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Stable closed form solution of a class of Riccati matrix differential equations

by

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In this paper, nonsymmetric Riccati matrix differential equations whose coefficients depend on a real parameter are studied. Conditions in order to obtain a closed form solution of the problem which is stable under coefficient perturbations are given.

Keywords: Nonsymmetric Riccati equations, perturbed equation, stable solution, closed form solution, block diagonalization.

1. Introduction

Initial value problems for matrix differential equations of the type

$$W'(t) = C - DW(t) - W(t)D^* - W(t)BW(t), \quad W(0) = W_0 \tag{1.1}$$

where $W(t), C, D, B$ and W_0 are $n \times n$ complex matrices and D^* denotes the adjoint matrix of D, arise in linear–quadratic optimal control problems [1] and filtering theory [3]. The nonsymmetric Riccati equation takes the form

$$W'(t) = C - DW(t) - W(t)A - W(t)BW(t), \quad W(0) = W_0 \tag{1.2}$$

where A is not necessarily equal to D^*, and it appears mainly in the invariant imbedding context [13].

The study of the perturbed Riccati equation

$$\begin{aligned} &W'(t) = C(\lambda) - D(\lambda)W(t) - W(t)A(\lambda) - W(t)B(\lambda)W(t) \\ &W(0) = W_0(\lambda) \end{aligned} \tag{1.3}$$

is interesting from the point of view of obtaining a good qualitative model of the physical problem. The measure of the matrix coefficients of Riccati equation are subject to some uncertainty because some parameters may be changing during operation or are difficult to measure. This motivates the study of solutions of the equation which are stable under perturbations of the matrix coefficients.

In a recent paper [10], a closed form solution for nonsymmetric Riccati equations of the type (1.2), where B is nonsingular, is given.

The aim of this paper is to show that the method proposed in [10] is stable under coefficient perturbations when some additional condition is assumed and the coefficients of the Riccati equation (1.3) are continuously differentiable functions of a real parameter λ.

The paper is organized as follows. In section 2 we state some auxiliary results that will be used in other sections. In section 3 we study the stability of the problem

$$\begin{aligned} &x^2(t) + A_1(\lambda)x^{(1)}(t) + A_0(\lambda)x(t) = 0, \\ &x(0) = E(\lambda), \quad x'(0) = F(\lambda) \end{aligned} \tag{1.4}$$

Finally, in section 4 we study the Riccati perturbed problem (1.3).

2. Preliminaries

For the sake of clarity in the presentation of the paper, we state some results that will be used in following sections.

Let A, B be matrices in $\mathcal{C}^{n\times n}$ and let us denote by $\|\cdot\|$ the 2–norm defined in [8], p.14, then from [2], [11], we have that

$$\|\exp(A) - \exp(B)\| \leq \exp(\|B\|)[exp(\|A - B\| - 1] \tag{2.1}$$

From the Banach lemma, if A is an invertible matrix and $\|B-A\| < (\|A^{-1}\|)^{-1}$, then B is invertible and

$$\|B^{-1} - A^{-1}\| \leq \|A^{-1}\| \, \|B^{-1}\| \, \|B - A\| \tag{2.2}$$

THEOREM 1 ([6]). *Let $F(\lambda)$ be a $\mathcal{C}^{n\times n}$ valued continuously differentiable matrix function of the real parameter $\lambda \in I(\lambda_0) =]\lambda_0 - \delta, \lambda_0 + \delta[$, where $F(\lambda_0)$ is given, M is an upper bound of $\|F'(\lambda)\|$ in $I(\lambda_0)$ and*

$$\|[F(\lambda_0)]^{-1}\| < [M(\lambda_0 + \delta)]^{-1}$$

then, it follows that

$$\|[F(\lambda)]^{-1}\| \leq \|[F(\lambda_0)]^{-1}\| [1 - \|[F(\lambda_0)]^{-1}\| (M(\lambda_0 + \delta))]^{-1}$$

THEOREM 2 ([9], PROBLEM 86). *For a matrix $A \in \mathcal{C}^{n\times n}$ let us denote by $\sigma(A)$ the set of all eigenvalues of A. If U is an open set of the complex plane containing $\sigma(A)$, there exists a positive number ε such that if $\|A - B\| < \varepsilon$, then $\sigma(B) \subseteq U$.*

Now we introduce the Gingold's condition that permits us to obtain a stable block diagonalization of a matrix function.

