i \in R^1, \bar{v}_i^* \in B, i \in \overline{0,n}\}.$$

Since $\dim \mathrm{Lin}\, B = n+1$, therefore $\mathrm{Lin}\, B$ as a finitely dimensional subspace of the normed space H is a closed set.

The problem we are concerned with is to find an element

$$\bar{w}^0 = a_0^0 \bar{v}_0^* + a_1^0 \bar{v}_1^* + \dots + a_n^0 \bar{v}_n^*$$

on the hyperplane $\mathrm{Lin}\, B$, which is situated the nearest (with respect to the norm $\|\cdot\|$) to the given element $\bar{w}^*$.

Hence we are to find such a set of real numbers $a_0^0, a_1^0, \dots, a_n^0$ that

$$\|\bar{w}^0 - \bar{w}^*\| = J_f(a_0^0, a_1^0, \dots, a_n^0) = \min\{J_f(a_0, a_1, \dots, a_n) : a_i \in R^1, i \in \overline{0,n}\}.$$

From the theorem on the orthogonal projection [17, Th.2,p.82] we infer the existence and uniqueness of the demanded vector $\bar{w}^0$ and the orthogonality of the difference $\bar{w}^0 - \bar{w}^*$ to every vector $\bar{v}_j^*$, $j \in \overline{0,n}$ in the basis B. Hence we have

$$(\bar{w}^0 - \bar{w}^* | \bar{v}_j^*) = \left(\sum_{i=0}^{n} a_i^0 \bar{v}_i^* - \bar{w}^* | \bar{v}_j^* \right) = 0, \quad j \in \overline{0,n}.$$

Herefrom and from the properties of the scalar product we obtain the system of linear equations for optimal coefficients $a_0^0, a_1^0, \dots, a_n^0$ of the equation (7):

$$\sum_{i=0}^{n} a_i^0 (\bar{v}_i^* | \bar{v}_j^*) = (\bar{w}^* | \bar{v}_j^*), \quad j \in \overline{0,n} \tag{16}$$

which is a Cramer system as the matrix $[(\bar{v}_i^* | \bar{v}_j^*)]_{(n+1)\times(n+1)}$ is non-singular (this is equivalent to the linear independence of vectors $\bar{v}_0^*, \bar{v}_1^*, \dots, \bar{v}_n^*$).

For further purposes it is convenient to represent system (16) in the form

$$\sum_{i=0}^{n} a_i^0 b_{ij} = c_j, \quad j \in \overline{0,n}, \tag{17}$$

where

$$b_{ij} = (\bar{v}_i^* | \bar{v}_j^*), \quad c_j = (\bar{w}^* | \bar{v}_j^*), \quad i,j \in \overline{0,n}. \tag{18}$$

4. Examples

A. We are given operational calculus in which

$$L^n := C^n([t_0, t_k], R^1), \quad n \in N_0$$

and

$$S := \frac{d}{dt}, \quad T_q := \int_q^t, \quad s_q := |_{t=q}, \quad q \in Q := [t_0, t_k] \subset R^1.$$

With the usual multiplication of functions, spaces L^n are such algebras that $L^n \subset L^{n-1}$, $n \in N$ whereas the derivative S satisfies the Leibniz condition and the operations s_q, $q \in Q$ are multiplicative.

Since Ker $\frac{d}{dt}$ is the space of constant functions in the interval $[t_0, t_k]$, isomorphic with R^1, so as a Hilbert space H we may admit the real space l^2_m with the scalar product

$$(\bar{a}|\bar{b}) := \sum_{\nu=1}^{m} a_\nu b_\nu, \quad \bar{a}, \bar{b} \in l^2_m \tag{19}$$

and the norm

$$\bar{a} = \sqrt{\sum_{\nu=1}^{m} a_\nu^2}, \quad \bar{a} \in l^2_m.$$

induced by it. In the operational calculus considered the dynamic system (7) is described by means of an ordinary linear differential equation of order n with constant coefficients

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) = u(t), \tag{20}$$

where $u = u(t)$ is the input signal and $y = y(t)$ is the output signal of the system to be identified.

The algorithm of identification of the system (20) comprises:

1. The algorithm of approximation of the input and output signals,
2. The algorithm of choice of the coefficients of the differential equation,
3. The algorithm of verification of the model.

Ad 1.

