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Analytic approximate solutions and error bounds for linear matrix differential equations appearing in control

by

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In this paper we construct analytic approximate solutions for initial value problems related to the matrix differential equation $X'(t) = A(t)X(t) + X(t)B(t)$, with twice continuously differentiable coefficient matrix functions. By means of one–step matrix methods and linear B–spline matrix functions which interpolate the numerical solution in a net of points, an approximate solution whose error is smaller than a prefixed $\varepsilon > 0$ in all the domain is constructed.

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1. Introduction

Linear matrix differential equations of the type

$$X'(t) = A(t)X(t) + X(t)B(t), \quad X(0) = C, \quad 0 \leq t \leq b \tag{1.1}$$

where the unknown $X(t)$ and the coefficients $A(t), B(t)$ are $r \times r$ complex matrices, elements of $C^{r\times r}$, arise in many fields of science and engineering mainly in optimization problems in linear control theory [1, 11, 13]. Equation (1.1) has been studied by several authors for the constant coefficient case, see [2, 3, 12], however for the variable coefficient case such equation has received little numerical treatment in the literature.

From [11], [1, p.109], the theoretical solution of equation (1.1) is given by $X(t) = Y(t)CZ(t)$ where $Y(t)$ is the solution of the matrix equation

$$Y'(t) = A(t)Y(t), \; Y(0) = I, \tag{1.2}$$

and $Z(t)$ is the solution of

$$Z'(t) = Z(t)B(t), \; Z(0) = I \tag{1.3}$$

Unfortunately the exact solution of problems (1.2) and (1.3) is not computable in an analytic way and this motivates the search of alternatives which provide analytic approximate solutions and error bounds for them in terms of data.

The paper is organized as follows. Section 2 deals with somewhat general results about one–step matrix methods for the numerical solution of matrix differential equations of the type

$$Y'(t) = f(t, Y(t)), \quad Y(0) = \Omega \in C^{r\times q}, \quad 0 \leq t \leq b \tag{1.4}$$

where $f : [0, b] \times C^{r\times q} \longrightarrow C^{r\times q}$ is bounded, continuous and satisfies the Lipschitz condition

$$\|f(t, P) - f(t, Q)\| \leq L\|P - Q\| \tag{1.5}$$

what guarantees the existence of a unique continuously differentiable matrix function $Y(t)$, solution of (1.4), [4, p.99]. Section 2 completes recent results given in [8] for linear k–step matrix methods with $k > 1$, including an upper bound for the global discretization error of linear one–step matrix methods. Using the results of section 2 and by means of spline matrix functions we construct analytic approximate solutions in all the domain and a global error bound in terms of the stepsize and the data is given. Given an admissible error $\varepsilon > 0$ we construct an approximate solution whose error is smaller than ε uniformly in all the interval.

If B is a matrix in $\mathcal{C}^{p\times q}$ we denote by $\|B\|$ the square root of the maximum of the set $\{|z|; z$ eigenvalue of $B^H B\}$ where B^H denotes the conjugate transpose of B, see [10, p.41]. We recall that from [5, p.15], it follows that

$$\begin{aligned} &\max\{|b_{ij}|; 1 \le i \le p, 1 \le j \le q\} \le \\ &\quad \le \|B\| \le (qp)^{\frac{1}{2}} \max\{|b_{ij}|; 1 \le i \le p, 1 \le j \le q\} \end{aligned} \tag{1.6}$$

The identity matrix in $\mathcal{C}^{r\times r}$ is denoted by I.

