Control
and Cybernetics
VOL. 20 (1991) No. 3

Closed form solution for coupled Riccati equation occurring in open loop Nash discrete time linear-quadratic games

by

L. Jódar, E. Navarro, J. L. Morera

Departamento de Matemática Aplicada
Universidad Politécnica, Valencia
P.O. Box 22.012 Valencia, Spain

This paper proposes a non recursive solution for coupled Riccati equations appearing in open-loop Nash discrete time linear quadratic terms. By means of appropriate algebraic transformations and the use of Kronecker products a closed form solution of such coupled system is obtained.

1. Introduction

When noncooperative control problems are considered a game theoretic approach is necessary: each decision maker tries to optimize his own cost function which may be more or less conflicting with others. An equilibrium solution must be sought and Nash strategy is a first natural choice. In this case , a player not can improve his payoff by deviating unilaterally for his Nash strategies lead to coupled Riccati equations.

Open loop Nash strategy for discrete time linear quadratic game is considered here.Take a deterministic linear-quadratic nonzero sum discrete time game of p players:

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} B_i u_i(k), \tag{1.1}$$

where $x(k) \in \mathbb{R}^n, u_i(k) \in \mathbb{R}^{r_i}, B_i \in \mathbb{R}^{n\times r_i}$, for $l \leq i \leq p$, and A is an invertible matrix in $\mathbb{R}^{n\times n}$. The performance index associated with player "i" ($l \leq i \leq p$) is

$$J_i = \frac{1}{2}x(N)^T K_{if} x(N) + \frac{1}{2}\sum_{k=0}^{n-1}(x(k)^T Q_i x(k) + \sum_{j=1}^{p} u_j(k)^T R_{ij} u_j(k)) \tag{1.2}$$

where all the weighting matrices are symmetric and R_{ii}, for $l \leq i \leq p$, are positive definite.

If an open loop Nash solution is sought for this problem, the necessary conditions to be satisfied are ([1],[2]):

$$u_i(k) = -R_{ii}^{-1} B_i^T \Psi_i(k+1) \quad , \quad l \leq i \leq p \tag{1.3}$$

and the costate vectors $\Psi_i(k)$ must satisfy

$$\Psi_i(k) = Q_i x(k) + A^T \Psi_i(k+1), \quad \Psi_i(N) = K_{if} x(N); \quad l \leq i \leq p \tag{1.4}$$

When we introduce the linear transformations:

$$\Psi_i(k) = K_i(k) x(k), \quad l \leq i \leq p \tag{1.5}$$

and use (1.3), the system (1.1) becomes

$$x(k+1) = \{I + S_1K_1(k+1) + S_2K_2(k+1) + \ldots + S_pK_p(k+1)\}^{-1} Ax(k) \tag{1.6}$$

where I is the identity matrix of order n and

$$S_i = B_i R_{ii}^{-1} B_i^T, \quad l \leq i \leq p$$

Coupled Riccati equations are obtained by substituting (1.5) and (1.6) into (1.4), i.e.,

$$\begin{aligned} K_i(k) &= Q_i + A^T K_i(k+1)\{I + \sum_{i=1}^{p} S_i K_i(k+1)\}^{-1} A; \\ K_i(N) &= K_{if},. \quad l \leq i \leq p \end{aligned} \tag{1.7}$$

These equations could be solved recursively from the terminal conditions since for $k = N$ they are decoupled [3]. However, this means that $K_i(k)$ will have to be stored along the whole horizon. Apart from this inconvenience, computation of $K_i(k)$ involves the inversion of the lot of possibly ill conditioned matrices that may produce bad numerical results due to error accumulation. Furthermore, if the boundary conditions are changed then the recursive algorithm has to be started all over again. In such causes it is clear that a closed form solution of (1.6) would be more versatile, would avoid error accumulation and may be used to study stability properties.

In a recent paper [4], an explicit solution of the problem (1.6) for the case of $p = 2$, and when the state weighting matrices in player's cost are proportional to each other, i.e., $Q_2 = \alpha Q_1$, where α is a scalar, is given. The aim of the present paper is to find a computable closed form solution of the system (1.6) for a more general case that the one considered in [4].

