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Optimal pole assignment and redundancy problem

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In the earlier paper of the same authors [1] an Optimal Pole Assignment (OPA) theorem has been enunciated. Using this theorem the Optimal Pole Region (OPR) has been delineated. A recursive procedure has been used to carry out optimal pole assignment. At each recursion one or two poles have been assigned. In this paper a method for multiple real pole assignment at each recursion is developed. In the process a physical interpretation for Riccati equation solution matrix P has been given. An algorithm for multiple pole assignment is presented. In carrying out optimal pole assignment we have assumed that $R = I_m$ and determined the optimal pole region.

In a subsequent article this assumption has been relaxed. Our investigation reveals that a general R alters the OPR and that it is possible to increase the OPR.

1. Introduction

It is well known that for a multi-input system there are many control laws which achieve the same closed loop pole configuration. It indicates thereby that apart from pole assignment, a state feedback could satisfy additional performance requirements such as minimization of a quadratic performance index. Such an optimal pole assignment (OPA) amalgamates the advantages of improved transient response of pole assignment and the feedback properties of linear quadratic design. In our earlier paper [1] an OPA theorem has been enunciated. Using this OPA theorem the Optimal Pole Region (OPR) for one or two poles has been delineated. A recursive procedure has been used to carry out the optimal pole assignment. In this paper OPA theorem has been extended to carry out multiple pole assignment at each recursion.

The optimal pole assignment by delineating OPR has been done with $R = I_m$. The OPR helps to locate the optimal poles and also to calculate the minimizing cost function Q.

For a given controller K, cost function (Q, R) is not unique. This is the redundancy problem. In the paper by Molinari [3] a theorem has been stated. This theorem states that the cost functions Q^1 and Q^2 for an optimal K are equivalent if there exists a symmetric matrix Y satisfying $A'Y + YA = Q^1 - Q^2$ and $YB = 0$. But nothing has been mentioned about the cost function R. In the paper by Martin [2] a linear algebraic equation has been established to determine the matrix pairs that are equivalent to (Q, R).

The redundancy problem subsequently dealt with in this paper differs from the works of Molinari and Martin. For a set of optimally assigned poles by delineating OPR, a method for generating equivalent controller K and corresponding cost function (Q, R) has been proposed. It has been found that optimal pole region alters and it is possible to increase the OPR.

2. OPA theorem

Consider the controllable system

$$\dot{X} = AX + BU \tag{1}$$

with control law

$$U = -KX \tag{2}$$

the closed loop system is

$$\dot{X} = \bar{A}X = (A - BK)X \tag{3}$$

wherein the dimensions of state and input vector are $n \times 1$ and $m \times 1$ respectively. The other system matrices are of compatible dimensions.

A reduced order controllable system can be decoupled from this system by the real Schur form (RSF) transformation for pole assignment. A recursive procedure is adopted for complete pole assignemnt [1]. Thus the reduced order controllable system at k–th recursion is

$$\dot{X}_{kL} = A_{kL}X_{kL} + B_{kL}U \tag{4}$$

with control law

$$U = -K_{kL}X_{kL} \tag{5}$$

the closed loop system becomes

$$\dot{X}_{kL} = \bar{A}_{kL}X_{kL} = (A_{kL} - B_{kL}K_{kL})X_{kL} \tag{6}$$

In this system X_{kL} and U are $p_k \times 1$ and $m \times 1$ state and input vectors respectively. The remaining matrices are of compatible dimensions. In addition, B_{kL} is of full rank.

Let the linear quadratic cost function be

$$J = \int_0^\infty (X_{kL}^T Q_{kL} X_{kL} + U^T RU)dt \tag{7}$$

where, $Q_{kL} = Q_{kL}^T \geq 0$ (positive semidefinite) and $R = R^T > 0$ (positive definite), the upper index T denoting the transposition. For this cost function to be minimum, the controller for the system is given by

$$K_{kL} = R^{-1}B_{kL}^T P_{kL} \tag{8}$$

where, $P_{kL} = P_{kL}^T > 0$ (positive definite) is the solution of the algebraic Ricatti equation. There is no loss of generality in assuming $R = I_m$ [2]. Hence

$$K_{kL} = B_{kL}^T P_{kL} \tag{9}$$

THEOREM 2.1 (OPA THEOREM) *For the controllable system with $p_k \leq m$, the poles of the closed loop system can be assigned so that $P_{kL} = P_{kL}^T > 0$ and $Q_{kL} = Q_{kL}^T \geq 0$ iff $(B_{kL}B_{kL}^T)^{-1}(A_{kL} - \bar{A}_{kL})$ is positive definite, and $-(B_{kL}B_{kL}^T)^{-1}(A_{kL} - \bar{A}_{kL})A_{kL} - A_{kL}^T(B_{kL}B_{kL}^T)^{-1}(A_{kL} - \bar{A}_{kL})$ is positive semidefinite.*

PROOF: See [1].

