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Threat Bargaining Problems with Incomplete Information [1]

by

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In this paper we provide a general framework for studying threat bargaining games with incomplete information. In this framework we obtain a characterization of the Kalai–Smorodinsky solution without any monotonicity assumption and the Nash solution without the independence of irrelevant alternatives assumption. The approach adds a dose of realism to the already existing literature on threat bargaining games.

1. Introduction

In many arbitration problems, the parties involved are fully aware of the true characteristics of the rival, but the procedure involved is one where each party

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makes a report on its status–quo position to the arbitrator, who on the basis of such a statement must arrive at a mutually acceptable decision. Further the arbitrator is unaware of the true characteristics of the players. Such situation abound in reality, where it is not the lack of information on the part of the players, but on the part of the arbitrator, which leads to strategic non cooperative behaviour. Such situations can be modelled as threat bargaining problems, as discussed in Lahiri (1988, 1989a, b); Owen (1982).

An additional complication to such problems is considered here, where each player has a belief regarding the acceptability to his opponent of an arbitrary outcome, which can be summarized by a probability distribution. Hence coupled with the strategic behaviour of the players in determining the final outcome of arbitration, there is an uncertainty about the solution being acceptable to the opponent. Each player's belief about an outcome being acceptable to his opponent may depend merely on what the opponent receives or may in addition be influenced by what the player himself was getting. Beliefs of the former type are naturally uncorrelated, whereas those of the latter type are correlated.

Throughout this paper we assume that the parameters determining the arbitrated outcome are known to the players. However, the arbitrator is unaware of the true status–quo point, and it is this ignorance which results in the strategic behaviour of the players. Arbitration proceeds on the basis of the stated value of the status–quo point. We show here first, that if the conditional distribution of the beliefs follow a certain specified form, then the only bargaining solution compatible with truthful revelation of the status–quo point is the Kalai–Smorodinsky (1975) bargaining solution. If on the other hand the beliefs are uncorrelated and uniformly distributed then the only bargaining solution compatible with truthful revelation of the status–quo point is the Nash (1950) bargaining solution.

2. Definitions

In a *pure bargaining problem* between a group of two participants there is a set of feasible outcomes, any one of which will result if it is specified by the unanimous agreement of all participants. In the event that no unanimous agreement is reached, a given disagreement outcome obtains. We shall assume that the utility space or the set of possible payoffs is R^2 i.e. a two person bargaining problem is a pair (H, d) of a subset H of R^2 and of a point $d \in H$. H is the *feasible set*, and d is the *disagreement* (or *threat*) point.

The class of bargaining problems we consider is given by the following definition:

DEFINITION 1 *The pair* $\Gamma = (H,d)$ *is a two-person fixed threat bargaining game if* $H \subseteq R^2$ *is compact, convex, comprehensive with nonempty interior,* $d \in H$, *and* H *contains at least one element* u *such that* $u \gg d$. *(Note:* $H \subseteq R^2$ *is said to be comprehensive if* $y \in R^2$, $x \geq y \geq d$ *for some* $x \in H$ *implies* $y \in H$*).*

DEFINITION 2 *The set of two-person fixed threat bargaining games is denoted by* W.

For the purpose of this paper we define a solution to bargaining problems in W as follows:

DEFINITION 3 *A solution is a function* $F: W \to R^2$ *satisfying*

(i) $f(H,d) \in H \quad \forall (H,d) \in W$ *(feasibility)*
(ii) $y \in H,\ y \geq F(H,d)$ *implies* $y = F(H,d)$ *(Pareto optimality)*
(iii) $F(H,d) \geq d \quad \forall (H,d) \in W$ *(individual rationality)*
(iv) If $(a_1,a_2) \gg 0$, $(b_1,b_2) \in R^2$, $H' = \{y \in R^2 // y_i = a_i x_i + b_i,\ i = 1,2,\ y = (y_1,y_2),\ x = (x_1,x_2) \in H\}$ *and* $d_i = a_i d_i + b_i$, $i = 1,2$, $d' = (d'_1, d'_2)$, *then* $f_i(H',d') = a_i F_i(H,d) + b_i$, $i = 1,2$. *(Independence with respect to affine utility transformations)*

The conditions we impose on a solution to bargaining problems are standard and are satisfied by the more well known solutions to bargaining problems (e.g. Nash (1950), Kalai–Smorodinsky (1975)).

We now make an assumption which is satisfied by most familiar solutions to bargaining problems and which will be required significantly by us.