2. One–step matrix methods for matrix differential equations

In a recent paper linear k–step matrix methods for $k \ge 2$ have been introduced in [8] but no error bound is given. Let us consider one–step methods of the form

$$Y_{n+1} - Y_n = h\{B_1 f_{n+1} + B_0 f_n\} \tag{2.1}$$

where B_0, B_1 are matrices in $\mathcal{C}^{r\times r}$ and $Y_n, f_n = f(t_n, Y_n) \in \mathcal{C}^{r\times q}$ with $t_n = nh \in [0, b]$, $h > 0$ and

$$B_0 + B_1 = I \tag{2.2}$$

Let us consider a matrix difference equation of the form

$$Z_{m+1} - Z_m = h\{B_{1,m}\|Z_{m+1}\| + B_{0,m}\|Z_m\|\} + \Lambda_m, \tag{2.3}$$

where $\Lambda_m, B_{1,m}, B_{0,m}$ are matrices in $\mathcal{C}^{r\times q}$, $h > 0$, m is a non–negative integer, and let $\{Z_m\}$ be a matrix sequence solution of (2.3). If we write equation (2.3) for $m = n - p - 1$, we have

$$Z_{n-p} - Z_{n-p-1} = h\{B_{1,n-p-1}\|Z_{n-p}\| + B_{0,n-p-1}\|\} + \Lambda_{n-p-1},$$

Considering the last equation for $p = 0, 1, \ldots, n-1$, and adding the resulting equations it follows that the sum of the left hand side is $S_n = Z_n - Z_0$ and the sum of the right hand side takes the value

$$\begin{aligned} &h\{B_{1,n-1}\|Z_n\| + (B_{0,n-1} + B_{1,n-2})\|Z_{n-1}\| + \ldots + B_{0,0}\|Z_0\|\} \\ &\quad + \Lambda_{n-1} + \Lambda_{n-2} + \ldots + \Lambda_0 \end{aligned}$$

Equating the last equation to $Z_n - Z_0$ and taking norms it follows that

$$\|Z_n\| \leq hB\|Z_n\| + hB_* \sum_{m=0}^{n-1} \|Z_m\| + N\Lambda + 2\|Z_0\| \tag{2.4}$$

where

$$\|B_{1,p}\| \leq B, \ \|B_{1,p}\| + \|B_{0,p}\| \leq B_*, \ \ \|\Lambda_p\| \leq \Lambda, \ \ 0 \leq p \leq N \tag{2.5}$$

$$h < B^{-1}, \ \Gamma_* = (1 - hB)^{-1}, \ L_* = \Gamma_* B_*, \ K_* = \Gamma_*(N\Lambda + 2\|Z_0\|) \tag{2.6}$$

Hence

$$(1 - hB)\|Z_n\| \leq hB_* \sum_{m=0}^{n-1} \|Z_m\| + (N\Lambda + 2\|Z_0\|)$$

and from (2.5), (2.6), it follows that

$$\|Z_n\| \leq hL_* \sum_{m=0}^{n-1} \|Z_m\| + K_* \tag{2.7}$$

From (2.7) and [7, p.246] one gets

$$\|Z_n\| \leq K_*(1 + hL_*)^n \quad , \quad 0 \leq n \leq N \tag{2.8}$$

Taking into account that $(1 + hL_*)^n \leq \exp(nhL_*)$, the following result has been proved.

THEOREM 1. *Let us consider the one-step matrix method (2.1)-(2.2) and the difference matrix equation (2.3). Let us consider the constants* Λ, B, B_*, K_* *and* L_* *defined by (2.5)-(2.6), then for any matrix sequence solution* Z_n *of (2.3) it follows that*

$$\|Z_n\| \leq K_* \exp(nhL_*) \quad , \quad 0 \leq n \leq N \tag{2.9}$$

Let us introduce the difference operator L associated to the method (2.1) and defined by

$$L(Y(t); h) = Y(t + h) - Y(t) - h(B_1 Y'(t + h) + B_0 Y'(t)) \tag{2.10}$$

where $Y(t)$ is an arbitrary $\mathcal{C}^{r \times q}$ valued continuously differentiable function in $[0, b]$. Expanding the test function $Y(t + jh)$ and its derivative $Y'(t + jh)$ as Taylor series about t, and collecting terms in (2.10) gives