Basing on the values of the input signals

$$\tilde{u}_{\nu,0} = \tilde{u}_\nu(t_0), \ldots, \tilde{u}_{\nu,k} = \tilde{u}_\nu(t_k), \quad \nu \in \overline{1,m}$$

and the output signals

$$\tilde{y}_{\nu,0} = \tilde{y}_\nu(t_0), \ldots, \tilde{y}_{\nu,k} = \tilde{y}_\nu(t_k), \quad \nu \in \overline{1,m}$$

obtained from measurements on the real system in the moments $t_0, \ldots, t_k$ the functions $u_\nu^* = u_\nu^*(t)$, $y_\nu^* = y_\nu^*(t)$ are determined by approximation so that

$$u_\nu^* \in C^0([t_0, t_k], R^1), \quad y_\nu^* \in C^n([t_0, t_k], R^1), \quad \nu \in \overline{1,m}.$$

These functions are used in calculation of the integrals (21). It may be advantageous to apply such approximation that the functions appearing in the formulas (21) be easily integrable. In [33, 34] for this purpose the interpolation of the input and output signals by means of so called basic splines of the third order [11, 25] is used.

Ad 2.

In the operational calculus considered, in the case when $q_1 = t_0$, $q_2 = t_k$, every function

$$f(t) \in C^n([t_0, t_k], R^1) - \text{Ker}\,\frac{d^n}{dt^n}$$

satisfying conditions

$$f^{(i)}(t_0) = f^{(i)}(t_k) = 0, \quad \nu \in \overline{0, n-1}$$

may be a modulating element.

Let $f(t)$ be the fixed modulating element. Moreover, let V_i^ν and W^ν denote the ν-th coordinates of the vectors $\bar{v}_i^*$ and $\bar{w}^*$, respectively. As $I_{q_1}^{q_2} = \int_{t_0}^{t_k}$, therefore from the form (13) of vectors $\bar{v}_i^*$, $\bar{w}^*$ we infer

$$\begin{aligned} V_i^\nu &= (-1)^i \int_{t_0}^{t_k} f^{(i)}(t) y_\nu^*(t) dt, \\ W^\nu &= \int_{t_0}^{t_k} f(t) u_\nu^*(t) dt, \quad i \in \overline{0,n}, \nu \in \overline{1,m}. \end{aligned} \tag{21}$$

Herefrom, from the scalar product (19) and on the basis of the formulas (18) we obtain the coefficients of the system of the equations (17):

$$b_{ij} = \sum_{\nu=1}^{m} V_i^\nu V_j^\nu, \qquad c_j = \sum_{\nu=1}^{m} W^\nu V_j^\nu, \quad i,j \in \overline{0,n}.$$

Solving the system (17) we get the following model of the dynamic system (20):

$$a_n^0 y^{(n)}(t) + a_{n-1}^0 y^{(n-1)}(t) + \cdots + a_0^0 y(t) = u(t), \tag{22}$$

where $a_0^0, \ldots, a_n^0$ are the optimal coefficients.

When calculating the integrals (21) by means of the Newton–Cotes quadratures [11] we need not make prior approximation of the input and output signals, because we use the Lagrange interpolation at once in these quadratures.

Ad 3.

The elaborated method of identification can be evaluated by means of the value of the functional (15) in the optimal point $(a_0^0, a_1^0, \ldots, a_n^0)$. In the considered operational calculus we have

$$J_f(a_0^0, a_1^0, \ldots, a_n^0) = \sqrt{\sum_{\nu=1}^{m} \left(\sum_{i=0}^{n} a_i^0 V_i^\nu - W^\nu \right)^2}. \tag{23}$$

Instead of (20) we consider now the equation

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) = u(t) + z(t). \tag{24}$$

Assume also that the noises z_ν, so called "equation errors", correspond to the signals u_ν, $\nu \in \overline{1,m}$, $m \geq n+1$. These noises are mean–square continuous on the interval $[t_0, t_k]$ and noncorrelated normal processes with the same expected value $m(t) = 0$ and the same (but unknown) autocorrelation function $K(t, \tau)$.

Let

$$\bar{r}_f^n := \sum_{i=0}^{n} a_i^0 \bar{v}_i^* - \bar{w}^*. \tag{25}$$

From the above and (24) it follows that the ν–th coordinates of the vector (25) has the form of the Riemann mean–square integral

$$r_{f,\nu}^n = \int_{t_0}^{t_k} f(t) z_\nu(t) dt, \quad \nu \in \overline{1,m}. \tag{26}$$

Random variables (26) have normal distribution [7, 12]. Because [26]

$$\operatorname{cov}(r_{f,\mu}^n, r_{f,\nu}^n) = \int_{t_0}^{t_k} \int_{t_0}^{t_k} f(t) f(\tau) K_{\mu,\nu}(t,\tau) dt d\tau = 0,$$

where $K_{\mu,\nu}(t,\tau) = 0$ is the crosscorrelation function of the processes $z_\mu(t), z_\nu(t)$, $\mu \neq \nu, \mu, \nu \in \overline{1,m}$, so $r_{f,\nu}^n, \nu \in \overline{1,m}$ are the independent random variables.