THEOREM 3 ([7]). *Let $C(\lambda)$ be a continuously differentiable $\mathcal{C}^{2n\times 2n}$ valued matrix function defined on the interval $I(\lambda_0) = [\lambda_0 - \delta, \lambda_0 + \delta]$ such that satisfies the condition*

> *The characteristic polynomial $\Phi(\lambda, z) = \det(C(\lambda) - zI)$ admits a decomposition $\Phi(\lambda, z) = \Phi_1(\lambda, z) \ldots \Phi_s(\lambda, z)$, where $\Phi_j(\lambda, z)$ for $1 \leq j \leq s$ are relatively prime monic polynomials in z of constant degree for all λ in $I(\lambda_0)$.* (2.3)

then for all $\lambda \in I(\lambda_0)$, $C(\lambda)$ is similar to a matrix function $J(\lambda) = diag[J_1(\lambda), \ldots, J_k(\lambda)]$, with $J_j(\lambda) \in \mathcal{C}^{m_j\times m_j}$, such that for an invertible matrix $M(\lambda) \in \mathcal{C}^{2n\times 2n}$ wih $M(\lambda) = (M_{ij}(\lambda))$ and $M_{ij}(\lambda) \in \mathcal{C}^{n\times m_j}$, for $1 \leq i \leq 2$, $1 \leq j \leq k$, with $m_1 + \ldots + m_k = 2n$, and m_j independent of $\lambda \in I(\lambda_0)$.

Finally, we recall that since two similar matrices have the same characteristic polynomial, if $C(\lambda)$ satisfies the condition (2.3) and there exists an invertible matrix function $P(\lambda)$ such that

$$P(\lambda)C(\lambda) = H(\lambda)P(\lambda),$$

then $H(\lambda)$ also satisfies the condition (2.3).

3. Stable solution of the equation $x'' + A_1x' + A_0x = 0$

We begin this section by considering the homogeneous differential system

$$(P_\lambda)\begin{cases} x^{(2)}(t) + A_1(\lambda)x^{(1)}(t) + A_0(\lambda)x(t) = 0 \\ x(0) = E(\lambda), \quad x'(0) = F(\lambda), \quad 0 \le t \le b \end{cases} \tag{3.1}$$

THEOREM 4 ([10]). *Let us suppose that the companion matrix* $C(\lambda)$

$$C(\lambda) = \begin{bmatrix} 0 & I \\ -A_0(\lambda) & -A_1(\lambda) \end{bmatrix} \tag{3.2}$$

is similar to the block diagonal matrix

$$J(\lambda) = diag[J_1(\lambda), \dots, J_k(\lambda)]$$

where $J_j(\lambda) \in \mathcal{C}^{m_j \times m_j}$ *for* $1 \le j \le k$ *and* $m_1 + \ldots + m_k = 2n$. *Let* $M(\lambda) = (M_{ij}(\lambda))$ *be the block partitioned invertible matrix in* $\mathcal{C}^{2n\times 2n}$ *with* $M_{ij}(\lambda) \in \mathcal{C}^{n\times m_j}$ *for* $1 \le i \le 2$, $1 \le j \le k$, *such that*

$$M(\lambda)J(\lambda) = C(\lambda)M(\lambda) \tag{3.3}$$

Then the unique solution $x(t,\lambda)$ *of system (3.1) is given by*

$$x(t,\lambda) = \sum_{j=1}^{k} M_{1j}(\lambda)\exp(tJ_j(\lambda))D_j(\lambda), \tag{3.4}$$

where $D_j(\lambda)$ *is*

$$\begin{bmatrix} D_1(\lambda) \\ \vdots \\ D_k(\lambda) \end{bmatrix} = [M(\lambda)]^{-1}\begin{bmatrix} E(\lambda) \\ F(\lambda) \end{bmatrix} \tag{3.5}$$

