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Three-level Stackelberg Strategies in Linear-Quadratic Descriptor Systems

by

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In this paper open-loop three-level Stackelberg strategies in deterministic, sequential decision-making problems for linear discrete-time descriptor systems and quadratic cost function are studied. Necessary conditions for existence of open-loop Stackelberg strategies are derived. Open-loop Stackelberg solutions (norm - minimizing) are calculated from the necessary conditions and the conditions for existence of a unique solution of the necessary conditions are given. An example is given to illustrate the results of the paper.

1. Introduction

A great deal of attention has been paid to methods of design and analysis of Stackelberg strategies in multi-level sequential decision-making problems [1,5,8-10]. During the last 20 years, there has been much interest in studying of the descriptor systems [3-4,6-7]. To the best knowledge of the present authors, there are no published results for multi-level sequential decision-making problems characterized by descriptor systems. In section 2, multi-level sequential decision-making problems characterized by quadratic cost functions and linear time-invariant discrete descriptor systems are considered, and necessary conditions for the existence of open-loop Stackelberg strategies are given. In section 3, open-loop Stackelberg solutions (norm-minimizing) are calculated from the necessary conditions and the conditions for existence of a unique solution of the necessary conditions are given. An example is given to illustrate the results of the paper in section 4.

2. Problem formulation and derivation of necessary conditions

Consider a tree–level Stackelberg problem for a linear descriptor system

$$Ex(k+1) = Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k) \quad Ex(0) = Ex_0 \quad (2.1)$$

with associated cost function for each decision maker P_i

$$\begin{aligned} J_i(u^1, u^2, u^3) &= 1/2 \sum_{k=0}^{K-1} [x(k)'Q^i x(k) + \sum_{j=1}^{3} u^j(k)'R^{ij}u^j(k)] \\ &+ 1/2x(K)'E'Q^i(K)Ex(K) \quad i = 1,2,3 \end{aligned} \quad (2.2)$$

where E is a square matrix with rank $(E) = r < n$, and $det[sE - A] \not\equiv 0$ for any $s \in R, x(k) \in R^n$ is a descriptor vector, $u^j(k) \in R^{rj}$ is control vector of P_j, the usual positive-(semi)definiteness conditions are imposed on $Q^i, Q^i(k), R^{ij}, i, j = 1, 2, 3$, as in the associated optimal control problem.

Now let us assume that the decision-making sequence is $\{P_1, P_2, P_3\}$, that is, decision maker P_3 is the leader and selects his strategy first; P_2 is the first follower and selects his strategy as the second; and P_1 is the second follower and selects his strategy as the last. Consequently, in making his decision, P_1 knows the control u^2 and u^3 of the other decision makers; P_2 knows u^3, and he knows that P_1 reacts according to declared functions u^2 and u^3; P_3 knows that P_2 reacts according to his declared control u^3, and he must take into account the reaction of P_1 to declared controls u^2 and u^3. The simplest problem is solved by P_1 (an optimal control problem); a more complex problem is solved by P_2 (a two-level Stackelberg problem); and the most complex problem is solved by P_3 (a three-level Stackelberg problem). The complete solution of the problem is obtained by the solution of the leader's control problem, since the leader must solve problems faced by both P_1 and P_2 to determine their reactions to a given

u^3, in order to select that control which is best with respect to J_3, taking these reactions of the followers into account.

Therefore, in order to solve three-level Stackelberg problem, we must first determine the rational reaction of P_1 to controls u^2 and u^3 which are declared by P_2 and P_3, respectively. Since the underlying information pattern is open-loop, the optimization problem faced by P_1 is reduced to an optimal control problem defined by (2.1) and (2.2), for $i = 1$, given u^2 and u^3. In order to solve this optimization problem we append the constraint (2.1) to the cost function J_1 using the Lagrange multiplier $p^1(k)$:

$$\begin{aligned} J_1(u^1,u^2,u^3) &= 1/2x(K)'E'Q^1(K)Ex(K) \\ &+ \sum_{k=0}^{k-1}[H_1(k) - p^1(k+1)'Ex(k+1)] \end{aligned}$$

where

$$\begin{aligned} H_1(k) &= p^1(k+1)'[Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k)] \\ &+ 1/2[x(k)'Q^1x(k) + \sum_{j=1}^{3} u^j(k)'R^{1j}u^j(k)] \end{aligned}$$

