Control and Cybernetics

vol. **22** (1993) No. 1/2

A method for improving weights in column aggregated linear programs

by

Sverre Storøy

Department of Informatics
University of Bergen
Bergen
Norway

The loss in accuracy due to aggregation of variables in linear programs, measured by the value of the objective function, has been studied by many authers, and good bounds are given. In the present paper we study how the weights used to aggregate the variables may be changed iteratively in order to improve the objective value. The methods are based on dual information from the aggregated problem and are rather simple to implement.

1. Introduction

There may be several reasons for performing aggregation on the data of large scale mathematical programming models. One reason can be that the certainty of the data is questionable, making computations on a detailed level not worthwhile. Another reason could be that the size of the problem is so large that it cannot be solved accurately in its full size. Yet another reason could be that the model has a natural organizational decomposition, and that the information that is relevant to the different parts of the organization is different with respect to level of detail. Jörnsten and Leisten (1990, 1992) studied aggregation due to this last presented reason, and very interesting results concerning the decomposition problem are given.

Bounds on the loss in accuracy when a simpler aggregate problem is solved have been developed for linear models by Zipkin (1980). These bounds have been improved by Mendelssohn (1980) and Knolmayer (1986) on the basis of a related work by Kallio (1977). Similar bounds are developed for aggregated integer models by Hallefjord and Storøy (1990).

All these bounds are related to the error in the objective function value.

The objective value (and the solution) of an aggregated problem is a function of the weights used when aggregating. The problem of determining "good" weights may be very complicated. In the survey paper by Rogers and al. (1991) references to some methods may be found. Their conclusion is, however, that "... methodology for better approximating this vector (i.e. the vector of optimal weights) continues to be an area of interesting research".

In the present paper we develop a procedure for improving a given set of weights. The weights are improved in the sense that the objective value of the aggregated problem corresponding to the improved weights is greater than the objective value corresponding to the original weights.

The procedure is based on standard postoptimal analysis of the optimal basis matrix of the aggregated problem.

2. Notation and preliminaries

Let the original problem be:

$$z^* = \max \underline{c}\,\underline{x} \tag{1}$$

$$\text{subject to} \quad A\underline{x} \leq \underline{b}, \underline{b} \geq \underline{0} \tag{2}$$

$$\underline{x} \geq \underline{0},$$

where $\underline{c} = (c_j)$ is an n–vector, $\underline{b} = (b_i)$ is an m–vector, $A = (a_{ij})$ is an $m \times n$ matrix and $\underline{x} = (x_j)$ is an n–vector of variables. We assume that (1)–(2) has a finite optimal solution.

Let $Q = \{S_k | k = 1, \ldots, K\}$ be an arbitrary partition of the column indices $\{1, \ldots, n\}$ and let $n_k = |S_k|$. Then by definition $\cup_{k=1}^{K} S_k = \{1, \ldots, n\}$ and $S_k \cap S_j = \emptyset$ for all $k \neq j$. Denote by A^k the $m \times n_k$ submatrix of A consisting of columns with indices in S_k, and let $\underline{c}^k$ be defined analogously so that $\underline{c} = (\underline{c}^1, \ldots, \underline{c}^K)$, with the subvectors rearranged if neccessary.

Now let $\underline{g}^k$ be an n_k–vector of nonnegative weights such that $\sum_{j \in S_k} g_j^k = 1$, $k = 1, \ldots, K$, and define $\bar{A} \equiv (A^1\underline{g}^1, \ldots, A^K\underline{g}^K)$ and $\bar{\underline{c}} \equiv (\underline{c}^1\underline{g}^1, \ldots, \underline{c}^K\underline{g}^K)$.

The column aggregate problem is then

$$\bar{z} = \max \underline{c}\,\underline{X} \tag{3}$$

$$\text{subject to} \quad \bar{A}\underline{X} \leq \underline{b} \tag{4}$$

$$\underline{X} \geq \underline{0}$$

where $\underline{X}$ is a K–vector of aggregated variables. We assume that Q and $\underline{g} = (\underline{g}^1, \ldots, \underline{g}^K)$ have been chosen so that (3)–(4) is feasible.

A column aggregate problem is always a restriction of the original problem, in the sense that a feasible aggregate solution always can be disaggregated to a **feasible** solution of the original problem. The simplest way is to use a fixed–weight dissaggregation, by defining the n_k–vector $\underline{x}^k \equiv \underline{g}^k X_k$, $k = 1, \ldots, K$.