Let us assume that the companion matrix $C(\lambda)$ defined by (3.2) satisfies the condition (2.3) with λ belonging to interval $I(\lambda_0) = [\lambda_0 - \delta, \lambda_0 + \delta]$. From [7], taking $M(\lambda) = (M_{ij}(\lambda))$ for $1 \le i \le 2$, $1 \le j \le k$, the solution of the matrix differential equation

$$Z'(\lambda) = (P'(\lambda)P(\lambda) - P(\lambda)P'(\lambda))Z(\lambda), \quad Z(\lambda_0) = M(\lambda_0), \tag{3.6}$$

where $P(\lambda)$ is a projection that commutes with $C(\lambda)$. Then, one satisfies (3.3) for some block diagonal matrix function $J(\lambda) = \text{diag}[J_1(\lambda), \ldots, J_k(\lambda)]$ and from theorem 4, we have

$$\begin{aligned} x(t,\lambda) - x(t,\lambda_0) &= \\ &= \sum_{j=1}^{k} \{M_{1j}(\lambda)\exp(tJ_j(\lambda))D_j(\lambda) - M_{1j}(\lambda_0)\exp(tJ_j(\lambda_0))D_j(\lambda_0)\} = \\ &= \sum_{j=1}^{k} M_{1j}(\lambda)\exp(tJ_j(\lambda))\{D_j(\lambda) - D_j(\lambda_0)\} + \\ &\quad + \sum_{j=1}^{k} M_{1j}(\lambda)\{\exp(tJ_j(\lambda)) - \exp(tJ_j(\lambda_0)\}D_j(\lambda_0) + \\ &\quad + \sum_{j=1}^{k} \{M_{1j}(\lambda) - M_{1j}(\lambda_0)\}\exp(tJ_j(\lambda_0))D_j(\lambda_0). \end{aligned} \tag{3.7}$$

From [5, p.110], $M(\lambda)$ satisfies for $|\lambda - \lambda_0| \leq \delta$ the inequality

$$\|M(\lambda)\| \leq \|M(\lambda_0)\| \exp(2PQ|\lambda - \lambda_0|) \leq \|M(\lambda_0)\| \exp(2PQ\delta) \tag{3.8}$$

where

$$P = \sup\{\|P(\lambda)\|,\ \ |\lambda - \lambda_0| \leq \delta\},\ \ Q = \sup\{\|P'(\lambda)\|,\ \ |\lambda - \lambda_0| \leq \delta\}. \tag{3.9}$$

Note that the problem (3.6) is equivalent to integral equation

$$Z(\lambda) = Z(\lambda_0) + \int_{\lambda_0}^{\lambda} F(s, Z(s))ds \tag{3.10}$$

where $F(s, Z(s))$ is the right–hand side of the differential equation (3.6), and using (3.9) we obtain

$$\|M(\lambda) - M(\lambda_0)\| \leq 2PQ\|M(\lambda_0)\| \exp(2PQ\delta)|\lambda - \lambda_0|, \text{ for } |\lambda - \lambda_0| \leq \delta. \tag{3.11}$$

From the mean value theorem [5], there exist constants $e > 0$ and $f > 0$ such that

$$\begin{aligned} &\|E(\lambda) - E(\lambda_0)\| \leq e|\lambda - \lambda_0|,\ \ \|F(\lambda) - F(\lambda_0)\| \leq f|\lambda - \lambda_0|, \\ &\text{for} \quad \lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]. \end{aligned} \tag{3.12}$$

From (2.1), for every $t \in [0, b]$ and $1 \leq j \leq k$, we have

$$\begin{aligned} &\|\exp(tJ_j(\lambda)) - \exp(tJ_j(\lambda_0))\| \leq \\ &\leq \exp(b\|J_j(\lambda_0)\|)\{\exp(b\|J_j(\lambda)) - J_j(\lambda_0)\|) - 1\}. \end{aligned} \tag{3.13}$$

From [5, p.114] and using that $M(\lambda)$ is a solution of (3.6)

$$\|M^{-1}(\lambda)\| \leq \|M^{-1}(\lambda_0)\| \exp(2PQ\delta), \text{ for } |\lambda-\lambda_0| \leq \delta. \tag{3.14}$$

Then, from (2.2) and (3.11), for $|\lambda-\lambda_0| \leq \delta$ it follows that

$$\begin{aligned} &\|M^{-1}(\lambda_0) - M^{-1}(\lambda)\| \leq \\ &\quad \leq 2PQ\|M(\lambda_0)\| \, \|M^{-1}(\lambda_0)\|^2 \exp(4PQ\delta)|\lambda-\lambda_0| \end{aligned} \tag{3.15}$$

From the condition of continuous differentiability of $C(\lambda)$

$$\begin{aligned} &C = \sup\{\|C(\lambda)\|, |\lambda-\lambda_0| \leq \delta\}, \\ &\|C(\lambda) - C(\lambda_0)\| \leq E|\lambda-\lambda_0|, \quad \text{for } \; |\lambda-\lambda_0| \leq \delta \end{aligned} \tag{3.16}$$