Thus p_k ($\leq m$) poles can be assigned optimally by delineating OPR using the above theorem and P_{kL} and Q_{kL} are given by [1]

$$P_{kL} = (B_{kL}B_{kL}^T)^{-1}(A_{kL} - \bar{A}_{kL}) > 0 \tag{10}$$

$$Q_{kL} = -(P_{kL}\bar{A}_{kL} - A_{kL}^T P_{kL}) \geq 0 \tag{11}$$

LEMMA *When p_k ($\leq m$) poles are optimally assigned using OPA theorem, leading principal minors P_{ii} of P_{kL} are directly proportional to pole shift.*

PROOF: The reduced order system matrices in RSF at k–th recursion are

$$A_{kL} = \begin{bmatrix} \alpha_{11} & \beta_{12} & \beta_{13} & \dots & \dots & \beta_{1p} \\ 0 & \alpha_{22} & \beta_{23} & \dots & \dots & \beta_{2p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \alpha_{pp} \end{bmatrix}$$

$$\bar{A}_{kL} = \begin{bmatrix} \bar{\alpha}_{11} & \bar{\beta}_{12} & \bar{\beta}_{13} & \dots & \dots & \bar{\beta}_{1p} \\ 0 & \bar{\alpha}_{22} & \bar{\beta}_{23} & \dots & \dots & \bar{\beta}_{2p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \bar{\alpha}_{pp} \end{bmatrix}$$

and

$$(B_{kL}B_{kL}^T)^{-1} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{12} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{1p} & b_{2p} & \dots & b_{pp} \end{bmatrix}$$

Then from (10) we have

$$
\begin{aligned}
P_{kL} &= \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{12} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{1p} & b_{2p} & \dots & b_{pp} \end{bmatrix} \begin{bmatrix} \alpha_{11}-\bar{\alpha}_{11} & \beta_{12}-\bar{\beta}_{12} & \dots & \dots & \beta_{1p}-\bar{\beta}_{1p} \\ 0 & \alpha_{22}-\bar{\alpha}_{22} & \dots & \dots & \beta_{2p}-\bar{\beta}_{2p} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \alpha_{pp}-\bar{\alpha}_{pp} \end{bmatrix} \\
&= \begin{bmatrix} p_{11} & p_{12} & p_{13} & \dots & p_{1p} \\ p_{12} & p_{22} & p_{23} & \dots & p_{2p} \\ \dots & \dots & \dots & \dots & \dots \\ p_{1p} & p_{2p} & \dots & \dots & p_{pp} \end{bmatrix} \\
&= \begin{bmatrix} P_{11} & & & & \\ & P_{22} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & P_{pp} \end{bmatrix}
\end{aligned} \tag{12}
$$

From (12) we get

$$P_{11} = p_{11} = b_{11}(\alpha_{11} - \bar{\alpha}_{11}) > 0 \tag{13}$$

and

$$p_{12} = b_{12}(\alpha_{11} - \bar{\alpha}_{11}) = b_{11}(\beta_{12} - \bar{\beta}_{12}) + b_{12}(\alpha_{22} - \bar{\alpha}_{22})$$

substituting for $(\beta_{12} - \bar{\beta}_{12})$ from this equation in (12) and simplifying we get

$$P_{22} = \left(\frac{b_{12}}{b_{11}} p_{11}\right)^2 - p_{12} + \frac{\Delta_{22}}{b_{11}} p_{11}(\alpha_{22} - \bar{\alpha}_{22}) \tag{14}$$

where

$$\Delta_{22} = (b_{11}b_{22} - b_{12}^2) > 0 \tag{15}$$

Now equations (13) and (14) are equations of a straight line. Similarly, it can be shown that the $\bar{\alpha}_{ii}$ is directly proportional to P_{ii}. This has been illustrated in Figure 1.

Therefore, the leading principal minors of P_{kL} are directly proportional to the pole shift.

This lemma leads us to the following definition.