An improved disaggregated solution is obtained by performing an **optimal disaggregation** of $\underline{X}$: Solve K subproblems given as

$$\begin{aligned} z^k(X_k) &= \max \underline{c}^k \underline{x}^k \\ \text{subject to} \quad A\underline{x}^k &\leq A^k \underline{g}^k X_k \\ \underline{x}^k &\geq \underline{0}, \end{aligned}$$

for $k = 1, \dots, K$.

In any case, after rearranging the $\underline{x}^k$ vectors into $\underline{x} = (\underline{x}^1, \dots, \underline{x}^K)$, we have that $\bar{z} \leq \underline{cx} \leq z^\star$. Normally the disaggregated solution $\underline{x}$ is not an extreme ponit of (2).

The partition Q and the weight vectors $\underline{g}$ are assumed to be predetermined. It is of theoretical interest to note that **optimal weightings** $\underline{g}^\star$ do exist for any partition Q such that $\bar{z} = z^\star$ (this is even true for IP–programs, Hallefjord and Storøy (1990)). Unfortunately, construction of $\underline{g}^\star$ requires an optimal solution of (1)–(2).

3. Improving weights

Let $\underline{X}_B$ be a non–degenerate optimal extreme point solution to the aggregated problem (3)–(4), and let $\underline{\bar{u}}$ be the corresponding dual solution. Then for any column vector $\underline{\bar{a}}_j$ of the optimal basis matrix $\bar{B}$ we have that

$$\underline{\bar{u}}\,\underline{\bar{a}}_j - \bar{c}_j = 0. \tag{5}$$

In terms of the original problem (1)–(2), (5) may be written as:

$$\begin{aligned} \underline{\bar{u}}[A^j \underline{g}^j] - \underline{c}^j \underline{g}^j &= 0, \\ \text{or as:} \quad [\underline{\bar{u}} A^j - \underline{c}^j]\underline{g}^j &= 0. \end{aligned}$$

Now $A^j = (\underline{a}_{j_1}, \underline{a}_{j_2}, \dots, \underline{a}_{j_{n_j}})$ and $\underline{c}^j = (c_{j_1}, c_{j_2}, \dots, c_{j_{n_j}})$. Substituting this in the formula above, we get:

$$(\underline{\bar{u}}\,\underline{a}_{j_1} - c_{j_1})g_1^j + \cdots + (\underline{\bar{u}}\,\underline{a}_{j_{n_j}} - c_{j_{n_j}})g_{n_j}^j = 0 \tag{6}$$

Assume that the optimal basis matrix of (1)–(2) (or at least one of its columns) has been aggregated into $\bar{B}$. If $\underline{\bar{u}}$ is not optimal for the given problem (1)–(2), we know that, for some j, at least one of the terms in (6), the i–th say, must be strictly negative:

$$\underline{\bar{u}}\,\underline{a}_{j_i} - c_{j_i} < 0.$$

Since all the weights in (6) are assumed to be strictly positive, then at least one term must also be strictly positive, e.g. the k–th:

$$\underline{\bar{u}}\,\underline{a}_{j_k} - c_{j_k} > 0.$$

Now define a new vector of weights, $\underline{\hat{g}}^j$, by increasing the weight of a negative term and reduciong the weight of a positive term accordingly, i.e.: set $\alpha > 0$ and

$$\left.\begin{array}{rrcl} & \hat{g}_i^j & = & g_i^j + \alpha, \\ & \hat{g}_k^j & = & g_k^j - \alpha, \\ \text{and} & \hat{g}_l^j & = & g_l^j, \quad l = 1, \ldots, n_{n_j}, \ l \neq i, l \neq k, \end{array}\right\} \tag{7}$$

such that $\hat{g}_l^i \geq 0$ for all l.

Then let $\underline{\hat{a}}_j = A^j \underline{\hat{g}}^j$ and $c_j = \underline{c}^j \underline{\hat{g}}^j$, and create a modified aggregated problem by replacing $\underline{\bar{a}}_j$ and $\bar{c}_j$ in (3)-(4) by $\underline{\hat{a}}_j$ and $\hat{c}_j$. Let $\hat{z}$ be the optimal objective value of the modified aggregated problem. We then state:

THEOREM 1 *For $\alpha > 0$ and sufficiently small, we have $\hat{z} > \bar{z}$.*