2. Algebraic transformations and the closed form solution

Let us consider a new index m defined by

$$m = N - k \tag{2.1}$$

and the change of variables by

$$\begin{aligned} \hat{x}(m) &= x(N-m) = x(k) \\ \hat{\Psi}_j(m) &= \Psi_j(N-m) = \Psi_j(k), \quad l \le i \le p \end{aligned} \tag{2.2}$$

By rearranging and combining equations (1.3)-(1.6), one gets the necessary conditions to be satisfied by an open loop Nash strategy in the form

$$\begin{bmatrix} \hat{x}(m+1) \\ \hat{\Psi}_1(m+1) \\ \vdots \\ \hat{\Psi}_p(m+1) \end{bmatrix} = M \begin{bmatrix} \hat{x}(m) \\ \hat{\Psi}_1(m) \\ \vdots \\ \hat{\Psi}_p(m) \end{bmatrix}; \quad \hat{\Psi}_j(0) = K_{jf}\hat{x}(0), \quad l \le j \le p \tag{2.3}$$

where

$$M = \begin{bmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 & \dots & A^{-1}S_p \\ Q_1A^{-1} & A^T + Q_1A^{-1}S_1 & Q_1A^{-1}S_2 & \dots & Q_1A^{-1}S_p \\ Q_2A^{-1} & Q_2A^{-1}S_1 & A^T + Q_2A^{-1}S_2 & \dots & Q_2A^{-1}S_p \\ \vdots & & & & \vdots \\ Q_pA^{-1} & Q_pA^{-1}S_1 & Q_pA^{-1}S_2 & \dots & A^T + Q_pA^{-1}S_p \end{bmatrix} \tag{2.4}$$

Now let us introduce the change of basis

$$\begin{bmatrix} \hat{x} \\ \hat{\Psi}_1 \\ \hat{\Psi}_2 \\ \hat{\Psi}_3 \\ \vdots \\ \hat{\Psi}_p \end{bmatrix} = T \begin{bmatrix} \hat{x} \\ \hat{\Psi}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \vdots \\ \hat{w}_p \end{bmatrix} \quad ; \quad T = \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & & 0 \\ 0 & L_2 & I & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & L_p & 0 & \dots & I \end{bmatrix} \tag{2.5}$$

for appropriate matrices $L_2, L_3, \dots, L_p$ in $\mathbb{R}^{n\times n}$ to be determined. Thus, problem (2.3) is equivalent to the following one

$$\begin{bmatrix} \hat{x}(m+1) \\ \hat{\Psi}_1(m+1) \\ \hat{w}_2(m+1) \\ \vdots \\ \hat{w}_p(m+1) \end{bmatrix} = S \begin{bmatrix} \hat{x}(m) \\ \hat{\Psi}_1(m) \\ \hat{w}_2(m) \\ \vdots \\ \hat{w}_p(m) \end{bmatrix} , \tag{2.6}$$

$$\hat{\Psi}_1(0) = K_{1f}\hat{x}(0) \; ; \quad \hat{w}_j(0) = (K_{jf} - L_jK_{1f})\hat{x}(0), \quad 2 \le j \le p \tag{2.7}$$

where

$$S = \begin{bmatrix} A^{-1} & A^{-1}(S_1 + \sum_{j=2}^{p} S_jL_j) \\ Q_1A^{-1} & A^T + Q_1A^{-1}(S_1 + \sum_{j=2}^{p} S_jL_j) \\ (Q_2 - L_2Q_1)A^{-1} & A^TL_2 - L_2A^T + (Q_2 - L_2Q_1)A^{-1}(S_1 + \sum_{j=2}^{p} S_jL_j) \\ \vdots & \\ (Q_p - L_pQ_1)A^{-1} & A^TL_p - L_pA^T + (Q_p - L_pQ_1)A^{-1}(S_1 + \sum_{j=2}^{p} S_jL_j) \end{bmatrix}$$

$$\begin{array}{ccc} A^{-1}S_2 & \dots & A^{-1}S_p \\ Q_1A^{-1}S_2 & \dots & Q_1A^{-1}S_p \\ A^T+(Q_2-L_2Q_1)A^{-1}S_2 & \dots & (Q_2-L_2Q_1)A^{-1}S_p \\ & & \vdots \\ Q_pA^{-1}S_2 & \dots & A^T+(Q_p-L_pQ_1)A^{-1}S_p \end{array}\right]$$