From the equality $J(\lambda) = M^{-1}(\lambda)C(\lambda)M(\lambda)$ we have

$$\begin{aligned} J(\lambda) - J(\lambda_0) = {} & [M^{-1}(\lambda) - M^{-1}(\lambda_0)]C(\lambda)M(\lambda) + \\ & + M^{-1}(\lambda_0)[C(\lambda) - C(\lambda_0)]M(\lambda) + \\ & + M^{-1}(\lambda_0)C(\lambda_0)[M(\lambda) - M(\lambda_0)]. \end{aligned} \tag{3.17}$$

Taking norms in the last expression and from (3.9), (3.11), (3.14) and (3.15), it follows that

$$\begin{aligned} \|J(\lambda) - J(\lambda_0)\| \leq {} & 2PQC\|M^{-1}(\lambda_0)\|^2\|M(\lambda_0)\|^2 \exp(6PQ\delta)|\lambda-\lambda_0| + \\ & + \|M^{-1}(\lambda_0)\|\|M(\lambda)\|\|C(\lambda) - C(\lambda_0)\| + \\ & + 2PQ\|M^{-1}(\lambda_0)\|\|C(\lambda_0)\|\|M(\lambda_0)\| \exp(2PQ\delta)|\lambda-\lambda_0|. \end{aligned} \tag{3.18}$$

From (3.18) we can write

$$\|J(\lambda) - J(\lambda_0)\| \leq L|\lambda-\lambda_0|, \text{for} |\lambda-\lambda_0| \leq \delta \tag{3.19}$$

where

$$\begin{aligned} L = {} & 2PQC\|M^{-1}(\lambda_0)\|^2\|M(\lambda_0)\|^2 \exp(6PQ\delta) + \\ & + E\|M^{-1}(\lambda_0)\| \, \|M(\lambda_0)\| \exp(2PQ\delta) + \\ & + 2PQC \, \|M^{-1}(\lambda_0)\| \, \|M(\lambda_0)\| \exp(2PQ\delta) \end{aligned} \tag{3.20}$$

Using the inequality

$$\exp(t) - 1 \leq |t| \exp(|t|) \tag{3.21}$$

(3.13) and (3.19) we have

$$\|\exp(-tJ_j(\lambda)) - \exp(-tJ_j(\lambda_0))\| \leq bL \exp(b(\|J_j(\lambda_0)\| + L\delta))|\lambda-\lambda_0| \tag{3.22}$$

Now from (3.8) and (3.14) we have

$$\begin{aligned} \|J_j(\lambda)\| &\leq \|M^{-1}(\lambda)C(\lambda)M(\lambda)\| \\ &\leq \|M(\lambda_0)\| \, \|M^{-1}(\lambda_0)\| \, C\exp(4PQ\delta). \end{aligned} \tag{3.23}$$

If $\Gamma = \max\{e, f\}$ and from (3.12), (3.14) and (3.15)

$$\begin{bmatrix} D_1(\lambda_0) & - & D_1(\lambda) \\ \vdots & & \vdots \\ D_k(\lambda_0) & - & D_k(\lambda) \end{bmatrix} = [M^{-1}(\lambda_0) - M^{-1}(\lambda)] \begin{bmatrix} E(\lambda_0) \\ F(\lambda_0) \end{bmatrix} + $$
$$+ [M^{-1}(\lambda)] \begin{bmatrix} E(\lambda_0) - E(\lambda) \\ F(\lambda_0) - F(\lambda) \end{bmatrix}$$

and

$$\|D_j(\lambda_0) - D_j(\lambda)\| \leq K|\lambda - \lambda_0|, \quad 1 \leq j \leq k, \quad |\lambda - \lambda_0| \leq \delta, \tag{3.24}$$

with

$$\begin{aligned} K = \{&2PQ\|M(\lambda_0)\| \, \|M^{-1}(\lambda_0)\|^2 \exp(4PQ\delta)\{\|E(\lambda_0)\| + \|F(\lambda_0)\|\} + \\ &+\|M^{-1}(\lambda_0)\| \exp(2PQ\delta)\Gamma\}, \end{aligned}$$

and

$$\|D_j(\lambda_0)\| \leq \|M^{-1}(\lambda_0)\| \, \{\|E(\lambda_0)\| + \|F(\lambda_0)\|\}. \tag{3.25}$$