DEFINITION *Since the leading principal minors of P_{kL} are proportional to pole shift in RSF plane, the algebraic Riccati equation solution matrix P is called pole shift matrix.*

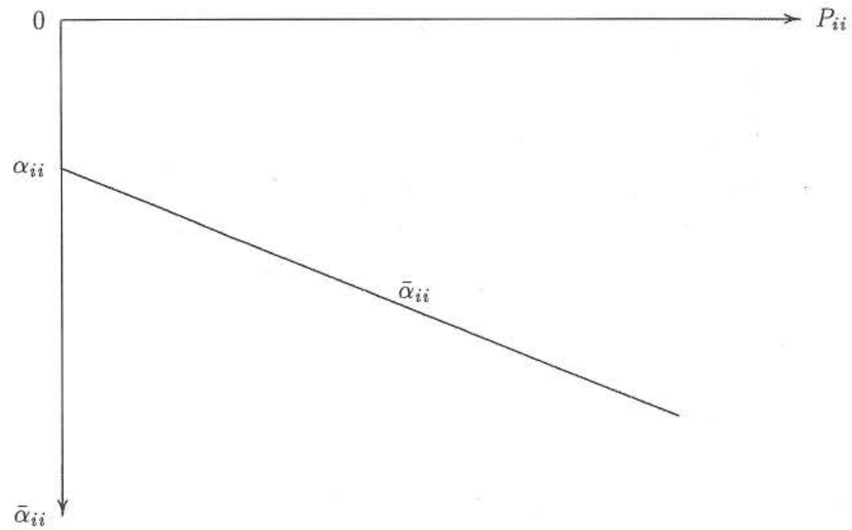

Figure 1. Relation between principal minor and closed loop poles

p_k ($\leq m$) closed loop poles $\{\bar{\alpha}_{ii}\}$ are assigned progressively such that $P_{ii} > 0$ and Q_{ii} (leading principal minor of Q_{kL}) ≥ 0. This has been illustrated by the numerical example involving one recursion.

3. OPA algorithm – the multiple pole assignment

1) Transform A and B to RSF

$$A_0 = U_0^T A U_0 \quad \text{and} \quad B_0 = U_0^T B$$

where U_0 is the unitary similarity transformation matrix that transforms A to RSF A_0.

2) Choose q, the number of recursions necessary to carry out pole assignment, and the order in which the poles are to be assigned. Set $k = 0$ and $\bar{A}_0 = A_0$.
3) Set $k = k + 1$
4) Obtain $A_k = U_k^T \bar{A}_{k-1} U_k$ and $B_k = U_k^T B_{k-1}$
5) Set $i = 0$
6) Set $i = i + 1$ and choose $P_{ii} = a > 0$
7) Calculate P_{ii} by solving $P_{ii} = a$

8) Solve the simultaneous equations for p_{1i}, p_{2i}, ..., p_{ii} and determine $\bar{\alpha}_{ii}$.
9) Draw the straight line joining α_{ii} and $\bar{\alpha}_{ii}$. Choose desired $\bar{\alpha}_{ii}$ on this straight line and read P_{ii}.
10) Determine p_{ii} from P_{ii}
11) If $i = 1$ go to step (13) otherwise go to step (12)
12) Solve the simultaneous equations for p_{1i}, p_{2i}, ..., $p_{(i-1)i}$ and determine $\bar{\beta}_{1i}$, $\bar{\beta}_{2i}$, ..., $\bar{\beta}_{(i-1)i}$
13) Set j=i
14) Set j=j+1
15) Calculate p_{ij}. If $j = p_k$ go to step (16) otherwise go to step (14)
16) Check for Q_{ii}; If $Q_{ii} \geq 0$ go to step (17) or else put $\bar{\alpha}_{ii} = \bar{\alpha}_{ii} - b$ $(b > 0)$, read P_{ii} on straight line and go to step (10)
17) If $i = p_k$ go to step (18) otherwise go to step (6)
18) Calculate K_{kL}
19) If $k = q$ go to step (20) otherwise go to step (3)
20) Calculate P, K, Q and $\bar{A}$ in the original system coordinates.