Note that if we take matrices $L_2, L_3, \dots, L_p$, such that

$$Q_j = L_jQ_1 \text{ and } A^TL_j = L_jA^T, \quad 2 \le j \le p \tag{2.8}$$

Then problem (2.6)-(2.7) takes the form

$$\begin{bmatrix} \hat{x}(m+1) \\ \hat{\Psi}_1(m+1) \\ \hat{w}_2(m+1) \\ \vdots \\ \hat{\Psi}_p(m+1) \end{bmatrix} = \left[\begin{array}{c|ccc} \multirow{2}{*}{W} & A^{-1}S_2 & \dots & A^{-1}S_p \\ & Q_1A^{-1}S_2 & \dots & Q_1A^{-1}S_p \\ \hline & A^T & \dots & 0 \\ 0 & \vdots & \ddots & \vdots \\ & 0 & \dots & A^T \end{array}\right] \begin{bmatrix} \hat{x}(m) \\ \hat{\Psi}_1(m) \\ \hat{w}_2(m) \\ \vdots \\ \hat{w}_p(m) \end{bmatrix} \tag{2.9}$$

$$\hat{\Psi}_1(0) = K_{1f}\hat{x}(0) \quad ; \quad \hat{\Psi}_j(0) = (K_{jf} - L_jK_{1f})\hat{x}(0), \quad 2 \le j \le p \tag{2.10}$$

where

$$W = \begin{bmatrix} A^{-1} & A^{-1}(S_1 + \sum_{j=2}^{p} S_jL_j) \\ Q_1A^{-1} & A^T + Q_1A^{-1}(S_1 + \sum_{j=2}^{p} S_jL_j) \end{bmatrix} \tag{2.11}$$

Solving (2.9) we obtain

$$\hat{w}_j(m) = (A^T)^m\hat{w}_j(0), \quad 2 \le j \le p \tag{2.12}$$

$$\begin{aligned} \begin{bmatrix} \hat{x}(m) \\ \hat{\Psi}_1(m) \end{bmatrix} &= W^m \begin{bmatrix} \hat{x}(0) \\ \hat{\Psi}_1(0) \end{bmatrix} + \\ \sum_{t=0}^{m-1} W^t \sum_{j=2}^{p} &\begin{bmatrix} A^{-1}S_j \\ Q_1A^{-1}S_j \end{bmatrix} (A^T)^{m-t-1}\hat{w}_j(0) \end{aligned} \tag{2.13}$$

and, taking into account (2.10), we get

$$\begin{bmatrix} \hat{x}(m) \\ \hat{\Psi}_1(m) \end{bmatrix} = G(m)\hat{x}(0) \tag{2.14}$$

where

$$G(m) = W^m \begin{bmatrix} I \\ K_{1f} \end{bmatrix} + \sum_{t=0}^{m-1} W^t \sum_{j=2}^{p} \begin{bmatrix} A^{-1}S_j \\ Q_1 A^{-1} S_j \end{bmatrix} (A^T)^{m-t-1}(K_{jf} - L_j K_{1f}) \tag{2.15}$$

for $m \geq 1$ and

$$G(0) = \begin{bmatrix} I \\ K_{1f} \end{bmatrix}$$

From (2.14) one gets

$$\begin{aligned} \hat{x}(m) &= [I, 0]G(m)\hat{x}(0) \\ \hat{\Psi}_1(m) &= [0, I]G(m)\hat{x}(0) \end{aligned}$$

and, assuming invertibility of $[I, 0]G(m)$, it follows that

$$\hat{x}(0) = \{[I, 0]G(m)\}^{-1}\hat{x}(m) \tag{2.16}$$

$$\hat{\Psi}_1(m) = [0, I]G(m)\{[I, 0]G(m)\}^{-1}\hat{x}(m) \tag{2.17}$$

From (1.4),(2.17),(2.1) and (2.2) one gets

$$K_1(k) = [0, I]G(N-k)\{[I, 0]G(N-k)\}^{-1} \tag{2.18}$$

and from (2.5),(2.10),(2.12) and (2.16) it follows that

$$\begin{aligned} \hat{\Psi}_j(m) &= L_j \hat{\Psi}_1(m) + \hat{w}_j(m) \\ &= \{L_j[0, I]G(m) + (A^T)^m(K_{jf} - L_j K_{1f})\}\{[I, 0]G(m)\}^{-1}\hat{x}(m) \end{aligned} \tag{2.19}$$