From the results of [3] or [6], we deduce that the necessary conditions, under which u^1 constitutes the rational reaction to given u^2 and u^3, take the form

$$Ex(k+1) = Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k) \quad Ex(0) = Ex_0 \tag{2.3a}$$

$$E'p^1(k) = Q^1x(k) + A'p^1(k+1) \quad E'p^1(K) = E'Q^1(K)Ex(K) \tag{2.3b}$$

$$0 = R^{11}u^1(k) + B^{1\prime}p^1(k+1) \tag{2.3c}$$

Now, let us consider the problem faced by P_2. In deciding on the rational reaction of the second follower P_2 to u^3, the rational reaction of P_1 to u^2 and u^3 must be taken into account. Thus what P_2 must do is to minimize the cost function (2.2) for $i = 2$ subject to (2.3). Toward this end, by introducing the Lagrange multipliers $p^2(k), n^1(k), m^1(k)$ and $n^1(K)$, one can get

$$\begin{aligned} J_2(u^1,u^2,u^3) &= 1/2x(K)'E'Q^2(K)Ex(K) \\ &+ n^1(K)'[E'Q^1(K)Ex(K) - E'p^1(K)] \\ &+ \sum_{k=0}^{k-1}[H_2(k) - p^2(k+1)'Ex(k+1) - n^1(k)'E'p^1(k)] \end{aligned}$$

where

$$\begin{aligned} H_2(k) &= p^2(k+1)'[Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k)] \\ &+ 1/2[x(k)'Q^2x(k) + \sum_{j=1}^{3} u^j(k)'R^{2j}u^j(k)] \\ &+ n^1(k)'[Q^1x(k) + A'p^1(k+1)] \\ &+ m^1(k)'[R^{11}u^1(k) + B^{1\prime}p^1(k+1)] \end{aligned}$$

By using the standard variational techniques, the necessary conditions that characterize u^2 being the rational reaction of P_2 to u^3 are obtained in the form

$$Ex(k+1) = Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k) \quad Ex(0) = Ex_0 \tag{2.4a}$$

$$E'p^1(k) = Q^1x(k) + A'p^1(k+1) \quad E'p^1(K) = E'Q^1(K)Ex(K) \tag{2.4b}$$

$$0 = R^{11}u^1(k) + B^{1\prime}p^1(k+1) \tag{2.4c}$$

$$E'p^2(k) = Q^2x(k) + A'p^2(k+1) + Q^1n^1(k)$$

$$E'p^2(K) = E'Q^2(K)Ex(K) + E'Q^1(K)En^1(K) \tag{2.4d}$$

$$En^1(k+1) = An^1(k) + B^1m^1(k) \quad En^1(0) = 0 \tag{2.4e}$$

$$0 = R^{21}u^1(k) + B^{1\prime}p^2(k+1) + R^{11}m^1(k) \tag{2.4f}$$

$$0 = R^{22}u^2(k) + B^{2\prime}p^2(k+1) \tag{2.4g}$$

Finally, consider the problem solved by P_3. P_3 minimizes his own function (2.2) for $i = 3$, and at the same time he must take account (2.4) which characterizes the rational reactions of P_1 and P_2 to u^3. Now by appending the constraints (2.4) to the cost function J_3 by means of the Lagrange multipliers $p^3(k), n^2(k), n^3(k), q(k), m^2(k), m^3(k), w(k), n^2(K)$ and $n^3(K)$, we obtain that

$$\begin{aligned} J_3(u^1, u^2, u^3) &= 1/2x(K)'E'Q^3(K)Ex(K) \\ &+ n^2(K)'[E'Q^1(K)Ex(K) - E'p^1(K)] \\ &+ n^3(K)'[E'Q^2(K)Ex(K) + E'Q^1(K)En^1(K) - E'p^2(K)] \\ &+ \sum_{k=0}^{k-1}[H_3(k) - p^3(k+1)'Ex(k+1) - n^2(k)'E'p^1(k) \\ &\quad - n^3(k)'E'p^2(k) - q(k+1)'En^1(k+1)] \end{aligned}$$

where

$$\begin{aligned} H_3(k) &= p^3(k+1)'[Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k)] \\ &+ 1/2[x(k)'Q^3x(k) + \sum_{j=1}^{3} u^j(k)'R^{3j}u^j(k)] \\ &+ n^2(k)'[Q^1x(k) + A'p^1(k+1)] \\ &+ n^3(k)'[Q^2x(k) + A'p^2(k+1) + Q^1n^1(k)] \\ &+ q(k+1)'[An^1(k) + B^1m^1(k)] \\ &+ m^2(k)'[R^{11}u^1(k) + B^{1\prime}p^1(k+1)] \\ &+ w(k)'[R^{21}u^1(k) + B^{1\prime}p^2(k+1) + R^{11}m^1(k)] \\ &+ m^3(k)'[R^{22}u^2(k) + B^{2\prime}p^2(k+1)] \end{aligned}$$