Illustrative example 1

Consider

$$\dot{X} = \begin{bmatrix} -4 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} U$$

By RSF transformation we get

$$A_0 = \begin{bmatrix} -4 & -0.707 & -2.121 \\ 0 & -2.0 & -1.0 \\ 0 & 0 & -1.0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 2.0 & 0 & 1 \\ -0.707 & 0 & -0.707 \\ 0.707 & 1.414 & 2.121 \end{bmatrix}$$

where

$$U_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix}$$

The open loop poles are at -4, -2, -1 and $q = 1$. Since $U_1 = I_3$ the identity matrix we have, $A_{1L} = A_1 = A_0$ and $B_{1L} = B_1 = B_0$. To shift the open loop pole at -4, choose $a = 0.5$

$$P_{1L} = (B_{1L}B_{1L}^T)^{-1}(A_{1L} - \bar{A}_{1L})$$

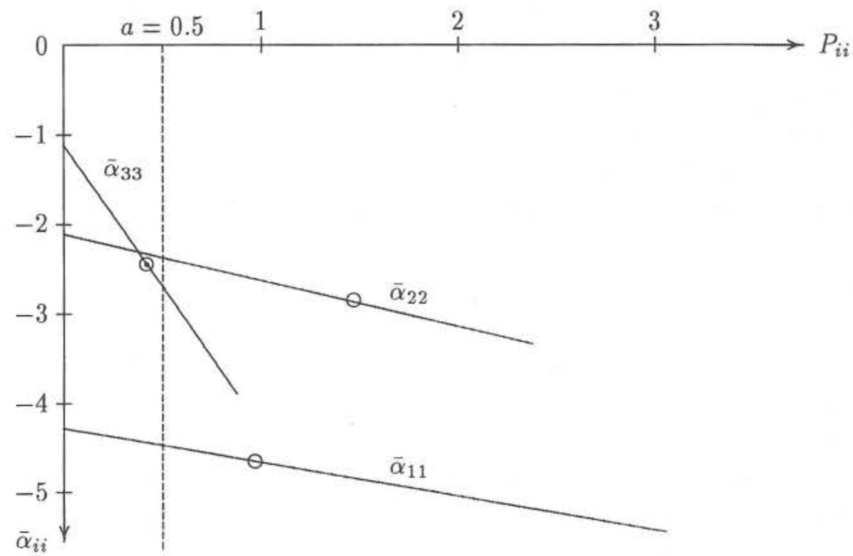

Figure 2. Closed loop poles of illustrative Example 1

$$= \begin{bmatrix} 3 & 7.777 & 0.707 \\ 7.777 & 22.5 & 2.5 \\ 0.707 & 2.5 & 0.5 \end{bmatrix} \begin{bmatrix} -4-\bar{\alpha}_{11} & -0.707-\bar{\beta}_{12} & 2.121-\bar{\beta}_{13} \\ 0 & -2-\bar{\alpha}_{22} & -1-\bar{\beta}_{23} \\ 0 & 0 & -1-\bar{\alpha}_{33} \end{bmatrix}$$

Now $P_{11} = p_{11} = 3(-4-\bar{\alpha}_{11}) = 0.5$ which implies $\bar{\alpha}_{11} = -4.167$.

Straight line is drawn through the points -4 and -4.167 in Figure 2.

Choosing $\alpha_{11} = -4.333$ from Figure 2 we get $P_{11} = p_{11} = 1$. Then

$$P_{1L} = \begin{bmatrix} 1 & 2.593 & 0.236 \\ 2.593 & p_{22} & p_{23} \\ 0.236 & p_{23} & p_{33} \end{bmatrix}, \quad P_{11} > 0$$

and

$$Q_{1L} = \begin{bmatrix} 8.333 & 17.128 & 1.728 \\ 17.128 & q_{22} & q_{23} \\ 1.728 & q_{23} & q_{33} \end{bmatrix}, \quad Q_{11} > 0$$

Now we shift the open loop pole at -2, for $P_{22} = 0.5$, $p_{22} = 7.222$ and $\bar{\alpha}_{22} = -2.214$.

A straight line is drawn through -2 and -2.214 in Figure 2. Choosing $\bar{\alpha}_{22} = -2.548$ we have $P_{22} = 1.278$ and $p_{22} = 8.0$. Solving for p_{12}, we have $\bar{\beta}_{12} =$

-0.151. Thus

$$P_{1L} = \begin{bmatrix} 1 & 2.593 & 0.236 \\ 2.593 & 8.0 & 0.976 \\ 0.236 & 0.976 & p_{33} \end{bmatrix}, \quad P_{22} > 0$$

and

$$Q_{1L} = \begin{bmatrix} 8.333 & 17.128 & 1.728 \\ 17.128 & 38.607 & 5.999 \\ 1.728 & 5.999 & q_{33} \end{bmatrix}, \quad Q_{22} > 0$$

Finally we shift the open loop pole at -1. For $P_{33} = 0.5$, we have $p_{33} = 0.551$ and $\bar{\alpha}_{33} = -3.739$. Choosing $\bar{\alpha}_{33} = -2.681$ on the straight line joining the points -1 and -3.739 in Figure 2, we get $P_{33} = 0.307$ and $p_{33} = 0.4$.