From (3.8), (3.23) and (3.24), if $1 \leq j \leq k$, $0 \leq t \leq b$ and $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, it follows that

$$\|\sum_{j=1}^{k} M_{1j}(\lambda)\exp(tJ_j(\lambda))\{D_j(\lambda) - D_j(\lambda_0)\}\| \leq q_1|\lambda - \lambda_0| \tag{3.26}$$

where

$$q_1 = k\|M(\lambda_0)\| \exp[2PQ\delta + bC\|M(\lambda_0)\| \, \|M^{-1}(\lambda_0)\| \exp(4PQ\delta)]K$$

From (3.8), (3.22) and (3.25) we have

$$\|\sum_{j=1}^{k} M_{1j}(\lambda)\{\exp(tJ_j(\lambda)) - \exp(tJ_j(\lambda_0))\}D_j(\lambda_0)\| \leq q_2|\lambda - \lambda_0| \tag{3.27}$$

where

$$\begin{aligned} q_2 = &Lbk\|M(\lambda_0)\| \exp[2PQ\delta + b(\|J_j(\lambda_0)\| + L\delta)]\|M^{-1}(\lambda_0)\| \\ &\cdot\{\|E(\lambda_0)\| + \|F(\lambda_0)\|\}. \end{aligned}$$

From (3.11) and (3.25), if $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ one gets that

$$\| \sum_{j=1}^{k} \{M_{1j}(\lambda) - M_{1j}(\lambda_0)\} \exp(tJ_j(\lambda_0)) D_j(\lambda_0) \| \leq q_3 |\lambda - \lambda_0| \tag{3.28}$$

where

$$q_3 = 2PQk \|M(\lambda_0)\| \exp[2PQ\delta + b(\|J_j(\lambda_0)\|)] \|M^{-1}(\lambda_0)\| \\ \cdot \{\|E(\lambda_0)\| + \|F(\lambda_0)\|\}$$

From (3.7) and (3.26)-(3.28), for $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, we obtain

$$\|x(t,\lambda) - x(,\lambda_0)\| \leq (q_1 + q_2 + q_3)|\lambda - \lambda_0| \tag{3.29}$$

for $0 \leq t \leq b$. Thus, if ε is an admissible error, one gets

$$\|x(t,\lambda) - x(t,\lambda_0)\| \leq \varepsilon, \text{ for } 0 \leq t \leq b, \quad |\lambda - \lambda_0| < \delta^* \tag{3.30}$$

where

$$\delta^* = \min\{\delta, (q_1 + q_2 + q_3)^{-1}\varepsilon\}. \tag{3.31}$$

Thus the following result has been established:

THEOREM 5. *Consider the problem* (P_λ) *where the matrices* $A_1(\lambda)$, $A_0(\lambda)$, $E(\lambda)$ *and* $F(\lambda)$ *are continuously differentiable functions of the real parameter* λ *and* $A_1(\lambda)$, $A_0(\lambda)$ *makes that the companion matrix* $C(\lambda)$ *defined in (3.2) satisfies the condition (2.3) of theorem 3. If* $\delta > 0$ *is the number defined in theorem 3 such that for* $|\lambda - \lambda_0| < \delta$, *the problem* (P_λ) *admists a unique solution* $x(t,\lambda)$ *defined by the (3.4). Moreover, if* ε *is an admissible error for the solution of the problem* (P_{λ_0}) *in the interval* $[0,b]$, *then taking* δ^* *defined in (3.31), the expression (3.30) is satisfied.*

4. Stability of the Riccati equation

Let us consider nonsymmetric Riccati matrix differential equation

$$W'(t) = C(\lambda) - D(\lambda)W(t) - W(t)A(\lambda) - W(t)B(\lambda)W(t), \; W(0) = W_0(\lambda) \tag{4.1}$$

where $W(t)$, $C(\lambda)$, $D(\lambda)$, $B(\lambda)$, $A(\lambda)$ and $W_0(\lambda)$ are $n \times n$ complex matrices. We use the explicit solution of (4.1) given in [10] for a fixed value of the parameter λ.