Therefore, the necessary conditions for the control u^3 constituting the open-loop Stackelberg solution of the leader p_3 take the form

$$Ex(k+1) = Ax(k) + B^1u^1(k) + B^2u^2(k) + B^3u^3(k) \tag{2.5a}$$

$$E'p^1(k) = Q^1x(k) + A'p^1(k+1) \quad E'p^1(K) = E'Q^1(K)Ex(K) \tag{2.5b}$$

$$0 = R^{11}u^1(k) + B^{1\prime}p^1(k+1) \tag{2.5c}$$

$$\begin{aligned} &E'p^2(k) = Q^2x(k) + A'p^2(k+1) + Q^1n^1(k) \\ &E'p^2(K) = E'Q^2(K)Ex(K) + E'Q^1(K)En^1(K) \end{aligned} \tag{2.5d}$$

$$En^1(k+1) = An^1(k) + B^1m^1(k) \quad En^1(0) = 0 \tag{2.5e}$$

$$0 = R^{21}u^1(k) + B^{1\prime}p^2(k+1) + R^{11}m^1(k) \tag{2.5f}$$

$$0 = R^{22}u^2(k) + B^{2\prime}p^2(k+1) \tag{2.5g}$$

$$\begin{aligned} &E'p^3(k) = Q^3x(k) + A'p^3(k+1) + Q^1n^2(k) + Q^2n^3(k) \\ &E'p^3(K) = E'Q^3(K)Ex(K) + E'Q^1(K)En^2(K) + E'Q^2(K)En^3(K) \end{aligned} \tag{2.5h}$$

$$En^2(k+1) = An^2(k) + B^1m^2(k) \quad En^2(0) = 0 \tag{2.5i}$$

$$En^3(k+1) = An^3(k) + B^2m^3(k) + B^1w(k) \quad En^3(0) = 0 \tag{2.5j}$$

$$E'q(k) = Q^1n^3(k) + A'q(k+1) \quad E'q(K) = E'Q^1(K)En^3(K) \tag{2.5k}$$

$$0 = R^{31}u^1(k) + B^{1\prime}p^3(k+1) + R^{11}m^2(k) + R^{21}w(k) \tag{2.5l}$$

$$0 = R^{32}u^2(k) + B^{2\prime}p^3(k+1) + R^{22}m^3(k) \tag{2.5m}$$

$$0 = B^{1'}q(k+1) + R^{11}w(k) \tag{2.5n}$$

$$0 = R^{33}u^3(k) + B^{3\prime}p^3(k+1) \tag{2.5o}$$

3. Characterization of the optimal solution

For any $n \times n$ matrix E with rank $(E) = r < n$, there exist $n \times n$ nonsingular matrices U and V and the $r \times r$ unit matrix I such that

$$UEV = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{3.1a}$$

Therefore, for convenience in the later derivation and without loss of generality, let us assume that E has the form (3.1a), and A, B^j and Q^j have the corresponding form

$$\{A|B^j|Q^j\} = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \middle| \begin{bmatrix} B_1^j \\ B_2^{j'} \end{bmatrix} \middle| \begin{bmatrix} Q_{11}^j & Q_{12}^j \\ (Q_{12}^j)' & Q_{22}^j \end{bmatrix} \right\} \tag{3.1b}$$