They have the same zero expected value and the same (but unknown) variance of the form [26]

$$\sigma^2 = \int_{t_0}^{t_k} \int_{t_0}^{t_k} f(t) f(\tau) K(t,\tau) dt d\tau.$$

Therefore $r^n_{f,\nu} \sim N(0,\sigma)$, $\nu \in \overline{1,m}$ and the random variable

$$\frac{1}{\sigma^2}R^n_f := \frac{1}{\sigma^2}(\bar{r}^n_f)^t \cdot \bar{r}^n_f = \frac{1}{\sigma^2}J^2_f(a^0_0, a^0_1, \dots, a^0_n), \tag{27}$$

where "t" denotes transposition, has the chi–square distribution with m–n degrees of freedom [9, 23].

It is not difficult to prove that the increase of the order of the equation (20) by one may possibly improve the exactness of identification, i.e. lower the value of the identification quality index (23) (cf. [31]).

Therefore if $n_1 < n_2$, then $R^{n_1}_f \geq R^{n_2}_f$ and the random variable

$$\frac{1}{\sigma^2}(R^{n_1}_f - R^{n_2}_f)$$

has the chi–square distribution with $(m - n_1) - (m - n_2) = n_2 - n_1$ degrees of freedom, but the random variables

$$\frac{1}{\sigma^2}(R^{n_1}_f - R^{n_2}_f), \quad \frac{1}{\sigma^2}R^{n_2}_f$$

are independent [23].

Due to this random variable

$$\rho_f = \frac{R^{n_1}_f - R^{n_2}_f}{R^{n_2}_f} \cdot \frac{m - n_2}{n_2 - n_1}$$

has the Snedecor's $F(k,l)$ distribution with $k = n_2 - n_1$, $l = m - n_2$ degrees of freedom.

This allows us to verify (with the same modulating function f), whether reduction of the identification quality index (23) with the increase of the order of the differential equation (20) from n_1 to n_2 (cf. [1, 10]) is significant.

When increasing the order n_1 of the equation (20) by one it is necessary to use at least $m = n_2 + 1 = n_1 + 2$ pairs (u^*_ν, y^*_ν) of input and output signals. For that case, for the level of significance $\alpha = 0.05$ we get $F(1,1) = 161$, or

$$P(\rho_f \geq 161) = 0.05.$$

Hence, if $n = n_1$ then the value ρ_f should be less than 161 with probability 0.95. In the opposite case it is necessary to use the model given by the differential equation of $n_2 = n_1 + 1$ order.

Assuming that the modulating function $f = f(t)$ has the constant sign in the interval $[t_0, t_k]$ and making use of the well-known inequality

$$\left| \frac{1}{m}\sum_{\nu=1}^{m} b_\nu \right| \leq \sqrt{\frac{1}{m}\sum_{\nu=1}^{m} b^2_\nu},$$

on the basis of (27) and (26), we obtain

$$\frac{1}{\sqrt{m}} K_f \left| \sum_{\nu=1}^{m} z_\nu(c_\nu) \right| \leq J_f(a_0^0, a_1^0, \ldots, a_n^0) \leq K_f \sum_{\nu=1}^{m} |z_\nu(c_\nu)|, \tag{28}$$

where $c_\nu, \nu \in \overline{1,m}$ are some fixed numbers from the interval $[t_0, t_k]$ and

$$K_f := \left| \int_{t_0}^{t_k} f(t)dt \right|. \tag{29}$$

Moreover, if $\lambda_\nu, \Lambda_\nu, \nu \in \overline{1,m}$ are such random variables that

$$P(\bigwedge_{t\in[t_o,t_k]} \lambda_\nu \leq z_\nu(t) \leq \Lambda_\nu) = 1,$$

then from (28) we obtain

$$\frac{1}{\sqrt{m}} K_f \left| \sum_{\nu=1}^{m} \lambda_\nu \right| \leq J_f(a_0^0, a_1^0, \ldots, a_n^0) \leq K_f \sum_{\nu=1}^{m} |\Lambda_\nu|, \tag{30}$$

but the inequalities (30) take place with the probability one. The smaller the number K_f given by the formula (29) the shorter the interval (30) covering the identification quality index (23).