$$L(Y(t); h) = C_0 Y(t) + C_1 h Y'(t) + \ldots + C_s h^s Y^{(s)}(t) + \ldots \tag{2.11}$$

where C_s is a matrix in $\mathcal{C}^{r\times r}$ that may be written in terms of the matrix coefficients

$$\begin{aligned} &C_0 = 0; \quad C_1 = I - (B_0 + B_1) = 0, \ldots, \\ &C_s = (s!)^{-1} I - ((s-1)!)^{-1} B_1, \qquad s = 2, 3, \ldots \end{aligned} \tag{2.12}$$

In an analogous way to the well known definition for the scalar case we say that the method (2.1) is of order p if, in (2.11), $C_0 = C_1 = \ldots = C_p = 0$ and $C_{p+1} \neq 0$. It is easy to prove like in the scalar case, see [9, p.49–52], [7, p.257], that

$$||L(Y(t_n); h)|| \leq h^{p+1} GD = Q, \tag{2.13}$$

where

$$G = ||C_{p+1}|| \ , \ D \geq \max\{||Y^{(p+1)}(t)||; 0 \leq t \leq b\} \tag{2.14}$$

The global truncation error of the method (2.1) at the point $t_n = nh$, denoted by e_n, is the difference $e_n = Y(t_n) - Y_n$, where $Y(t_n)$ is the value of the theoretical solution of (1.4) at t_n and Y_n is the approximate value provided by the method (2.1).

If we write (2.1) in the form

$$Y_{n+1} - Y_n - h\{B_1 f_{n+1} + B_0 f_n\} = 0$$

and substracting from this expression the quantity $L(Y(t_n); h)$ it follows that

$$\begin{aligned} &e_{n+1} - e_n - h\{B_1(Y'(t_n + h) - f_{n+1}) + B_0(Y'(t_n) - f_n)\} = \\ &= e_{n+1} - e_n - h\{B_1(f(t_{n+1}, Y(t_{n+1})) - f_{n+1}) + \\ &\quad + B_0(f(t_n, Y(t_n)) - f_n))\} + L(Y(t_n); h) \end{aligned} \tag{2.15}$$

Now let us consider the sequence of matrices in $\mathcal{C}^{r\times q}$ defined by

$$P_n = \begin{cases} (f(t_n, Y(t_n)) - f(t_n, Y_n))||e_n||^{-1}, & \text{if} \quad e_n \neq 0 \\ 0, & \text{if} \quad e_n = 0 \end{cases} \tag{2.16}$$

From (2.15)–(2.16) we have

$$e_{n+1} - e_n = h\{B_1 P_{n+1} ||e_{n+1}|| + B_0 P_n ||e_n||\} + L(Y(t_n); h)$$

Let us denote $Z_n = e_n, \Lambda_n = L(Y(t_n); h)$ and let us suppose that the method (2.1) is of the order $p \geq 1$. From (2.13) it follows that $||\Lambda_n|| = ||L(Y(t_n); h)|| \leq$

$h^{p+1}GD$, where G and D are defined by (2.14). Note that from (1.5) and (2.16) it follows that $\|P_n\| \leq L$, $\|B_1 P_{n+1}\| \leq L\|B_1\|$ and taking

$$B^* = \|B_0\| + \|B_1\|, \quad N = b/h, \tag{2.17}$$

from THEOREM 1 and the previous comments the following result has been established

THEOREM 2. *Let us consider a one-step method of the type (2.1)-(2.2), of order* $p \geq 1$. *Let* h, Γ^* *be defined by*

$$h < (L\|B_1\|)^{-1}, \quad \Gamma^* = (1 - hL\|B_1\|)^{-1} \tag{2.18}$$

and G, D *defined by (2.14), then the global discretization error* e_n *is upper bounded by*

$$\|e_n\| \leq \Gamma^* h^P GDt_n \exp(L\Gamma^* B^* t_n) \tag{2.19}$$

Now we apply the previous results to a concrete matrix differential equation that will be important in the following.