From (1.4),(2.1),(2.2) and (2.19) one gets

$$\begin{aligned} K_j(k) &= \{L_j[0, I]G(N-k) \\ &\quad +(A^T)^{N-k}(K_{jf} - L_j K_{1f})\}\{[I, 0]G(N-k)\}^{-1} \\ &\quad 2 \leq j \leq p \end{aligned}$$

or

$$\begin{aligned} &K_j(k) = L_j K_1(k) + (A^T)^{N-k}(K_{jf} - L_j K_{1f})\{[I, 0]G(N-k)\}^{-1} \\ &2 \leq j \leq p \end{aligned} \tag{2.20}$$

Now we are going to consider the existence and the computation of solutions L_j, of (2.8). If A, B are matrices in $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{k\times s}$, respectively, then the

Kronecker product of A and B, denoted to $A \otimes B$, is defined as the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

If $A \in \mathbb{R}^{m \times n}$, we denote

$$A_{.j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, l \leq j \leq n; \mathrm{vec} M = \begin{bmatrix} M_{.1} \\ \vdots \\ M_{.n} \end{bmatrix}$$

and for matrices M,N and P of suitable dimensions, from the column lemma [5,p.410], we get

$$\mathrm{vec}(MNP) = (P^T \otimes M)\mathrm{vec}N \tag{2.21}$$

Taking into account (2.21) and by application of Kronecker products in each equation appearing in the system (2.8), it follows that the solvability of (2.8) is equivalent to the one of the linear system

$$C\mathrm{vec}L_j = \mathrm{vec}[0, Q_j] \tag{2.22}$$

for $2 \leq j \leq p$, where

$$C = \begin{bmatrix} I \otimes A^T - A \otimes I \\ Q_1 \otimes I \end{bmatrix} \tag{2.23}$$

If we denote by C^+ the Moore-Penrose pseudoinverse of C, then from theorem 2.3.2 of [7,p.24], the system (2.22) is solvable if and only if,

$$CC^+\mathrm{vec}[0, Q_j] = \mathrm{vec}[0, Q_j] \tag{2.24}$$

Furthermore, under condition (2.24) the general solution of (2.22) for L_j is given by

$$\mathrm{vec}L_j = C^+\mathrm{vec}[0, Q_j] + (I - C^+C)z \tag{2.25}$$

where I denotes the identity matrix in $\mathbb{R}^{n^2 x n^2}$ and z is an arbitrary vector in $\mathbb{R}^{n^2}$. We recall that an effective procedure for computing C^+ may be found by using MATLAB,[6].

From the previous comments the following result can be derived:

THEOREM 1 *Let us consider the coupled Riccati system (1.7) and let us suppose that matrices A and Q_j, for $l \leq j \leq p$, satisfy conditions (2.24) for $2 \leq j \leq p$, and let $L_2, L_3, \ldots, L_p$ be a solution of (2.8). If $G(m)$ is defined by (2.15) where W is given by (2.11), and if the matrix $[I,0]G(m)$ is invertible, then the solution of (2.7) is defined by (2.18),(2.20).*

Remark 1. Note that the case of $p = 2$ and $Q_2 = \alpha Q_1$, where α is a scalar, a solution of the corresponding system (2.8) is $L_2 = \alpha I$. Thus, this case is a particular case of the one considered in the above theorem and the result of [4] is an easy consequence of it. However, it is interesting to point out that the solvability of (2.8) does not imply that matrices Q_i are proportional. In fact if we consider the case of $p = 2$ and the problem (2.7) where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

an easy computation shows that matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies the system $Q_2 = L_2 Q_1, L_2 A^T = A^T L_2$ but matrices Q_1 and Q_2 not can be proportional.

ACKNOWLEDGEMENTS. This paper has been partially supported by the D.G.I.C.Y.T grant PS87-0064 and NATO grant 900040

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Analiza rozwiązania sprzężonego równania Riccatiego dla dyskretnych liniowo–kwadratowych gier Nasha (bez sprzężenia zwrotnego)

W artykule przedstawiono analityczne rozwiązanie sprzężonych równań Riccatiego występujących w rozwiązaniach dyskretnych liniowo-kwadratowych grach typu Nasha bez sprzężenia zwrotnego. W tym celu wykorzystano odpowiednie przekształcenia algebraiczne oraz sumę i iloczyn Kroneckera dla macierzy.

Аналитическое решение сопряженного уравнения Риккати для дискретных линейно–квадратных игр Нэша (без обратной связи)

В статье представлено аналитическое решение сопряженных уравниений Риккати, появляющихся при определении дискретных линейно–квадратных игр типа Нэша без обратной связи. Для этой цели используются соответствующие алгебраические преобразования, а также сумма и произведение Кронекера для матриц.