The exactness of the elaborated identification method characterizes in an essential manner also the relative errors

$$\mathcal{E}_{f,\nu} y(t_i) = \left| \frac{\tilde{y}_\nu(t_i) - y_\nu(t_i)}{\tilde{y}_\nu(t_i)} \right| \cdot 100\%, \quad i \in \overline{0,k}, \nu \in \overline{1,m}, \tag{31}$$

where $\tilde{y}_{\nu,i} = \tilde{y}_\nu(t_i)$ denote the measured values of the output signals and $y_\nu(t_i)$ are the output signals obtained from the model (20) as the responses of the system to the input signals $u_\nu(t_i) = u_\nu^*(t_i)$ in the time moments t_i.

It is obvious that the expression (31) has no sense for $\tilde{y}_{\nu,i} = 0$. In order to determine the relative errors (31) we must first solve the initial value problems

$$a_n^0 y_\nu^{(n)}(t) + a_{n-1}^0 y_\nu^{(n-1)}(t) + \cdots + a_0^0 y_\nu(t) = u_\nu^*(t)$$

$$y_\nu(t_0) = \tilde{y}_{\nu,0}, \quad y_\nu^{(i)}(t_0) = \frac{d^i y_\nu^*(t_0)}{dt^i}, \quad \nu \in \overline{1,m}, i \in \overline{1,n-1}.$$

The numerical examples given below concern computation of the optimal coefficients of differential equation with various modulating functions. To this end it is assumed that the general form of differential equation describing the dynamic system and the approximated input signals u_ν^* and output signals y_ν^*

are given. Next, according to the foregoing algorithm, the program[1] for choice of the coefficients and for verification of the obtained model for various modulating functions was elaborated. For the numerical calculation of integrals and for solution of the systems of algebraic linear equations the Simpson method and the Cramer method ($n = 1$) or the Gauss elimination ($n > 1$) were applied, respectively. This program was used for the differential equations of the first and the third order. The results are shown in Table 1 and Table 2. These tables, apart from the form of the differential equation, contain the forms of u_ν^* and y_ν^* of the approximated input and output signals, the general forms of modulating functions, the beginning t_0 and the end t_k of the observation interval $[t_0, t_k]$, the number k denoting the number of parts into which the integration interval $[t_0, t_k]$ was divided for the Simpson method, the values of optimal coefficients a_i^0, the identification quality index J_f and the number K_f. Moreover, Table 1 includes the values of approximated functions y_ν^* and functions y_ν obtained from the model in time moments $t_i = 0,2i$, $i \in \overline{0,5}$ and the relative errors

$$\mathcal{E}_{f,\nu} y(t_i) = \left| \frac{y_\nu^*(t_i) - y_\nu(t_i)}{y_\nu^*(t_i)} \right| \cdot 100\%,$$

which do not exceed 1.42%.

The general method of identification of the dynamic system by means of the modulating element, described here, may be used for identification of certain nonstationary systems and systems with distributed parameters.

B. Assume that we are to identify the coefficients $b_0, b_1, \ldots, b_n$ of the system described by the nonhomogeneous Euler differential equation of order n:

$$b_n t^n y^{(n)}(t) + b_{n-1} t^{n-1} y^{(n-1)}(t) + \cdots + b_1 t y'(t) + b_0 y(t) = u(t). \tag{32}$$

To this end we have to write equation (32) in the form (7). In this connection let us take into consideration the model of operational calculus in which the algebras L^n, $n \in N_0$ are determined as in Example A with $0 < t_0 < t_k$. Moreover, let

$$Sx := t\frac{dx}{dt}, \quad T_q w := \int_q^t \frac{w(\tau)}{\tau} d\tau, \quad s_q x := x(q),$$

where $w = w(t) \in L^0$, $x = x(t) \in L^1$, $q \in Q := [t_0, t_k]$.

[1]For all the numerical examples in this paper, there exist programs in Basic 1.1 for Amstrad CPC 6128.

$a_1y' + a_0y = u$		$u_1^* = t$		$y_1^* = 1.2e^{-t} + t - 0.99$	
$t_0 = 0, \quad t_k = 1$		$u_2^* = \sin t$		$y_2^* = 0.5(e^{-t} + \sin t - \cos t)$	
$k = 128$		$f(t) = (t - t_0)(t - t_k)$		$f(t) = (t - t_k)\sin(t - t_0)$	
a_0^0		0.941761543		0.942082502	
a_1^0		1.01938683		1.01897995	
t	y_1^*	y_1	$\mathcal{E}_1 y$	y_1	$\mathcal{E}_1 y$
0.0	0.21	0.21	0	0.21	0
0.2	0.192476904	0.193037091	0.291041141	0.193019772	0.282043183
0.4	0.214384055	0.214763384	0.17693901	0.214745529	0.168610487
0.6	0.268573964	0.268651827	2.89913E − 02	0.268642986	2.56994E − 02
0.8	0.349194757	0.349276514	0.023413033	0.34928097	2.46891E − 02
1.0	0.45145533	0.452126916	0.148760238	0.452145187	0.152807353
t	y_2^*	y_2	$\mathcal{E}_2 y$	y_2	$\mathcal{E}_2 y$
0.0	0	0	−	0	−
0.2	1.86668E − 02	1.84021E − 02	1.41802559	1.84086E − 02	1.38320441
0.4	6.93387E − 02	6.86742E − 02	0.958339276	6.86957E − 02	0.927332077
0.6	0.144059248	0.143311721	0.51890247	0.143350853	0.491738632
0.8	0.234989173	0.23476664	9.46993E − 02	0.23482169	7.12726E − 02
1.0	0.33452406	0.335591898	0.3192111	0.335657979	0.338964855
K_f		0.166666667		0.158529015	
J_f		2.68324E − 10		2.11879E − 10	