For facility of notation, we define the following matrices

$$\bar{R}^{11} = \begin{bmatrix} 0 & A_{22} & B_2^1 \\ A_{22}' & Q_{22}^1 & 0 \\ B_2^{1'} & 0 & R^{11} \end{bmatrix} \quad \bar{Q}_{22}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_{22}^2 & 0 \\ 0 & 0 & R^{21} \end{bmatrix} \quad \bar{B}_2^2 = \begin{bmatrix} B_2^2 \\ 0 \\ 0 \end{bmatrix} \tag{3.2a}$$

$$\bar{R}^{22} = \begin{bmatrix} 0 & \bar{R}^{11} & \bar{B}_2^2 \\ \bar{R}^{11'} & \bar{Q}_{22}^2 & 0 \\ \bar{B}_2^{2'} & 0 & R^{22} \end{bmatrix} \quad \bar{Q}_{22}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{Q}_{22}^3 & 0 \\ 0 & 0 & R^{32} \end{bmatrix} \quad \bar{B}_2^3 = \begin{bmatrix} B_2^3 \\ 0 \\ 0 \end{bmatrix} \tag{3.2b}$$

$$\bar{R}^{33} = \begin{bmatrix} 0 & \bar{R}^{22} & \bar{B}_2^3 \\ \bar{R}^{22'} & \bar{Q}_{22}^3 & 0 \\ \bar{B}_2^{3'} & 0 & R^{33} \end{bmatrix} \quad \tilde{Q}_{22}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_{22}^3 & 0 \\ 0 & 0 & R^{31} \end{bmatrix} \tag{3.2c}$$

$$\bar{B}_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{12}\ B_1^1\ B_1^2\ B_1^3)$$

$$\bar{B}_2 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{12}\ B_1^1\ 0\ 0\ 0\ 0\ 0)$$

$$\bar{B}_3 = (0\ A_{12}\ B_1^1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$\bar{B}_4 = (0\ 0\ 0\ 0\ A_{12}\ B_1^1\ B_1^2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \tag{3.2d}$$

$$\bar{S}_1 = (A_{21}'\ Q_{12}^1\ 0\ 0\ Q_{12}^2\ 0\ 0\ 0\ 0\ 0\ 0\ Q_{12}^3 0\ 0\ 0)$$

$$\bar{S}_2 = (0\ 0\ 0\ A_{21}'\ Q_{12}^1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$\bar{S}_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{21}'\ Q_{12}^1\ 0\ 0\ 0)$$

$$\bar{S}_4 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ A'_{21}\ Q^1_{12}\ 0\ 0\ Q^2_{12} 0\ 0\ 0) \quad (3.2e)$$

$$\begin{aligned} \bar{u}(k)' \quad = \quad & (p^3_2(k+1), n^2_2(k)', m^2(k)', q_2(k+1)', n^3_2(k)', w(k)', \\ & m^3(k)', p^2_2(k+1)', n^1_2(k)', m^1(k)', p^1_2(k+1)', x_2(k)', \\ & u^1(k)', u^2(k)', u^3(k)') \end{aligned} \quad (3.2f)$$

Using this new notation, the necessary conditions (2.5) can be rewritten as follows:

$$x_1(k+1) = A_{11}x_1(k) + \bar{B}_1\bar{u}(k) \quad x_1(0) = x_{10} \quad (3.3a)$$

$$n^1_1(k+1) = A_{11}n^1_1(k) + \bar{B}_2\bar{u}(k) \quad n^1_1(0) = 0 \quad (3.3b)$$

$$n^2_1(k+1) = A_{11}n^2_1(k) + \bar{B}_3\bar{u}(k) \quad n^2_1(0) = 0 \quad (3.3c)$$

$$n^3_1(k+1) = A_{11}n^3_1(k) + \bar{B}_4\bar{u}(k) \quad n^3_1(0) = 0 \quad (3.3d)$$

$$\begin{aligned} & p^1_1(k) = Q^1_{11}x_1(k) + A'_{11}p^1_1(k+1) + \bar{S}_3\bar{u}(k) \\ & p^1_1(K) = Q^1_{11}(K)x_1(K) \end{aligned} \quad (3.3e)$$

$$\begin{aligned} & p^2_1(k) = Q^2_{11}x_1(k) + Q^1_{11}n^1_1(k) + A'_{11}p^2_1(k+1) + \bar{S}_4\bar{u}(k) \\ & p^2_1(K) = Q^2_{11}(K)x_1(K) + Q^1_{11}(K)n^1_1(K) \end{aligned} \quad (3.3f)$$

$$\begin{aligned} & p^3_1(k) = Q^3_{11}x_1(k) + Q^1_{11}n^2_1(k) + Q^2_{11}n^3_1(k) + A'_{11}p^3_1(k+1) + \bar{S}_1\bar{u}(k) \\ & p^3_1(K) = Q^3_{11}(K)x_1(K) + Q^2_{11}(K)n^3_1(K) + Q^1_{11}(K)n^2_1(K) \end{aligned} \quad (3.3g)$$