Solving for p_{13} and p_{23} we get $\bar{\beta}_{13} = 1.599$ and $\bar{\beta}_{23} = -0.676$. Then

$$P_{1L} = \begin{bmatrix} 1 & 2.593 & 0.236 \\ 2.593 & 8.0 & 0.976 \\ 0.236 & 0.976 & 0.4 \end{bmatrix}, \quad P_{33} > 0$$

and

$$Q_{1L} = \begin{bmatrix} 8.333 & 17.128 & 1.728 \\ 17.128 & 38.607 & 5.999 \\ 1.728 & 5.999 & 2.231 \end{bmatrix}, \quad Q_{33} > 0$$

We thus have

$$\bar{A}_{1L} = \begin{bmatrix} -4.333 & -0.151 & 1.599 \\ 0 & -2.548 & -0.676 \\ 0 & 0 & -2.681 \end{bmatrix}$$

and

$$K_{1L} = B_{1L}^T P_{1L} = \begin{bmatrix} 0.333 & 0.219 & 0.064 \\ 0.333 & 1.380 & 0.566 \\ -0.333 & -0.993 & 0.394 \end{bmatrix}$$

Now, referred to the original system coordinates

$$\bar{A} = \begin{bmatrix} -4.333 & 1.024 & 1.238 \\ 0 & -2.952 & -0.405 \\ 0 & 0.271 & -2.276 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.333 & 0.2 & -0.109 \\ 0.333 & 1.376 & -0.576 \\ -0.333 & -0.424 & 0.981 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.0 & 1.2 & -1.667 \\ 1.2 & 5.176 & -3.8 \\ -1.667 & -3.8 & 3.224 \end{bmatrix} > 0$$

and

$$Q = \begin{bmatrix} 8.333 & 13.333 & -10.889 \\ 13.333 & 26.418 & -18.188 \\ -10.889 & -18.188 & 14.420 \end{bmatrix} > 0$$

4. The redundancy problem

With $R_1 = I_m$, $p_k(\leq m)$ poles can be optimally assigned by delineating OPR using Theorem 1 (OPA theorem). From (10) and (11) P^1_{kL} and Q^1_{kL} are given by

$$P^1_{kL} = (B_{kL}B^T_{kL})^{-1}(A_{kL} - \bar{A}_{kL}) > 0 \tag{16}$$

$$Q^1_{kL} = -P^1_{kL}\bar{A}_{kL} - A^T_{kL}P^1_{kL} \geq 0 \tag{17}$$

Equivalent positive definite cost function $R_2 = R_2^T \neq I_m$ can be found by using Theorem 2 stated below. It also helps in finding equivalent functions P^2_{kL} and Q^2_{kL}.

THEOREM 4.1 *For optimally assigned $p_k(\leq m)$ poles, $R_2 = R_2^T > 0$ is equivalent cost function iff SP^1_{kL} is positive definite and $SQ^1_{kL} + (SA^T_{kL} - A^T_{kL}S)P^1_{kL}$ is positive semidefinite where*

$$S = (B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL}) \tag{18}$$

Hence equivalent p^2_{kL} and Q^2_{kL} can be calculated.

PROOF: We have, substituting for S from (18)

$$SP^1_{kL} = (B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL})P^1_{kL}$$

From (16) $(B_{kL}B^T_{kL})P^1_{kL} = (A_{kL} - \bar{A}_{kL})$, therefore

$$SP^1_{kL} = (B_{kL}R_2^{-1}B^T_{kL})^{-1}(A_{kL} - \bar{A}_{kL}) = P^2_{kL} > 0 \tag{19}$$