Theorem 6 ([10]). *Let us consider problem (4.1) where $B(\lambda)$ is invertible and let $B^{1/2}(\lambda)$ be a square root of $B(\lambda)$. Let $A_0(\lambda)$, $A_1(\lambda)$ be matrices defined by*

$$\begin{aligned} A_0(\lambda) &= -B^{\frac{1}{2}}(\lambda)[C(\lambda)+D(\lambda)B^{-1}(\lambda)A(\lambda)]B^{\frac{1}{2}}(\lambda) \\ A_1(\lambda) &= -B^{\frac{1}{2}}(\lambda)[B^{-1}(\lambda)A(\lambda)-D(\lambda)B^{-1}(\lambda)]B^{-\frac{1}{2}}(\lambda) \end{aligned} \tag{4.2}$$

Then, the solution $W(t,\lambda)$ of the problem (4.1) in a neighbourhood $\xi(0)$ of $t=0$ is given by the expression

$$W(t,\lambda) = B^{-1/2}(\lambda)V'(t,\lambda)[V(t,\lambda)]^{-1}B^{-1/2}(\lambda) - B^{-1}(\lambda)A(\lambda) \tag{4.3}$$

where $V(t,\lambda)$ is the solution of

$$\begin{aligned} &x^{(2)}(t) + A_1(\lambda)x^{(1)}(t) + A_0(\lambda)x(t) = 0 \\ &x(0) = B^{1/2}(\lambda), \quad x'(0) = B^{1/2}(\lambda)W_0(\lambda) + B^{-1/2}(\lambda)A(\lambda). \end{aligned} \tag{4.4}$$

Consider the Riccati equation (4.1) where the coefficients $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ are $\mathcal{C}^{n\times n}$ valued continuously differentiable matrix functions of the real parameter $\lambda \in I(\lambda_0) =]\lambda_0-\delta_0, \lambda_0+\delta_0[$, where $B(\lambda_0)$ is invertible.

Since $B(\lambda)$ is differentiable, from the mean value theorem [5], there exists constants $\beta>0$, $\delta_1>0$ such that for $|\lambda-\lambda_0|<\delta_1$, one gets

$$\|B(\lambda)-B(\lambda_0)\| \le \beta|\lambda-\lambda_0| < \frac{1}{\|B(\lambda_0)^{-1}\|}, \tag{4.5}$$

Then from the Banach lemma (2.2), $B(\lambda)$ is also invertible for $|\lambda-\lambda_0|<\delta_1$

$$\|B(\lambda_0)^{-1} - B(\lambda)^{-1}\| \le \|B(\lambda_0)^{-1}\|K_1\beta|\lambda-\lambda_0| \tag{4.6}$$

where K_1 is an upper bound of $\|B(\lambda)^{-1}\|$ for $|\lambda-\lambda_0|<\delta_1$.

Taking into account that

$$\begin{bmatrix} 0 & I \\ -A_0(\lambda) & -A_1(\lambda) \end{bmatrix} = P(\lambda)\begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -D(\lambda) \end{bmatrix}[P(\lambda)]^{-1} \tag{4.7}$$

where

$$P(\lambda) = \begin{bmatrix} B^{-1/2}(\lambda) & 0 \\ B^{-1/2}(\lambda)A(\lambda) & B^{1/2}(\lambda) \end{bmatrix}$$

is invertible in $\mathcal{C}^{2n\times 2n}$ because $B^{-1/2}(\lambda)$ is invertible in $\mathcal{C}^{n\times n}$.

Let us suppose that

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -D(\lambda) \end{bmatrix}$$

satisfies the condition (2.3) of theorem 3, then from (4.7)

$$\begin{bmatrix} 0 & I \\ -A_0(\lambda) & -A_1(\lambda) \end{bmatrix}$$

also satisfies the condition (2.3) of theorem 3. Thus, if $V(t,\lambda)$ is the solution of the equation (4.4). Then, by section 3, there exist real positive numbers δ, q_1 and q_2 such that

$$\begin{aligned} &||V'(t,\lambda_0) - V'(t,\lambda)|| \leq q_1|\lambda - \lambda_0| \\ &||V(t,\lambda_0) - V(t,\lambda)|| \leq q_2|\lambda - \lambda_0|. \end{aligned} \tag{4.8}$$

for $\lambda \in]\lambda_0 - \delta, \lambda_0 + \delta[$.