Table 1. Identification of an ordinary differential equation of the first order; true values of the parameters are $a_0 = 1$, $a_1 = 1$.

	$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = u$					
$t_0 = 0$ $t_k = 3$ $k = 128$	$u_1^* = 1 \quad y_1^* = 2e^{-t} - 0.3e^{-2t} - 2.6e^{-0.5t} + 1$ $u_2^* = t \quad y_2^* = -2e^{-t} + 0.2e^{-2t} + 5.3e^{-0.5t} + t - 3.5$ $u_3^* = \sin t \quad y_3^* = -0.5e^{-t} - 0.1e^{-2t} + 0.8e^{-0.5t} - 0.2(\sin t + \cos t)$ $u_4^* = t^2 \quad y_4^* = 10e^{-t} + 20.5e^{-2t} - 48e^{-0.5t} + t^2 - 7t + 17.5$					
f	a_0^0	a_1^0	a_2^0	a_3^0	K_f	J_f
$[(e^{t-t_0} - 1)(t - t_k)]^3$	0.999940742	3.49986087	3.49995092	0.999956023	309.544901	2.28151E − 05
$[(t - t_0)(t - t_k)]^3$	0.999942334	3.50013854	3.49990281	1.00001798	15.621428	1.82995E − 05
$[(t - t_k)\sin(t - t_0)]^3$	0.999984559	3.50002321	3.49999136	0.999995593	5.29407461	2.30532E − 06
$[(t - t_0)\sin(t - t_k)]^3$	1.00000229	3.49999883	3.50000773	0.999997673	5.29407461	2.28405E − 06
$[\sin(t - t_0)\sin(t - t_k)]^3$	1.00000022	3.50000401	3.49999943	1.00000053	0.964180587	6.62586E − 08

Table 2. Identification of an ordinary differential equation of the third order; true values of the parameters are $a_0 = 1$, $a_1 = 3.5$, $a_2 = 3.5$, $a_3 = 1$.

It is easy to verify that the derivative S satisfies the Leibniz condition whereas the limit conditions $s_q, q \in Q$ are multiplicative. Using the mathematical induction we can prove that

$$t^r x^{(r)} = S(S-I)(S-2I)\dots(S-(r-1)\cdot I)x, \tag{33}$$

where $Ix := x$, $x = x(t) \in L^r$, $r \in N$ [5].

The equality (33) can be rewritten in the form

$$t^r x^{(r)} = (S^r + \alpha_1 S^{r-1} + \dots + \alpha_{r-1} S)x. \tag{34}$$

The coefficients $\alpha_1, \alpha_2, \dots, \alpha_{r-1}$ of the expression (34) were determined using Vietá formulas [20].

Namely, for the polynomial

$$w(z) = z^r + \alpha_1 z^{r-1} + \dots + \alpha_{r-1} z$$

of real or complex variable z, which have roots $0, 1, \dots, r-1$, we have

$$\alpha_j = (-1)^j \sum_{\substack{i_1, i_2, \dots, i_j = 1 \\ i_1 < i_2 < \dots < i_j}}^{r-1} i_1 i_2 \dots i_j, \quad j \in \overline{1, r-1}. \tag{35}$$

Obviously, the operation

$$w(S) = S^r + \alpha_1 S^{r-1} + \dots + \alpha_{r-1} S$$

corresponds (one–to–one) to the polynomial $w(z)$.