Furthermore, substituting for Q^1_{kL} and S from (17) and (18) respectively

$$\begin{aligned} & SQ^1_{kL} + (SA^T_{kL} - A^T_{kL}S)P^1_{kL} \\ = \; & (B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL})(-P^1_{kL}\bar{A}_{kL} - A^T_{kL}P^1_{kL}) \\ & +(B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL})A^T_{kL}P^1_{kL} \\ & -A^T_{kL}(B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL})P^1_{kL} \\ = \; & -(B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL})P^1_{kL}\bar{A}_{kL} \\ & -A^T_{kL}(B_{kL}R_2^{-1}B^T_{kL})^{-1}(B_{kL}B^T_{kL})P^1_{kL} \end{aligned}$$

substituting again from (16)

$$\begin{aligned} & SQ^1_{kL} + (SA^T_{kL} - A^T_{kL}S)P^1_{kL} \\ = \; & -(B_{kL}R_2^{-1}B^T_{kL})^{-1}(A_{kL} - \bar{A}_{kL})\bar{A}_{kL} \\ & -A^T_{kL}(B_{kL}R_2^{-1}B^T_{kL})^{-1}(A_{kL} - \bar{A}_{kL}) \\ = \; & -P^2_{kL}\bar{A}_{kL} - A^T_{kL}P^2_{kL} \;\; = \;\; Q^2_{kL} \;\; \geq 0 \end{aligned} \tag{20}$$

Thus, R_2 is equivalent cost function for the optimally assigned $p_k(\leq m)$ poles iff

$$SP^1_{kL} > 0 \quad \text{and} \quad SQ^1_{kL} + (SA^T_{kL} - A^T_{kL}S)P^1_{kL} \geq 0$$

and equivalent P^2_{kL} and Q^2_{kL} can be calculated using equations (19) and (20) respectively.

All the equivalent cost functions (Q, R) for a set of optimally assigned poles can be determined using the above stated theorem. Illustrative examples are given in Section 5.

5. Illustrative examples

It is possible to shift a complex conjugate pair of poles to a pair of complex conjugate poles or real poles in the OPR. A pair of real poles can be shifted to real poles or to complex conjugate poles in the OPR. All the equations for optimal pole regions derived in our previous paper [1] remain the same but define the new optimal pole regions because of the presence of R. The OPR delineated using these equations is illustrated in Figure 3. The hatched area represents the OPR.

EXAMPLE 2. Consider

$$A_{1L} = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \quad B_{1L} = \begin{bmatrix} -0.707 & 0.0 \\ 0.707 & 1.414 \end{bmatrix}$$

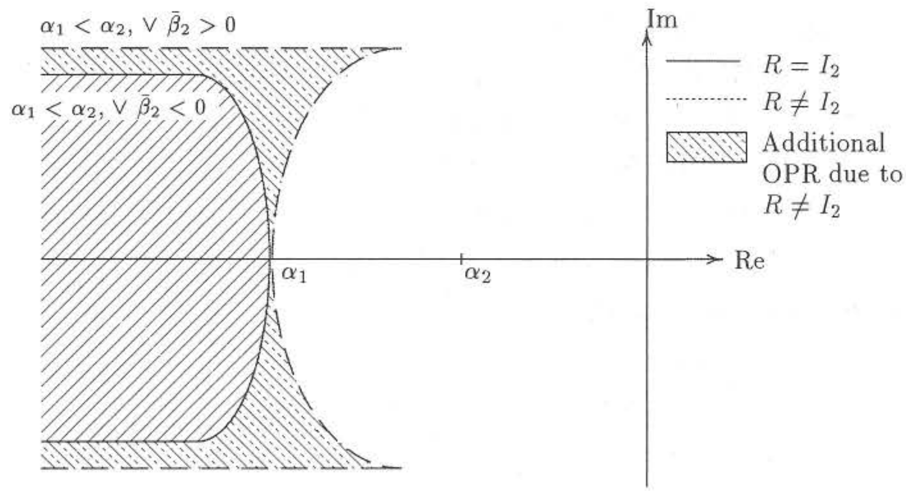

Figure 3. OPR shifting of real poles to complex poles

a) With $R_1 = I_2$ [1]
OPR of $\bar{\beta}$:

$$\bar{\beta}_{\max} = 0.89$$

Let $\bar{\beta} = 0.5$ with this value of $\bar{\beta}$, $\bar{\beta}_2 = 0.342$ and $\bar{\beta}_1 = -0.731$ with the value of $\bar{\beta}_2$, the OPR of $\bar{\alpha}$ is