EXAMPLE 1. Let us consider the one-step matrix method (2.1)-(2.2), where $B_0 = B_1 = I/2$,

$$Y_{n+1} - Y_n = \frac{h}{2}\{f_{n+1} + f_n\} \tag{2.20}$$

From (2.12) it follows that $C_0 = C_1 = C_2 = 0$ and $C_3 = -I/12$. Thus (2.20) defines a one-step method of order $p = 2$. The constants appearing in THEOREM 2 take the values

$$G = \|C_3\| = 1/12, \quad B = \|B_0\| + \|B_1\| = 2, \quad \Gamma^* = (1 - hL/2)^{-1}$$

and $D \geq \max\{\|Y^{(3)}(t)\|;\ 0 \leq t \leq b\}$. The inequality (2.19) takes the form

$$\|e_n\| \leq h^2 \frac{Dt_n}{12}(1 - hL/2)^{-1} \exp(t_n L(1 - hL/2)^{-1}) \tag{2.21}$$

Let us consider the matrix differential equation (1.2) where $A(t)$ is a 2-times continuously differentiable matrix function. Taking derivatives for the solution $Y(t)$ of (1.2), it follows that

$$\begin{aligned} Y^{(2)}(t) &= A'(t)Y(t) + (A(t))^2 Y(t), \\ Y^{(3)}(t) &= A^{(2)}(t)Y(t) + A'(t)A(t)Y(t) + (A(t))^3 Y(t) + \\ &\quad + (A'(t)A(t) + A(t)A'(t))Y(t) = \\ &= A^{(2)}(t)Y(t) + 2A'(t)A(t)Y(t) + A(t)A'(t)Y(t) + (A(t))^3 Y(t) \end{aligned}$$

From [4, p.114], the theoretical solution $Y(t)$ of (1.2) satisfies

$$\|Y(t)\| \le \exp(tk_0)\ ,\quad 0 \le t \le b, \tag{2.22}$$

and if we denote by k_i, for $i = 0, 1, 2$, the positive constants satisfying

$$k_i \ge \max\{\|A^{(i)}(t)\|;\ 0 \le t \le b\},\quad i = 0, 1, 2, \tag{2.23}$$

we have

$$\max\{\|Y^{(3)}(t)\|; 0 \le t \le b\} \le \exp(bk_0)\{k_0^3 + 3k_1k_0 + k_2\} = D \tag{2.24}$$

If $h < 2/k_0$, then from the previous comments it follows that the global discretization error e_n at $t_n = nh$, when one approximates the exact value of the solution of (1.2) by the value Y_n obtained by means of (2.20), satisfies

$$\begin{aligned}\|e_n\| \le h^2 t_n \exp(bk_0)(1 - k_0h/2)^{-1}\{k_0^3 + 3k_1k_0 + k_2\}\cdot \\ \cdot \exp(t_n(1 - k_0h/2)^{-1}k_0)/12\end{aligned} \tag{2.25}$$

Since $h < 1/k_0$ implies

$$1 - k_0h/2 > 1/2 \tag{2.26}$$

then for $h < 1/k_0$, (2.25) takes the form

$$\|e_n\| \le \frac{h^2 t_n}{6} \exp(bk_0)\{k_0^3 + 3k_1k_0 + k_2\} \exp(2k_0t_n) \tag{2.27}$$