Taking the formulas (35) and (34) into account we reduce the Euler equation (32) into the operational form

$$a_n S^n y + a_{n-1} S^{n-1} y + \dots + a_1 S y + a_0 y = u \tag{36}$$

in which the coefficients a_i are represented by the coefficients b_i in such a way that if we know the coefficients a_i we can determine the coefficients b_i.
For example, for the Euler equation of the second order

$$b_2 t^2 y''(t) + b_1 t y'(t) + b_0 y(t) = u(t) \tag{37}$$

which corresponds to the operational equation

$$a_2 S^2 y + a_1 S y + a_0 y = u \tag{38}$$

we have

$$b_0 = a_0, \quad b_1 = a_1 + a_2, \quad b_2 = a_2.$$

Hence the problem of identification of the dynamic system described by the Euler equation (32) may be reduced to the problem of identification of the equation (36) with the derivative $S = t\frac{d}{dt}$. As a Hilbert space H we may take the space l_m^2 considered in the Example A, because also in this case $\operatorname{Ker} t\frac{d}{dt} \simeq R^1$. The modulating element (with $q_1 = t_0, q_2 = t_k$) may be an arbitrary function

$$f(t) \in C^n([t_0, t_k], R^1) - \operatorname{Ker}\left(t\frac{d}{dt}\right)^n$$

satisfying the conditions

$$f^{(i)}(t_0) = f^{(i)}(t_k) = 0, \quad i \in \overline{0, n-1}.$$

In the discussed model of the operational calculus the ν–th coordinates of the vectors $\bar{v}_i^*$ and $\bar{w}^*$ are represented by the formulas

$$\begin{aligned} V_i^\nu &= (-1)^i \int_{t_0}^{t_k} \left[\left(t\frac{d}{dt}\right)^i f(t)\right] \frac{y_\nu^*(t)}{t} dt, \\ W^\nu &= \int_{t_0}^{t_k} \frac{f(t)u_\nu^*(t)}{t} dt, \quad i \in \overline{0, n}, \nu \in \overline{1, m}. \end{aligned}$$

Besides this, the algorithm of identification of the system (36) is similar to the algorithm in Example A.

The results of identification of equation (38) and its equivalent Euler equation (37) are shown in Table 3.

C. In the case of the operational calculus with the derivative

$$Sx(z,t) := \frac{\partial x(z,t)}{\partial z} + \frac{\partial x(z,t)}{\partial t},$$

the integrals

$$T_q w(z,t) := \int_q^t w(z - t + \tau, \tau) d\tau$$

and the limit conditions

$$s_q x(z,t) := x(z - t + q, q),$$

where

$$q \in Q := [t_0, t_k], \quad w = w(z,t) \in L^0 := C^1(R^1 \times [t_0, t_k], R^1),$$

$$x = x(z,t) \in L^1 = \{x \in L^0 : Sx \in L^0\}$$

$t_0 = 1$ $t_k = 3$ $k = 128$	$S = t\frac{d}{dt}$, $a_2S^2y + a_1Sy + a_0y = u \longleftrightarrow b_2t^2y'' + b_1ty' + b_0y = u$ $u_1^* = \ln t$ $y_1^* = -0.08t^{-2} + 0.33t - 0.5\ln t - 0.25$ $u_2^* = t^{-1}$ $y_2^* = 0.33t^{-2} + 0.17t - 0.5t^{-1}$ $u_3^* = t^2$ $y_3^* = 0.08t^{-2} - 0.33t + 0.25t^2$			
f	a_0^0 b_0^0	a_1^0 b_1^0	a_2^0 b_2^0	J_f
$[\ln \frac{t}{t_0} \cdot \ln \frac{t}{t_k}]^2$	−1.99996276 −1.99996276	0.999983003 1.999985463	1.00000246 1.00000246	4.83354E − 09
$[(t - t_0)(t - t_k)]^2$	−2.00000075 −2.00000075	1.00000029 2.000000158	0.999999868 0.999999868	1.69903E − 09

Table 3. Identification of an Euler differential equation of the second order; true values of the parameters are $a_0 = b_0 = -2$, $a_1 = 1$, $b_1 = 2$, $a_2 = b_2 = 1$.

we may identify the coefficients of the system described by the partial quasi–linear differential equation

$$a_1\left(\frac{\partial y(z,t)}{\partial z}+\frac{\partial y(z,t)}{\partial t}\right)+a_0 y(z,t)=u(z,t) \tag{39}$$

which may be written in the operational form

$$a_1 Sy + a_0 y = u.$$

It is easy to verify that with usual multiplication of functions of two variables the derivative S satisfies the Leibniz condition whereas the limit conditions $s_q, q \in Q$ are multiplicative. Moreover, we have

$$\operatorname{Ker} S = \{c(z,t) \in L^1 : c(z,t) = \varphi(z-t), \quad \varphi \in C^2(R^1, R^1)\}.$$

Notice that an arbitrary function $c(z,t) \in \operatorname{Ker} S$, considered only in rectangle $P = [z_0, z_k] \times [t_0, t_k]$, may be treated as an element belonging to the Hilbert space $H_1 = L^2(P, R^1)$. In this connection, identifying the system (39) in P, for the Hilbert space H may take the direct sum $H = \bigoplus_{\nu=1}^{m} H_1$, which follows from (14).