$$\bar{\alpha} \leq -2.068$$

The value $\bar{\alpha} = -3$ satisfies the necessary constraints and

$$\bar{A}^1_{1L} = \begin{bmatrix} -3 & -0.731 \\ 0.342 & -3 \end{bmatrix} \quad P^1_{1L} = \begin{bmatrix} 2.329 & 0.329 \\ 0.329 & 0.866 \end{bmatrix}$$

$$K^1_{1L} = \begin{bmatrix} -1.414 & 0.379 \\ 0.465 & 1.224 \end{bmatrix} \quad Q^1_{1L} = \begin{bmatrix} 11.538 & 3.35 \\ 3.35 & 4.03 \end{bmatrix}$$

b) Let

$$R_2 = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

This R_2 satisfies conditions (1) and (2) of Theorem 2.
OPR of $\bar{\beta}$:

$$\bar{\beta}_{\max} = 1.1547$$

And for chosen $\bar{\beta}$ of 0.5, $\bar{\beta}_2 = 0.197$, $\bar{\beta}_1 = -1.267$ and OPR of $\bar{\alpha}$ is

$$\alpha \leq -1.934$$

Selected $\bar{\alpha} = -3$ satisfies other constraints also. Thus

$$\bar{A}^2_{1L} = \begin{bmatrix} -3 & -1.267 \\ 0.197 & -3 \end{bmatrix} \quad P^2_{1L} = \begin{bmatrix} 3.197 & -1.197 \\ -1.197 & 1.732 \end{bmatrix}$$

$$K^2_{1L} = \begin{bmatrix} -1.414 & -0.378 \\ 0.568 & 1.603 \end{bmatrix} \quad Q^2_{1L} = \begin{bmatrix} 16.221 & -1.932 \\ -1.932 & 4.214 \end{bmatrix}$$

REMARKS:

1. OPR of $\bar{\alpha}$ alters with R and it is possible to increase OPR.
2. Generally $\bar{\beta}_{\max}$ range also alters it is possible to increase $\bar{\beta}_{\max}$ range.
3. Though closed loop poles remain the same with different R's off-diagonal elements of $\bar{A}_{kL}$ are altered.

EXAMPLE 3. Consider

$$\dot{X} = \begin{bmatrix} -4 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} U$$

a) with $R_1 = I_2$ [1], the closed loop system matrices are

$$\bar{A}^1 = \begin{bmatrix} -4.318 & 3.274 & -0.312 \\ 0.403 & -4.236 & -0.825 \\ 0.244 & -0.099 & -2.981 \end{bmatrix}$$

The eigenvalues of open loop system are -4, -2 and -1. The assigned eigenvalues of $\bar{A}^1$ are $(-3 \pm j0.5)$ and (-5.53).

$$P^1 = \begin{bmatrix} 0.123 & -0.316 & -0.087 \\ -0.316 & 2.741 & -0.505 \\ -0.087 & -0.505 & 1.33 \end{bmatrix}$$

$$K^1 = \begin{bmatrix} 0.159 & -1.137 & 1.156 \\ -0.403 & 2.236 & 0.825 \end{bmatrix}$$

and

$$Q^1 = \begin{bmatrix} 1.167 & -3.012 & -0.883 \\ -3.012 & 2.741 & -1.596 \\ -0.883 & -1.596 & 5.025 \end{bmatrix}$$

b) Let

$$R_2 = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

Since $m = 2$ the poles are assigned in two recursions ($q = 2$). In first recursion $(-2, -1)$ are shifted to complex conjugate pair. In second recursion (-4) is shifted.

By RSF transformation of A we get

$$A_0 = \begin{bmatrix} -4 & -0.707 & 2.12 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 2 & 0 \\ -0.707 & 0 \\ 0.707 & 1.414 \end{bmatrix}$$

where

$$U_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & 0.707 \\ 0 & -0.707 & 0.707 \end{bmatrix}$$

Since $U_1 = I_3$ the identity matrix

$$A_{1L} = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B_{1L} = \begin{bmatrix} -0.707 & 0 \\ 0.707 & 1.414 \end{bmatrix}$$

This has been solved in example 2(b), and the chosen optimal closed loop poles are $(-3 \pm j0.5)$ as before

$$\bar{A}_1 = \begin{bmatrix} -4 & 2.121 & 2.876 \\ 0 & -3 & -1.267 \\ 0 & 0.197 & -3 \end{bmatrix}$$