ASSUMPTION (FUD) *Let* $(H,d) \in W$ *and* $P(H_d) = \{(x_1,x_2) \in H // x = (x_1,x_2)$,
$x_i \geq d_i,\ i = 1,2$ *and* $y_i \geq x_i\ y \in H$ *implies* $y = x\}$. *Then* $\forall (x_1,x_2) \in P(H_d), \exists d'_1 \geq d_1$, *or* $d'_2 \geq d_2$ *such that*

(i) $F(H; d'_1, d_2) = (x_1,x_2)$ *or*
(ii) $F(H; d_1, d'_2) = (x_1,x_2)$

(fullness through unilateral deviations).

This assumption requires that unilateral deviation from the given disagreement payoffs yield any Pareto Optimal and individually rational outcome. As mentioned earlier this property is satisfied by all the more well known solutions to bargaining problems, including some of those which may not satisfy some of the conditions of Definition 3 (e.g the *Proportional Solution* of Kalai (1977)).

Our analysis requires the notion of a *true bargaining problem*, which in view of the above and following Anbar and Kalai (1978) may be defined as follows:

DEFINITION 4 *A true bargaining problem H is a compact, convex subset of the unit square containing* $(0,0)$, $(1,0)$ *and* $(0,1)$.

The interpretation of such a bargaining game is that the *true disagreement point* of the players have been set equal to $(0,0)$ and the game has been normalized in such a way that the utility demands of the players belong to the closed interval $[0,1]$. Let us call the set of all true bargaining problems $\bar{W}$.

Every member $H \in W$ defines uniquely a monotone non–increasing concave function $\wp_H: [0,1] \to [0,1]$ by $\wp_H(x_1) = \max\{x_2 // (x_1, x_2) \in H\}$. Conversely every monotone non–increasing concave function $\wp: [0,1] \to [0,1]$ such that $\wp(0) = 1$ determines uniquely a set $H_\wp \in \bar{W}$ by $H_\wp = \{(x_1, x_2) | 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq \wp(x_1)\}$. For every such function $\wp$ we define the (generalized) inverse $\wp^{-1}: [0,1] \to [0,1]$ by $\wp^{-1} = \max\{x_1 // (x_1, x_2) \in H_\wp\}$.

Let $G_i: [0,1] \times [0,1] \to [0,1]$ be the conditional distribution function which summarizes the belief of player i about player $j \neq i$ (i's opponent) accepting a utility outcome, given player i's utility outcome, $i = 1, 2$. Thus, $G_1(x_2|x_1)$ is player 1's assessment of the probability of player 2 accepting a utility outcome x_2 or less, given that player 1's utility outcome is x_1.

The non–cooperative game we have in mind is the following. The underlying *true bargaining problem* $H \in \bar{W}$ being given each player i announces a disagreement utility d_i. The pair (H, d), $d = (d_1, d_2)$ is a fixed threat bargaining problem in W. Based on the information announced by the players the arbitrator using a solution F selects an outcome $F(H, d)$ which each player accepts with a probability determined by G_1 and G_2 respectively. In the event that the outcome is rejected, by any one or both the players, the participants settle down for their true disagreement payoffs $0 = (0,0)$.

Let $(d_1, d_2) \in H$ be the announced disagreement payoffs of the respective players. If F is the solution being used by the arbitrator, the expected payoff of the player 1 is

$$P_1(d_1, d_2) = F_1(H; d_1, d_2) \cdot G_1(F_2(H; d_1, d_2) | F_1(H; d_1, d_2)).$$

The expected payoff of the player 2 is

$$P_2(d_1, d_2) = F_2(H; d_1, d_2) \cdot G_2(F_1(H; d_1, d_2)|F_2(H; d_1, d_2)).$$

DEFINITION 5 *A threat bargaining game with incomplete information equipped with the solution F is an ordered triplet (H, F, G) where*

(i) $H \in \bar{W}$ *is a true bargaining problem*
(ii) $F: W \to R^2$ *is a bargaining solution*
(iii) $G = (G_1, G_2)$ *is a pair of conditional probability distribution functions on* $[0, 1]$.

The notion of an equilibrium that we adopt in this paper is given by the following definition.