If we consider the problem (1.3) and define the constants

$$q_i \ge \max\{\|B^{(i)}(t)\|; 0 \le t \le b\}\ ,\quad i = 0, 1, 2 \tag{2.28}$$

and we denote by $v_n = Z(t_n) - Z_n$ the global discretization error when one approximates the exact value $Z(t_n)$ of the solution of (1.3) by the numerical solution Z_n computed by

$$Z_{n+1} - Z_n = \frac{h}{2}\{g_{n+1} + g_n\}\ , \tag{2.29}$$

where $g_n = G(t_n, Z_n)$ and $G(t, Z) = ZB(t)$, then for values of h such that

$$h < 1/q_0, \tag{2.30}$$

it follows that

$$\|v_n\| \le \frac{h^2 t_n}{6} \exp(bq_0)\{q_0^3 + 3q_1q_0 + q_2\} \exp(2q_0t_n) \tag{2.31}$$

3. Analytic approximate solutions and error bounds

We begin this section with some results about interpolating B–splines. If we are interested in the construction of an approximation of a function which interpolates the exact values $y_0, y_1, \ldots, y_N$ at the knots $t_0, t_1, \ldots, t_N$, then the linear B–spline is defined by

$$\hat{s}(t) = \sum_{n=-1}^{N-1} B_{1n}(t) y_{n+1} \tag{3.1}$$

where $t \in [0, b]$, $t_{n+1} - t_n = h, h = b/N$ and

$$B_{1n} = h^{-1} \begin{cases} (t - t_n) & for \quad t_n \leq t < t_{n+1} \\ (t_{n+2} - t) & for \quad t_{n+1} \leq t < t_{n+2} \end{cases} \tag{3.2}$$

with $B_{1n}(t) = 0$ for $t < t_n$ and $t_{n+2} \leq t$. In addition, B_{1n} is non–negative satisfying $B_{1n}(t) + B_{1,n-1}(t) = 1$ for all $t \in [t_n, t_{n+2}]$, see [6, p.247–248]. If we consider the linear B–spline constructed in terms of approximate values $\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_N$, then we have a new approximating function

$$\hat{S}(t) = \sum_{n=-1}^{N-1} B_{1n}(t) \hat{y}_{n+1} \tag{3.3}$$

such that

$$\begin{aligned} ||\hat{s}(t) - \hat{S}(t)| &\leq \max\{|y_{n+1} - \hat{y}_{n+1}|; -1 \leq n \leq N-1\} \sum_{n=-1}^{N-1} |B_{1n}(t)| = \\ &= \max\{|y_{n+1} - \hat{y}_{n+1}|; -l \leq n \leq N\}, \quad \text{for } 0 = t_0 \leq t \leq t_N = b \end{aligned} \tag{3.4}$$

Now let us consider the matrix case, given the exact values $Y(t_0), Y(t_1), \ldots, Y(t_N)$, of a $\mathcal{C}^{r \times r}$ valued function defined in $t_0 = 0, t_1, \ldots, t_N = b$, but unknown in the rest of the interval, we are interested in construction of a linear interpolating B–spline matrix function $W(t)$ knowing not the exact values $Y(t_n)$ but the approximate values Y_n of $Y(t_n)$, for $n = 0, 1, \ldots, N$. Let us consider the linear B–spline matrix functions defined by

$$\begin{aligned} &V(t) = \sum_{n=-1}^{N-1} Y(t_{n+1}) B_{1n}(t), \quad W(t) = \sum_{n=-1}^{N-1} Y_{n+1} B_{1n}(t), \\ &\quad 0 = t_0 \leq t \leq t_n = b. \end{aligned} \tag{3.5}$$

Then, taking norms, we obtain

$$\|V(t) - W(t)\| \le \max\{\|Y(t_n) - Y_n\|;\ \ 0 \le n \le N\} \tag{3.6}$$

If $f(t)$ is a scalar function and $\hat{s}(t)$ is the linear B–spline defined by (3.1) with $y_n = f(t_n)$ for $0 \le n \le N$, and assuming that $f(t)$ is twice continuously differentiable function in the interval [0,b], then from [6, p.257], it follows that

$$\max_{0\le t\le b} |f(t) - \hat{s}(t)| \le \frac{h^2}{8} \max_{0\le t\le b} |f^{(2)}(t)| \tag{3.7}$$