Given $B(\lambda_0)$, let us consider a number $\alpha \in [0, 2\pi[$, such that if we denote by H_α the half-line $\{-re^{i\alpha} : r \geq 0\}$, one gets $H_\alpha \cap \sigma(B(\lambda_0)) \neq \emptyset$. Since $\sigma(B(\lambda))$ is contained in the open set $D_\alpha = \mathcal{C} \sim H_\alpha$, by theorem 2, there exists $\delta_2 > 0$ such that $\sigma(B(\lambda)) \subseteq D_\alpha$, for $|\lambda - \lambda_0| < \delta_2$. Let us denote by $\log_\alpha$ the holomorphic determination of the logarithm defined in D_α. Then from the holomorphic functional calculus [4], a continuous determination of the square root of $B(\lambda)$ is defined by

$$[B(\lambda)]^{1/2} = \exp(1/2 \log_\alpha(B(\lambda))) \tag{4.9}$$

Since $\log_\alpha$ is holomorphic, from (2.2), (4.5) and (4.6), it follows that

$$\begin{aligned} &||B(\lambda_0)^{-1/2} - B(\lambda)^{-1/2}|| \leq \\ &\quad \leq ||B(\lambda_0)^{-1/2}||(\exp(-1/2K_2\beta||\lambda - \lambda_0||) - 1) \end{aligned} \tag{4.10}$$

for some $K_2 > 0$ and for $|\lambda - \lambda_0| < \min(\delta_1, \delta_2)$.

From (2.2) and (4.8), there exists a number $\delta_3 > 0$ such that

$$||V(t,\lambda_0) - V(t,\lambda)|| \leq q_2|\lambda - \lambda_0| < \frac{1}{||V(t,\lambda_0)^{-1}||} \tag{4.11}$$

and

$$||V(t,\lambda_0)^{-1} - V(t,\lambda)^{-1}|| \leq ||V(t,\lambda_0)^{-1}||K_3 q_2|\lambda - \lambda_0| \tag{4.12}$$

where K_3 is an upper bound of $||V(t,\lambda)^{-1}||$ for $|\lambda - \lambda_0| < \delta_3$ and $t \in \xi(0)$.

From theorem 1, there exists a positive number δ_4 such that

$$\|B(\lambda)^{-1/2}\| \leq \|B(\lambda_0)^{-1/2}\|(1 - \|B(\lambda_0)^{-1/2}\|\delta_4 M_1)^{-1} \tag{4.13}$$

and

$$\|B(\lambda)^{-1}\| \leq \|B(\lambda_0)^{-1}\|(1 - \|B(\lambda_0)^{-1}\|\delta_4 M_2)^{-1} \tag{4.14}$$

where M_1 and M_2 are upper bounds of $\frac{dB(\lambda)^{1/2}}{d\lambda}$ and $\frac{dB(\lambda)}{d\lambda}$.

Let $\delta = \min\{\delta_1, i = 1, \ldots, 4\}$ and let $U(\lambda_0) =]\lambda_0 - \delta, \lambda_0 + \delta[$. Then from the differentiability of $A(\lambda)$, there exists a constant $q_3 > 0$ such that

$$\|A(\lambda_0) - A(\lambda)\| \leq q_3|\lambda - \lambda_0|. \tag{4.15}$$

From the continuity of $V'(t, \lambda)$, there exists a constant $q_4 > 0$ such that

$$\|V'(t, \lambda)\| \leq q_4, \text{for}(t, \lambda) \in \xi(0) \times U(\lambda_0) \tag{4.16}$$

Now, we consider the difference

$$\begin{aligned} &W(t, \lambda_0) - W(t, \lambda) = \\ &\quad B(\lambda_0)^{-1/2}V'(t, \lambda_0)V(t, \lambda_0)^{-1}B(\lambda_0)^{-1/2} - B(\lambda_0)^{-1}A(\lambda_0) - \\ &\quad -B(\lambda)^{-1/2}V'(t, \lambda)V(t, \lambda)^{-1}B(\lambda)^{-1/2} - B(\lambda)^{-1}A(\lambda). \end{aligned} \tag{4.17}$$