DEFINITION 6 *An equilibrium for a threat bargaining games with incomplete information equipped with a solution F, i.e. (H, F, G) is an ordered pair $(d_1^*, d_2^*) \in H$ such that*

(i) $P_1(d_1^*, d_2^*) \geq P_1(d_1, d_2^*) \quad \forall d_1 \in [0, 1]$
(i) $P_2(d_1^*, d_2^*) \geq P_1(d_1^*, d_2) \quad \forall d_2 \in [0, 1]$

This is the familiar Nash equilibrium which by dint of its self enforceability finds a distinguished placed as a solution concept. In the case of threat bargaining problems, the relationship between a Nash equilibrium and well known solutions to bargaining problems have been studied in Lahiri (1988, 1989b).

3. Main Theorem

In this section we shall try to impose conditions under which truthfull revelation of disagreement utility will be guaranteed by a bargaining solution.

The main theorems of this paper are following:

THEOREM 1 *Let*

$$G_1(x_2|x_1) = \begin{cases} \frac{\min\{x_1, x_2\}}{x_1} & \text{if } x_1 > 0 \\ 1 & \text{if } x_1 = 0 \end{cases}$$

and

$$G_2(x_1|x_2) = \begin{cases} \frac{\min\{x_1, x_2\}}{x_2} & \text{if } x_2 > 0 \\ 1 & \text{if } x_2 = 0 \end{cases}$$

Then $(0,0)$ *is an equilibrium of the threat bargaining game with incomplete information* (H,F,G) *equipped with a solution* F *if and only if* F *is the Kalai–Smorodinsky (1975) solution i.e.*

$$F(S) = \arg\max_{0\le x_1\le 1}\{\min(x_1,\varphi_s(x_1))\} = \arg\max_{0\le x_2\le 1}\{\min(x_2,\varphi_s^{-1}(x_2))\}$$

$\forall s \in \bar{W}$.

PROOF: Given G_1 and G_2,

$$P_1(d_1,d_2) = \min\{F_1(H;d_1,d_2), \varphi_H(F_1(H;d_1,d_2))\}$$

$$P_2(d_1,d_2) = \min\{F_2(H;d_1,d_2), \varphi_H^{-1}(F_2(H;d_1,d_2))\}.$$

Observe that by property *(i)* of a solution $F_2(H;d_1,d_2) = \varphi_H(F_1(H;d_1,d_2))$ and $F_1(H;d_1,d_2) = \varphi_H^{-1}(F_2(H;d_1,d_2))$.

Suppose $F = (F_1,F_2)$ is the Kalai–Smorodinsky (1975) solution.

$$P_1(0,0) = \min\{F_1(H;0,0), \varphi_H(F_1(H;0,0))\} \ge \min\{x_1, \varphi_H(x_1)\}$$

$\forall 0 \le x_1 \le 1$, by the definition of the solution.

Since $P_1(H;d_1,0) = \min\{x_1,\varphi_H(x_1)\}$ for some $x_1 \in [0,1]$, we get

$$P_1(0,0) \ge P_1(d_1,0) \quad \forall d_1 \in [0,1].$$

By a similar argument it follows that

$$P_2(0,0) \ge P_2(0,d_2) \quad \forall d_2 \in [0,1].$$

Hence $(0,0)$ is an equilibrium for (H,F,G).

Conversely suppose that $(0,0)$ is an equilibrium for $H,F,G)$, but F is not Kalai–Smorodinsky (1975) solution. Let $(x_1^*, \varphi_H(x_1^*))$ be the Kalai–Smorodinsky solution outcome for $H \in \bar{W}$. By assumption (FUD) and without loss of generality $\exists d_1' \ge 0$, such that

$$F(H;d_1',0) = (x_1^*, \varphi_H(x_1^*))$$

Hence

$$P_1(d_1',0) = \min\{x_1^*, \varphi_H(x_1^*)\} > \min\{F_1(H;0,0), \varphi_H(F_1(H;0,0))\}$$
$$= P_1(0,0),$$

contradicting that $(0,0)$ is an equilibrium. Hence the theorem ■

Let us now assume that the beliefs of each player regarding the acceptability to his opponent of an arbitrated outcome does not depend on what he himself gets. This is a special case of our general framework.