In this space we define the scalar product

$$(\bar{\alpha}|\bar{\beta}) := \sum_{\nu=1}^{m} \int_{t_0}^{t_k} \int_{z_0}^{z_k} \alpha_\nu(z,t)\beta_\nu(z,t)\, dzdt, \quad \bar{\alpha}, \bar{\beta} \in H \tag{40}$$

with the introduced norm

$$||\bar{\alpha}|| = \sqrt{\sum_{\nu=1}^{m} \int_{t_0}^{t_k} \int_{z_0}^{z_k} [\alpha_\nu(z,z)]^2\, dzdt}, \quad \bar{\alpha} \in H.$$

From the form of the limit conditions in this operational calculus it follows that the modulating element (with $q_1 = t_0, q_2 = t_k$) may be a function of variable t such that

$$f(t) \in C^2([t_0, t_k], R^1) - \operatorname{Ker} \frac{d}{dt}, \quad f(t_0) = f(t_k) = 0.$$

Let. $u_\nu^* = u_\nu^*(z,t)$, $y_\nu^* = y_\nu^*(z,t)$, $\nu \in \overline{1,m}$ be the approximated input and output signals of the system (39) in the rectangle P, on the basis of which we shall identify its coefficients a_0 and a_1. From the definitions of the integrals and limit conditions it follows that the ν–th coordinates of vectors $\bar{v}_i^*, \bar{w}^*$ are the functions of variables z, t and have the form

$$\begin{aligned} V_i^\nu(z,t) &= (-1)^i \int_{t_0}^{t_k} f^{(i)}(\tau) y_\nu^*(z-t+\tau,\tau)d\tau \\ W^\nu(z,t) &= \int_{t_0}^{t_k} f(\tau)u_\nu^*(z-t+\tau,\tau)d\tau, \quad i=0,1,\nu \in \overline{1,m}. \end{aligned} \tag{41}$$

Herefrom and from the form (40) of the scalar product, on the basis of (18) we obtain the formulas for the coefficients of the system of equations (17):

$$\begin{aligned} b_{ij} &= \sum_{\nu=1}^{m} \int_{t_0}^{t_k} \int_{z_0}^{z_k} V_i^\nu(z,t)V_j^\nu(z,t)dzdt, \\ c_j &= \sum_{\nu=1}^{m} \int_{t_0}^{t_k} \int_{z_0}^{z_k} W^\nu(z,t)V_j^\nu(z,t)dzdt, \quad i,j=0,1. \end{aligned} \tag{42}$$

The value of the identification quality index J_f is calculated from the formula

$$\begin{aligned} J_f(a_0^0,a_1^0) = \|a_0^0\bar{v}_0^* + a_1^0\bar{v}_1^* - \bar{w}^*\| &= \\ = \left(\sum_{\nu=1}^{m}\left[\sum_{i,j=0}^{1} a_i^0 a_j^0 (V_i^\nu|V_j^\nu) - 2\sum_{i=0}^{1} a_i^0 (W^\nu|V_i^\nu) + (W^\nu|W^\nu)\right]\right)^{\frac{1}{2}}, \end{aligned} \tag{43}$$

where $(\cdot|\cdot)$ is the scalar product in H_1 with the form

$$(\alpha|\beta) = \int_{t_0}^{t_k}\int_{z_0}^{z_k} \alpha(z,t)\beta(z,t)dzdt, \quad \alpha,\beta \in H_1$$

whereas the functions V_i^ν, W^ν are defined by formulas (41).

We assume $z_0 = t_0$, $z_k = t_k$ in the numerical example whose results are shown in Table 4. In this case the values of definite integrals in formulas (42) and (43) given in the form

$$K = \int_{t_0}^{t_k}\int_{t_0}^{t_k}\left[\int_{t_0}^{t_k} \gamma(z,t,\tau)d\tau \cdot \int_{t_0}^{t_k} \sigma(z,t,\tau)d\tau\right] dzdt$$

are approximated by

$$K \approx \sum_{i,j=1}^{k}\left[\sum_{l=1}^{k} \gamma(\xi_i,\xi_j,\xi_l) \cdot \sum_{l=1}^{k} \sigma(\xi_i,\xi_j,\xi_l)\right] h^4,$$

where

$$h = \frac{t_k - t_0}{k}, \quad \xi_i = \frac{t_{i-1}+t_i}{2}, \quad t_j = t_0 + jh, \quad i \in \overline{1,k}, \quad j \in \overline{0,k}.$$