$$\begin{aligned} & q_1(k) = Q^1_{11}n^3_1(k) + A'_{11}q_1(k+1) + \bar{S}_2\bar{u}(k) \\ & q_1(K) = Q^1_{11}(K)n^3_1(K) \end{aligned} \quad (3.3h)$$

$$\begin{aligned} 0 \quad = \quad & \bar{S}'_1x_1(k) + \bar{S}'_2n^1_1(k) + \bar{S}'_3n^2_1(k) + \bar{S}'_4n^3_1(k) + \bar{B}'_3p^1_1(k+1) \\ + \quad & \bar{B}'_4p^2_1(k+1) + \bar{B}'_1p^3_1(k+1) + \bar{B}'_2q_1(k+1) + \bar{R}^{33}\bar{u}(k) \end{aligned} \quad (3.3i)$$

If the following matrices are defined

$$\bar{A} = diag(A_{11}, A_{11}, A_{11}, A_{11}) \quad \bar{R} = \bar{R}^{33} \quad (3.4a)$$

$$\bar{B}' = (\bar{B}'_1\ \bar{B}'_2\ \bar{B}'_3\ \bar{B}'_4) \quad (3.4b)$$

$$\bar{S}' = (\bar{S}'_1\ \bar{S}'_2\ \bar{S}'_3\ \bar{S}'_4) \quad (3.4c)$$

$$\bar{Q} = \begin{bmatrix} Q_{11}^3 & & Q_{11}^1 & Q_{11}^2 \\ 0 & 0 & 0 & Q_{11}^1 \\ Q_{11}^1 & 0 & 0 & 0 \\ Q_{11}^2 & Q_{11}^1 & 0 & 0 \end{bmatrix} \tag{3.4d}$$

then the system (3.3) can be written in the compact form

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) \quad \bar{x}(0)' = (x_{10}' \ 0 \ 0 \ 0) \tag{3.5a}$$

$$\bar{p}(k) = \bar{Q}\bar{x}(k) + \bar{A}'\bar{p}(k+1) + \bar{S}\bar{u}(k) \quad \bar{p}(K) = \bar{Q}(K)\bar{x}(K) \tag{3.5b}$$

$$0 = \bar{S}'\bar{x}(k) + \bar{B}'\bar{p}(k+1) + \bar{R}\bar{u}(k) \tag{3.5c}$$

where $\bar{x}(k)$ and $\bar{p}(k)$ are defined by

$$\bar{x}(k)' = (x_1(k)', n_1^1(k)', n_1^2(k), n_1^3(k)') \tag{3.5d}$$

$$\bar{p}(k)' = (p_1^3(k)', q_1(k)', p_1^1(k)', p_1^2(k)') \tag{3.5e}$$

Now, the two-point boundary value probem (3.5) is solved in the usual way by defining the linear transformation

$$\bar{p}(k) = P(k)\bar{x}(k) \tag{3.6a}$$

then, from (3.5a), we can get

$$\bar{p}(k+1) = P(k+1)\bar{A}\bar{x}(k) + P(k+1)\bar{B}\bar{u}(k) \tag{3.6b}$$

Substituting this into (3.5c) gives

$$[\bar{R} + \bar{B}'P(k+1)\bar{B}]\bar{u}(k) = -[\bar{S}' + \bar{B}'P(k+1)\bar{A}]\bar{x}(k) \tag{3.6c}$$

This equation may not have an exact solution; however, a least-square (norm-minimizing) solution is

$$\bar{u}(k) = -[\bar{R} + \bar{B}'P(k+1)\bar{B}]^{-1}[\bar{S}' + \bar{B}'P(k+1)\bar{A}]\bar{x}(k) \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5b), and assuming that it is true for $\bar{x}(k)$, we can obtain that $P(k)$ must satisfy the matrix Riccati equation

$$\begin{aligned} P(k) = \bar{Q} + \bar{A}'P(k+1)\bar{A} - [\bar{S} + \bar{A}'P(k+1)\bar{B}]^{-1}[\bar{R} + \bar{B}'P(k+1)\bar{B}] \\ [\bar{S}' + \bar{B}'P(k+1)\bar{A}] \end{aligned} \tag{3.8a}$$

$$P(K) = \bar{Q}(K) \tag{3.8b}$$

Substituting (3.7b) into (3.5a) gives

$$\bar{x}(k+1) = \bar{Z}(k)\bar{x}(k) \quad \bar{x}(0)' = (x'_{10}\ 0\ 0\ 0) \tag{3.9a}$$

with $\bar{Z}(k)$ being defined as follows:

$$\bar{Z}(k) = \bar{A} - \bar{B}[\bar{R} + \bar{B}'P(k+1)\bar{B}]^{-1}[\bar{S}' + \bar{B}'P(k+1)\bar{A}] \tag{3.9b}$$