If $Y(t)$ is a $\mathcal{C}^{r\times r}$ valued function and we apply (3.7) and (1.6), assuming twice continuously differentiably $Y(t)$, the linear B–spline matrix function $V(t)$ defined by (3.5) satisfies

$$\max_{0\le t\le b} \|Y(t) - V(t)\| \le \frac{rh^2}{8} \max_{0\le t\le b} \|Y^{(2)}(t)\| \tag{3.8}$$

Now we are interested in the construction of an analytic approximate solution of equation (1.1). First of all note that the numerical solution Y_n of (2.20) corresponding to the problem (1.2) comes from the relationship

$$Y_{n+1} - Y_n = \frac{h}{2}(A(t_{n+1})Y_{n+1} + A(t_n)Y_n)\ ;\ Y_0 = I$$

$$(I - \frac{h}{2}A(t_{n+1}))Y_{n+1} = (I + \frac{h}{2}A(t_n))Y_n,\ Y_0 = I, n \ge 1 \tag{3.9}$$

If $h < 1/k_0$, where k_0 is defined by (2.23), then from the Perturbation lemma, [10, p.45], the matrix coefficients of Y_{n+1} and Y_n in (3.9) are invertible and one gets

$$Y_0 = I,\ Y_n = \prod_{j=0}^{n-1}\{(I - \frac{h}{2}A(t_{n-j}))^{-1}(I + \frac{h}{2}A(t_{n-j-1}))\}\ ,\ n \ge 1 \tag{3.10}$$

In an analogous way, if $h < 1/q_0$, where q_0 is defined by (2.28), the numerical solution Z_n of equation (2.29) corresponding to problem (1.3) comes from the relationship

$$Z_{n+1} - Z_n = \frac{h}{2}(Z_{n+1}B(t_{n+1}) + Z_nB(t_n));\ Z_0 = I$$

$$Z_{n+1}(I - \frac{h}{2}B(t_{n+1})) = Z_n(I + \frac{h}{2}B(t_n));\ Z_0 = I,\ n \ge 1$$

whose solution takes the form

$$Z_0 = I, \quad Z_n = \prod_{j=0}^{n-1} \{(I + \frac{h}{2}B(t_j))(I - \frac{h}{2}B(t_{j+1}))^{-1}\} \ , \ n \geq 1 \tag{3.11}$$

Takıng into account that the exact solution $X(t)$ of problem (1.1) is given by $X(t) = Y(t)CZ(t)$, from (3.10) and (3.11), we have numerical approximate solution of $X(t)$ at $t_n = nh$, given by

$$X_n = Y_n C Z_n, \quad X_0 = C, \quad 1 \leq n \leq N, \tag{3.12}$$

where Y_n is given by (3.10) and Z_n by (3.11), for $1 \leq n \leq N$ with $Nh = b$. Starting from the approximate values of $X(t)$ at $t_n = nh$, given by X_n defined by (3.12), we construct the linear B–spline matrix function

$$W(t) = \sum_{n=-1}^{N-1} X_{n+1} B_{1n}(t), \quad 0 \leq t \leq b, \tag{3.13}$$

where $B_{1n}(t)$ is defined by (3.2). If we denote by $V(t)$ the theoretical linear B–spline matrix function

$$V(t) = \sum_{n=-1}^{N-1} X(t_{n+1}) B_{1n}(t), \quad 0 \leq t \leq b \tag{3.14}$$

interpolating the exact values $X(t_n)$ at t_n of the solution of problem (1.1), then from (3.6) it follows that

$$\|V(t) - W(t)\| \leq \max\{\|X(t_n) - X_n\|; 0 \leq n \leq N\} \ , \quad 0 \leq t \leq b \tag{3.15}$$