$z_0 = t_0 = 0$	$a_1(\frac{\partial y}{\partial z} + \frac{\partial y}{\partial t}) + a_0 y = u$		
$z_k = t_k = 1$	$u_1^* = 1.9z^2 - 3.1z^2t + 4.2zt$		$y_1^* = z^2t$
$k = 10$	$u_2^* = 8.9z - 12.2t + 2.1$		$y_2^* = -2.9z + 3.8t$
f	a_0^0	a_1^0	J_f
$(t-t_0)(t-t_k)$	-3.06736518	2.0436578	$6.12507\mathrm{E}-03$
$\sin(t-t_0)\sin(t-t_k)$	-3.06728641	2.03768893	$5.51663\mathrm{E}-03$

Table 4. Identification of a partial differential equation of the first order; true values of the parameters are $a_0 = -3$, $a_1 = 2$.

The disscused example of identification may be generalized easily to the case of the system with distributed parameters described by means of a partial differential equation

$$a_1\left(b_1\frac{\partial y}{\partial t_1} + b_2\frac{\partial y}{\partial t_2} + \cdots + b_r\frac{\partial y}{\partial t_r}\right) + a_0 y = u$$

using the operational calculus with the derivatives $S = \sum_{i=1}^{r} b_i \frac{\partial}{\partial t_i}$ [6].

5. Conclusions

The application of the operational calculus made it possible to elaborate the method of the modulating element in the algebraic approach with the use of the elements of the functional analysis. The method is so general that it may be used for identification of the continuous dynamic systems with compensating constants and also for some of the systems with distributed parameters and the nonstationary systems (in the classical sense). From the examples A, B it follows that every dynamic stationary system in the classical sense is stationary in the operational sense, but not to the contrary. And from the example C it follows that some of the distributed parameter systems in the classical sense are the compensating constant systems in the operational sense.

The identification problem solved here is reduced to the choice of the best model from the given class of equations of the form (0). The choice of the best model depends on the determination of such values $a_0^0, a_1^0, \ldots, a_n^0$ of the coefficients $a_0, a_1, \ldots, a_n$ for which the identification quality index takes the minimum value. The values $a_0^0, a_1^0, \ldots, a_n^0$ define the best model only for the given identifying series (u_ν^*, y_ν^*), $\nu \in \overline{1,m}$, $m \geq n+1$ and for the given modulating element

f. Generally, for other identifying series or for other modulating elements we get other parameters of the best model. It is in this sense that we mean here the best model.

Due to this we can put the question: what ordering relation occurs between the numbers

$$J_f(a_0^0, a_1^0, \ldots, a_n^0) \quad \text{and} \quad J_F(A_0^0, A_1^0, \ldots, A_n^0),$$

where $f, F \in L^n$ are the modulating elements of the system (0) in the moments $q_1, q_2 \in Q$, with the same identifying series?
The problem of choice of the optimal modulating element is, in the present authors' opinion, rather difficult and for the present it is an open question, but in the case when $S = \frac{d}{dt}$ the formula (30) indicates a certain dependence of identification quality index on the modulating function. Moreover, for the modulating functions contained in Table 1 and Table 2 the following implication holds:

$$(K_f < K_F) \Rightarrow (J_f < J_F).$$

The idea of the discussed method of verification of the order of equation describing the dynamics of the system in the model of the operational calculus with the derivative $S = \frac{d}{dt}$ may be easily transformed to an arbitrary model of the operational calculus which satisfies the assumptions of the identification method elaborated in this paper.

Other methods of identification of the generalized linear dynamic differential systems by means of modulating elements are discussed in [30, 33, 34].

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Identyfikacja uogólnionego układu dynamicznego metodą elementu modulującego

W artykule opracowano metodę identyfikacji uogólnionego układu dynamicznego różniczkowego za pomocą elementu modulującego. Rozważany układ dynamiczny opisany jest w rachunku operatorów Bittnera abstrakcyjnym liniowym równaniem różniczkowym o stałych współczynnikach. Stosując różne modele rachunku operatorów podano przykłady zastosowań przedstawionej metody ogólnej.

Идентификациа обобщенной динамической системы методом модулирующего элемента

В статье разработан метод идентификации обобщенной дифференциальной динамической системы с помощью модулирующего элемента. Рассматриваемая динамическая система описана здесь в операционном исчислении Биттнера при помощи абстрактного линейного дифференциального уравнения с постоянными коэффициентами. Используя разные модели операционного исчисления указываются примеры применения представленного общего метода.