For $k = 2$, by RSF transformation, A_2 and B_2 are given by

$$A_2 = \begin{bmatrix} -3.061 & -0.186 & 2.087 \\ -1.366 & -2.938 & 2.858 \\ 0 & 0 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.431 & 0.447 \\ 1.717 & 0.165 \\ 0.054 & 1.329 \end{bmatrix}$$

where

$$U_2 = \begin{bmatrix} 0.284 & 0.935 & -0.212 \\ -0.905 & 0.334 & 0.264 \\ 0.316 & 0.117 & 0.94 \end{bmatrix}$$

Then

$$A_{2L} = [-4] \quad \text{and} \quad B_{2L} = [0.054 \;\; 1.329]$$

Since $\bar{\alpha}_3 \leq \bar{\alpha}_3$ with the chosen value of $\bar{\alpha}_3 = -5.53$ we get

$$P_{2L} = [0.61], \quad Q_{2L} = [5.813] \quad \text{and} \quad K_{2L} = \begin{bmatrix} -0.778 \\ 1.183 \end{bmatrix}$$

Referred to the original system coordinates

$$\bar{A}^2 = \begin{bmatrix} -4.33 & 4.86 & 1.27 \\ 0.252 & -4.54 & -1.294 \\ 0.083 & 0.399 & -2.649 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.027 & -0.11 & -0.062 \\ -0.11 & 1.709 & -0.484 \\ -0.062 & -0.484 & 3.8 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} 0.165 & -1.929 & 0.361 \\ -0.251 & 2.542 & 1.297 \end{bmatrix}$$

and

$$Q^2 = \begin{bmatrix} 0.261 & -1.048 & -0.589 \\ -1.048 & 12.494 & -3.637 \\ -0.589 & -3.637 & 13.474 \end{bmatrix}$$

6. Conclusions

A multiple real pole assignment procedure using OPA theorem has been presented in this paper. It has been shown that principal minors P_{ii} are directly proportional to pole shift $\bar{\alpha}_{ii}$. Hence P is called pole shift matrix. This OPA procedure enables the designer to locate optimally the closed loop poles to meet the closed loop system specifications.

Further, it has been shown in this paper for optimally assigned poles that using our second theorem it is possible to calculate the equivalent controller

K and the corresponding cost function (Q, R). In addition, the optimal pole region increases with increase of cost function R. Thus it is possible to use the entire region to the left of open loop poles (to the left of mirror image in case of unstable poles) for optimal pole assignment by using a general R instead of a unit matrix. This paper enables to identify the set of all cost functions that are being minimized for a set of chosen optimal poles. This in turn provides the freedom for the designer to choose an optimal feedback matrix K.

Further research work is being carried out to establish the relation between complex poles of the closed system and elements of P_{kL}. Also constraints on p_{ii} such that $Q_{ii} \geq 0$ are being investigated. In addition the effect of R on OPR in the redundancy problem is being investigated.

References

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Optymalne przesuwanie biegunów a zagadnienie nadmiarowości

We wcześniejszej pracy autorów [1] podano twierdzenie o optymalnym przesuwaniu biegunów. Korzystając z tego twierdzenia określono obszar optymalnych biegunów. Opisano rekurencyjną procedurę optymalnego przesuwania biegunów. W każdym kroku rekurancyjnym przesuwa się jeden lub dwa bieguny. W tej pracy przedstawia się algorytm przesuwania wielu biegunów rzeczywistych w każdym kroku. Podaje się fizyczną interpretację macierzy P będącej rozwiązaniem równania Riccatiego. Przy określaniu obszaru optymalnych biegunów początkowo zakłada się, że $R = I_m$, a następnie to ograniczenie jest osłabione. Rozważania wykazują, że wybor R wpływa na obszar optymalnych biegunów i że w ten sposób można ten obszar powiększyć.

Оптимальное перемещение полюсов а вопрос избыточности

В более ранней работе авторов [1] дана теорема об оптимальном перемещении полюсов. На основе теоремы определена область оптимальных полюсов. Описана рекуррентная процедура оптимального перемещения полюсов. На каждом рекуррентном шагу перемещается один либо два полюса. В данной работе представлен алгоритм перемещения многих действительных полюсов на каждом шаге. Рассмотрена физическая интерпретация матрицы P, являющейся решением уравенства Риккати. При определении области оптимальных полюсов в начале предполагается, что $R + I_m$, а затем это ограничение ослабляется. Исследования показывают, что выбор R влияет на область оптимальных полюсов и что можно таким образом эту область увеличить.