In order to obtain an upper bound of the right hand side of (3.15), note that

$$\begin{aligned} X(t_n) - X_n &= Y(t_n)CZ(t_n) - Y_nCZ_n = \\ &= (Y(t_n)-Y_n)CZ(t_n)+Y(t_n)C(Z(t_n)-Z_n)-(Y(t_n)-Y_n)C(Z(t_n)-Z_n) \end{aligned} \tag{3.16}$$

Since from [4, p.114], we have that

$$\|Y(t)\| \leq \exp(tk_0) \ \text{ and } \ \|Zt)\| \leq \exp(tq_0) \ \text{ for } \ 0 \leq t \leq b, \tag{3.17}$$

taking norms in (3.16) and using (3.17) one gets

$$\begin{aligned} &\|X(t_n) - X_n\| \leq \exp(t_n q_0)\|e_n\|\|C\| + \exp(t_n k_0)\|v_n\|\|C\| + \\ &+\|C\|\|e_n\|\|v_n\| \end{aligned} \tag{3.18}$$

where $e_n = Y(t_n) - Y_n$ and $v_n = Z(t_n) - Z_n$, for $0 \le n \le N$, $t_n = nh$. From (2.27), (2.31) and (3.18) it follows that

$$\begin{aligned} \|X(t_n) - X_n\| \le \|C\|\frac{h^2 b}{6}\{&\exp(3bq_0)(q_0^3 + 3q_1q_0 + q_2)+ \\ &+\exp(3k_0b)(k_0^3 + 3k_1k_0 + k_2)\} + \\ &+\|C\|\frac{h^4b^2}{36}\exp(3b(k_0+q_0))(q_0^3 + 3q_1q_0 + q_2)(k_0^3 + 3k_1k_0 + k_2), \end{aligned} \tag{3.19}$$

for $0 \le n \le N$ and for values of $h < 1/q_0$, $h < 1/k_0$.

Note that if $X(t)$ is the theoretical solution of (1.1) and $A(t), B(t)$ are twice continuously differentiable functions in $[0, b]$, then,

$$\begin{aligned} X^{(2)}(t) &= A'(t)X(t) + A(t)X'(t) + X'(t)B(t) + X(t)B'(t) = \\ &= A'(t)X(t)+(A(t))^2X(t)+A(t)X(t)B(t)+A(t)X(t)B(t)+ \\ &\quad +X(t)(B(t))^2 + X(t)B'(t) \end{aligned} \tag{3.20}$$

Taking into account (2.23), (2.28), the relationship $X(t) = Y(t)CZ(t)$ and (3.20), (3.17), it follows that

$$\|X^{(2)}(t)\| \le \|C\| \exp(b(k_0+q_0))\{k_1+k_0^2+2q_0k_0+q_0^2+q_1\}, \quad 0 \le t \le b \tag{3.21}$$

From (3.8) and (3.14), it follows that

$$\|X(t) - V(t)\| \le h^2\alpha, \quad 0 \le t \le b, \tag{3.22}$$

where

$$\alpha = \frac{r}{8}\|C\| \exp(b(k_0 + q_0))\{k_1 + k_0^2 + 2q_0k_0 + q_0^2 + q_1\} \tag{3.23}$$

From (3.14),(3.15),(3.19),(3.22) and (3.23), the difference between the theoretical solution $X(t)$ of problem (1.1) and the linear B-spline matrix function $V(t)$ interpolating the approximate values X_n given by (3.10)-(3.12), is uniformly upper bounded in $[0, b]$ by the inequality

$$\|X(t) - V(t)\| \le \beta h^2 + \gamma h^4, \quad 0 \le t \le b, \tag{3.24}$$

where

$$\begin{aligned} \beta = \alpha + \frac{\|C\|b}{6}\{&\exp(3bq_0)(q_0^3 + 3q_1q_0 + q_2) \\ &+\exp(3k_0b)(k_0^3 + 3k_1k_0 + k_2)\} \end{aligned} \tag{3.25}$$

$$\gamma = \frac{\|C\|b^2}{36} \exp(3b(k_0 + q_0))(q_0^3 + 3q_1q_0 + q_2)(k_0^3 + 3k_1k_0 + k_2) \tag{3.26}$$

α is defined by (3.23) and $h < \min(1/k_0, 1/q_0)$.