Let us denote

$$\begin{aligned} &\|V'(t, \lambda_0)\| = Q_1, \|V(t, \lambda_0)^{-1}\| = Q_2, \\ &\|B(\lambda_0)^{-1/2}\| = Q_3, \|A(\lambda_0)\| = Q_4, \end{aligned} \tag{4.18}$$

and let us take norms in (4.17). Then, from, (4.6), (4.10), (4.12)–(4.18) and the inequality $\exp(t) - 1 \leq |t| \exp(t)$, for $|\lambda - \lambda_0| < \delta$ it follows that

$$\begin{aligned} &\|W(t, \lambda_0) - W(t, \lambda)\| \leq \\ &\leq \|[B(\lambda_0)^{-1/2} - B(\lambda)^{-1/2}]V'(t, \lambda_0)V(t, \lambda_0)^{-1}B(\lambda_0)^{-1/2}\| + \\ &\quad +\|B(\lambda)^{-1/2}[V'(t, \lambda_0) - V'(t, \lambda)]V(t, \lambda_0)^{-1}B(\lambda_0)^{-1/2}\| + \\ &\quad +\|B(\lambda)^{-1/2}V'(t, \lambda)[V(t, \lambda_0)^{-1} - V(t, \lambda)^{-1}]B(\lambda_0)^{-1/2}\| + \\ &\quad +\|B(\lambda)^{-1/2}V'(t, \lambda_0)V(t, \lambda_0)^{-1}[B(\lambda_0)^{-1/2} - B(\lambda)^{-1/2}]\| + \\ &\quad +\|[B(\lambda_0)^{-1} - B(\lambda)^{-1}]A(\lambda_0)\| + \|B(\lambda)^{-1}[A(\lambda_0) - A(\lambda)]\| \leq \\ &\leq \Gamma|\lambda - \lambda_0|. \end{aligned} \tag{4.19}$$

where

$$\Gamma = Q_1Q_2Q_3^2\exp(-1/2K_2\beta\delta)1/2K_2\beta +$$
$$+Q_2Q_3^2(1-Q_3\delta_4M_1)^{-1}q_1 + (1-Q_3\delta_4M_1)^{-1}Q_2Q_3^2K_3q_2q_4 +$$
$$+(1-Q_3\delta_4M_1)^{-1}Q_3^2q_4K_3\exp(-1/2K_2\beta\delta)1/2K_2\beta +$$
$$+Q_3Q_4K_1\beta + Q_3(1-Q_3\delta_4M_2)^{-1}q_3. \quad (4.20)$$

THEOREM 7. *Let us consider the previous notation and the Riccati equation (4.1) where $B(\lambda_0)$ is invertible and the matrix*

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -D(\lambda) \end{bmatrix}$$

satisfies the condition (2.3) in a neighbourhood of $\lambda = \lambda_0$. Then, there exist an interval $U(\lambda_0) =]\lambda_0 - \delta, \lambda_0 + \delta[$ and a neighbourhood $\xi(0)$ of $t = 0$ such that for $(t,\lambda) \in U(\lambda_0) \times \xi(0)$ the solution $W(t,\lambda)$ of (4.1) is given by

$$W(t,\lambda) = B(\lambda)^{-1/2}\{\sum_{j=1}^{k} M_{2j}(\lambda)\exp(tJ_j(\lambda))D_j(\lambda)\}$$
$$\{\sum_{j=1}^{k} M_{1j}(\lambda)\exp(tJ_j(\lambda))D_j(\lambda)\}^{-1}B(\lambda)^{-1/2} - B(\lambda)^{-1}A(\lambda)$$

where the matrices $D_j(\lambda) \in \mathcal{C}^{m_j x n}$, for $1 \leq j \leq k$, are uniquely determined by $W(0) = W_0(\lambda)$ and the expression

$$\begin{bmatrix} D_1(\lambda) \\ \vdots \\ D_k(\lambda) \end{bmatrix} = [M(\lambda)]^{-1} \begin{bmatrix} B^{1/2}(\lambda) \\ B^{1/2}(\lambda)W_0(\lambda) + B^{-1/2}(\lambda)A(\lambda) \end{bmatrix},$$

and satisfies (4.19), (4.20).

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Stabilne rozwiązanie analityczne dla pewnej klasy macierzowych równań różniczkowych Riccatiego

W pracy rozważa się niesymetryczne macierzowe równania różniczkowe Riccatiego, których współczynniki zależą od rzeczywistego parametru. Podano warunki, przy których otrzymuje się rozwiązanie analityczne, stabilne względem zaburzeń współczynników.

Устойчивое аналитическое решение для некоторого класса матричных дифференциальных уравнении Риккати

В работе рассматриваются несимметричные матричные дифференциальные уравниения Риккати, коэффициенты которых зависят от действительного параметра. Приведены условия при которых получаем аналитическое решение, устойчивое при изменяющихся коэффициентах.