THEOREM 2 *Let G_1, G_2 be distributed uniformly, i.e.*

$$G_i(x) = x \quad \forall\, 0 \le x \le 1, \quad i = 1, 2.$$

Then $(0,0)$ is an equilibrium of the threat bargaining game with incomplete information (H, F, G) equipped with a solution F if and only if F is the Nash bargaining solution i.e.

$$F(S,d) = \arg \min_{\substack{(x_1,x_2)\in S \\ x_1 \ge d_1,\; x_2 \ge d_2}} (x_1 - d_1)(x_2 - d_2) \quad \forall (S,d) \in W$$

PROOF: Since G_1 and G_2 are uniformly distributed,

$$P_1(d_1, d_2) = F_1(H; d_1, d_2) \cdot \varphi_H(F_1(H; d_1, d_2))$$

$$P_2(d_1, d_2) = F_2(H; d_1, d_2) \cdot \varphi_H^{-1}(F_2(H; d_1, d_2))$$

Observe that by property *(i)* of a solution $F_2(H; d_1, d_2) = \varphi_H(F_1(H; d_1, d_2))$ and $F_1(H; d_1, d_2) = \varphi_H^{-1}(F_2(H; d_1, d_2))$. Suppose $F = (F_1, F_2)$ is the Nash bargaining solution.

$$P_1(0,0) = F_1(H; 0, 0) \cdot \varphi_H(F_1(H; 0, 0)) \ge x_1 \varphi_H(x_1) \quad \forall\, 0 \le x_1 \le 1$$

by definition of the Nash bargaining solution. Since $P_1(H; d_1, 0) = x_1 \varphi_H(x_1)$ for some $x_1 \in [0,1]$, we get

$$P_1(0,0) \ge P_1(d_1, 0) \quad \forall d_1 \in [0,1].$$

By a similar argument it follows that

$$P_2(0,0) \ge P_2(0, d_2) \quad \forall d_2 \in [0,1].$$

Hence $(0,0)$ is an equilibrium for (H, F, G).

Conversely suppose that $(0,0)$ is an equilibrium for (H, F, G), but F is not the Nash bargaining solution. Let $(x_1^*, \varphi_H(x_1^*))$ be the Nash bargaining solution outcome for $H \in \bar{W}$. By assumption (FUD) and without loss of generality $\exists d_1' \ge 0$, such that

$$F(H; d_1', 0) = (x_1^*, \varphi_H(x_1^*))$$

Hence

$$P_1(d_1', 0) = x_1^* \varphi_H(x_1^*) > F_1(H; 0, 0)\varphi_H(F_1(H; 0, 0)) = P_1(0,0)$$

contradicting that $(0,0)$ is an equilibrium. Hence the theorem. ■

A characterization of the family of nonsymmetric Nash bargaining solution (see, Harsanyi and Selton (1972), Kalai (1977b)) is embodied in the following theorem.

THEOREM 3 *Let,* $G_1(x) = x^k$, $0 \leq x \leq 1$ *and* $G_2(x) = x^{1//k}$, $0 \leq x \leq 1$, $k > 0$, *be the distribution functions embodying the beliefs of the two players. Then* $(0,0)$ *is an equilibrium for the threat bargaining game with incomplete information* (H, F, G) *equipped with a solution* F *if and only if,* F *is a non-symmetric Nash bargaining solution i.e.*

$$F(S; d_1, d_2) = \arg \min_{\substack{(x_1,x_2)\in S \\ x_1 \geq d_1,\ x_2 \geq d_2}} (x_1 - d_1)^k (x_2 - d_2) \quad \forall (S, d) \in W.$$

PROOF: Analogous to the proof of Theorem 2.

4. Conclusion

Now we shall briefly summarize all that we have achieved in this paper.

To begin with we have extended the framework of threat bargaining games to include within it the study of threat bargaining games with incomplete information. This extension adds a dose of realism to our analysis.

Second, we achieved a characterization of the Kalai–Smorodinsky (1975) solution without a monotonicity assumption, which is an interesting problem in its own right.

Third, we achieved a characterization of the Nash bargaining solution (or more generally, the family of non-symmetric Nash bargaining solutions) without the debatable Independence of Irrelevant Alternatives assumption, a problem which has come to occupy a central place in bargaining game theory. We show that provided the beliefs of the players in a threat bargaining game with incomplete information are distributed uniformly, the only bargaining solution compatible with truthful revelation of disagreement payoffs is the Nash bargaining solution.

Fourth, our framework is perfectly general in that given definition – 3 (of a solution to bargaining problems) our results continue to hold in the class W of bargaining problems. In this case a true bargaining problems would be $(H, d) \in W$ and truthful revelation of disagreement payoffs would imply agent i announcing d_i as his disagreement payoff. The support of the distribution function of the beliefs of player i would naturally be $[d_j, \max\{x_j // (x_i, x_j) \in H\}]$ where $j \neq i$. $\bar{W}$ was invoked merely for notational convenience and ease of analysis.

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