Up to now, norm-minimizing open-loop Stackelberg solutions $u^1(k), u^2(k)$ and $u^3(k)$ have been derived, which are 13th, 14th and 15th subvectors of $\bar{u}(k)$, respectively, according to (3.2f). In fact, if $[R + B'P(k+1)B]$ is nonsingular for $k = K-1, K-2, \ldots, 0$, then, from (3.6c) it follows that the solution is unique.

Now we shall conclude the above discussion with a theorem which gives conditions under which the necessary conditions in the section 2 admit a unique solution.

THEOREM 1 *Under the assumptions (a) the matrix E has the form (3.1a); (b) the matrices $L(k) = \bar{R} + \bar{B}'P(k+1)\bar{B}$ are all nonsingular for $k = 0, \ldots, K-1$, the necessary conditions (2.5), i.e. (3.5), admit a unique solution which is given by*

$$\bar{u}(k) = -[\bar{R} + \bar{B}'P(k+1)\bar{B}]^{-1}[\bar{S}' + \bar{B}'P(k+1)\bar{A}]\bar{x}(k) \tag{3.10}$$

with $P(k)$ and $\bar{x}(k)$ being given by (3.8) and (3.9), respectively.

PROOF: The statement of Theorem 3.1 has already been verified in the discussion prior to Theorem 3.1.

4. Illustrative example

Let the system and the cost functions for a three-level Stackelberg problem be

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u^1(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u^2(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u^3(k)$$

$$x_1(0) = x_{10}$$

$$J_j = \sum_{k=0}^{2} \{1/2x(k)'x(k) + 1/2[u^j(k)]^2\} + 1/2x(3)'x(3) \quad j = 1, 2, 3$$

where $x(k)' = [x_1(k), x_2(k)]$.

After developing a computer program, the matrices $L(k)$ being invertible for $k = 0, 1, 2$, and the following results were obtained.

$$\begin{array}{lll} u^1(0) = -x_{10} & u^1(1) = 0.2590284x_1(1) & u^1(2) = 3.650466x_1(2) \\ u^2(0) = 0 & u^2(1) = 0.1244947x_1(1) & u^2(2) = 1.002068x_1(2) \\ u^3(0) = 0 & u^3(1) = -1.383523x_1(1) & u^3(2) = -5.652534x_1(2) \end{array}$$

5. Conclusion

This paper develops explicit expressions for three-level open-loop Stackelberg strategies for sequential decision making problems characterized by linear discrete–time descriptor system and quadratic cost function. The results of the note can be extended in a straightforward manner to multi-level Stackelberg problems. But the burden of computing multi-level open-loop Stackelberg strategies will be considerably increased, and so will the time of computation.

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Trzypoziomowe strategie Stackelberga w liniowo–kwadratowych systemach deskryptorowych

W artykule rozwija się strategie Stackelberga w deterministycznych sekwencyjnych zadaniach podejmowania decyzji dla liniowych systemów deskryptorowych z czasem dyskretnym kwadratową funkcją kosztu. Wyprowadzono warunki konieczne istnienia strategii Stackelberga bez pamiętania. Rozwiązania minimalizujące Stackelberga są wyznaczane z warunków koniecznych i podano warunki istnienia jednoznacznego rozwiązania dla warunków koniecznych. Przytoczono przykład ilustrujący wyniki zawarte w artykule.

Трехуровневая стратегия Стакельберга в линейноквадратных дескрипторных системах

В статье рассматривается стратегия Стакельберга в детерминированных последовательных задачах принятия решения для линейных дескрипторных систем, дискретных по времени, с квадратной функцией затрат. Формулируются необходимые условия существования незамкнутой стратегии Стакельберга. Решения Стакельберга по минимизации определяются из необходимых условий и приводятся условуя существования однозначного решения для необходимых условий. Дается пример иллюстрирующий результаты приведенные в статье.