Given an admissible error $\varepsilon > 0$ if we choose a positive number h such that

$$h < \min(1/k_0, 1/q_0),\ b/h \text{ is integer and } \quad \beta h^2 + \gamma h^4 \leq \varepsilon, \tag{3.27}$$

then taking $N = b/h$, considering the sequence $\{X_n\}_{n=0}^N$ defined by (3.10)–(3.12), and $V(t)$, the linear B–spline matrix function defined by

$$\begin{aligned} V(t) &= h^{-1}\{(t_{n+1} - t)X_n + (t - t_n)X_{n+1}\}; \\ t_n &= nh,\ t \in [t_n, t_{n+1}],\ 0 \leq n \leq N-1 \end{aligned} \tag{3.28}$$

defines an approximate solution of problem (1.1) whose error $E(t) = X(t) - V(t)$ satisfies

$$\|X(t) - V(t)\| \leq \varepsilon, \quad \text{uniformly for } \ 0 \leq t \leq b \tag{3.29}$$

Taking logarithms the inequality $\beta h^2 + \gamma h^4 \leq \varepsilon$ is equivalent to the condition

$$\ln(h) \leq \frac{1}{4}\{\ln(\varepsilon) - \ln(\beta + \gamma)\} \tag{3.30}$$

From the previous comments the following result has been established

THEOREM 3. *Let us consider the problem (1.1) where the matrix functions* $A(t), B(t)$ *are twice continuously differentiable in* $[0, b]$, *and let* k_i *and* q_i *be defined by (2.23) and (2.28) respectively, for* $i = 0, 1, 2$. *Given an admissible error* $\varepsilon > 0$, *let* β *and* γ *be defined by (3.25) and (3.26) and let* h *be a positive real number satisfying* $h < \min(1/k_0, 1/q_0)$ *and (3.30) with* b/h *integer. Let* $b/h = N$ *and let* $\{Y_n\}\{Z_n\}$ *be the finite sequences defined by (3.10) and (3.11) respectively, for* $0 \leq 1 \leq N$, *then the linear B–spline matrix function* $V(t)$ *defined by (3.28) which interpolates the sequence* $\{X_n\}_{n=0}^N$ *given by (3.12) is an analytic approximate solution of problem (1.1) whose error is uniformly upper bounded by* ε *in all the domain* $[0, b]$.

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Przybliżone rozwiązanie analityczne i oszacowanie błędu dla liniowych macierzowych równań różniczkowych występujących w zagadnieniach sterowania

W artykule przedstawiono przybliżone rozwiązania analityczne dla zadań z warunkiem początkowym związanych z macierzowym równanieniem różniczkowym $X'(t) = A(t)X(t)+X(t)B(t)$, w którym funkcje macierzowe parametrów są klasy C^2. Używając do interpolacji rozwiązania na siatce punktów macierzowych funkcji skokowych i liniowych splajnów uzyskuje się rozwiązanie przyliżone, którego błąd jest mniejszy od założonego $\epsilon > 0$ w całym obszarze.

Приближенные аналитические решения и оценка погрешности для линейных матричных дифференциальных уравнений появляющихся в управлении

В статье представлены приближенные аналитические решения для задач с начальными условиями, связанных с матричным дифференцальным уравнением $X'(t) = A(t)X(t) + X(t)B(t)$, в котором матричные функции параметров являются класса C^2. Используя для интерполяции решения на сетке точек матричных функций скачкообразные и линейные сплайны, получаем приближенное решение, погрешность которого меньше заданной $\epsilon > 0$ для целой области.