PROOF: Since $\underline{X}_B$ is assumed to be nondegenerate, it is possible to select $\alpha > 0$ and sufficiently small, such that the modified basis matrix $\hat{B}$ (where $\underline{\bar{a}}_j$ in $\bar{B}$ is replaced by $\underline{\hat{a}}_j$) is a feasible basis for the modified problem. Then since $\underline{\bar{u}}\underline{\hat{a}}_j - \hat{c}_j < 0$, it follows from classical postoptimal analysis (see e.g Chavatal (1983)) that the corresponding objective value will increase when $\underline{\bar{a}}_j$ and $\bar{c}_j$ are replaced by $\underline{\hat{a}}_j$ and $\hat{c}_j$, respectively. □

Numerical example:

Consider Zipkin's example from Zipkin (1980)

$$Z^* = \max 2.5x_1 + 3x_2 + 4x_3 + 5x_4$$

$$\begin{array}{rcl} \text{subject to } 4x_1 + 5x_2 + 7x_3 + 10x_4 & \leq & 54 \\ x_1 + 2x_2 + x_3 + 2x_4 & \leq & 10 \\ x_j & \geq & 0 \text{ all } i. \end{array}$$

The optimal solution is $x_1^* = 16/3$, $x_2^* = x_4^* = 0$, $x_3^* = 14/3$, $Z^* = 32$. Suppose we construct an aggregated problem by letting $K = 2$, $S_1 = \{1, 2\}$, $S_2 = \{3, 4\}$ and use the weightings $(\underline{g}^1)^T = (\underline{g}^2)^T = (1/2, 1/2)$. The aggregate problem is then as follows:

$$\bar{z} = \max 2.75X_1 + 4.5X_2$$

$$\begin{array}{rcl} \text{subject to } 4.5X_1 + 8.5X_2 & \leq & 54 \\ 1.5X_1 + 1.5X_2 & \leq & 10 \\ X_1, X_2 & \geq & 0. \end{array}$$

The optimal solution is $\bar{X}_1 = 2/3$, $\bar{X}_2 = 6$, $\bar{z} = 28\ 5/6$, and the optimal dual solution is $\bar{u}_1 = 21/48$, $\bar{u}_2 = 25/48$. The optimal basis matrix is $\bar{B} = \begin{bmatrix} 4.5 & 8.5 \\ 1.5 & 1.5 \end{bmatrix}$.

Consider $\underline{\bar{a}}_2 = \begin{bmatrix} 8.5 \\ 1.5 \end{bmatrix} = \frac{1}{2}\underline{a}_3 + \frac{1}{2}\underline{a}_4$, and calculate the terms in (6):

$$\underline{\bar{u}}\,\underline{a}_3 - c_3 = [21/48, 25/48] \begin{bmatrix} 7 \\ 1 \end{bmatrix} - 4 = -\frac{5}{12}$$

$$\underline{\bar{u}}\,\underline{a}_4 - c_4 = [21/48, 25/48]\begin{bmatrix} 10 \\ 2 \end{bmatrix} - 5 = \frac{5}{12}$$

In accordance with (6) and Theorem 1 a better aggregate problem is found if $\underline{\bar{a}}_2$ is modified by increasing g_1^2 and decreasing g_2^2. Set $\alpha = 0.3$. Then $\hat{g}_1^2 = 0.8$ and $\hat{g}_2^2 = 0.2$, we get:

$$\underline{\hat{a}}_2 = 0.8\underline{a}_3 + 0.2\underline{a}_4 = \begin{bmatrix} 7.6 \\ 1.2 \end{bmatrix}$$

and $\hat{c}_2 = 0.8c_3 + 0.2c_4 = 4.2$

The new basis matrix $\hat{B} = \begin{bmatrix} 4.5 & 7.6 \\ 1.5 & 1.2 \end{bmatrix}$ is the optimal basis for the new problem, and the new solution is: $\hat{X}_1 = 1.876$, $\hat{X}_2 = 6.0$, $\hat{z} = 30\ 1/3$, with corresponding dual solution $\hat{u} = [1/2, 1/3]$. Thus a strictly better aggregated problem has been constructed.

4. "Steplength" determination

In the numerical example above we selected $\alpha = 0.3$ as the "steplength" for weights modifications without any justification of why this value of α would be good or not. For this small example it is rather straightforward to see that $\alpha = 0.5$ is even better, (i.e replace $\underline{\bar{a}}_2$ with $\underline{a}_3$ and $\bar{c}_2$ with c_3).

In general we need criteria for calculating the steplength α such that Theorem 1 be valid.

Consider $\underline{\bar{a}}_j$ in (5). When $n_j = 2$, there are exactly one negative and one positive terms in (6). However, when $n_j > 2$, there might be several negative and positive terms. It is then natural to select **the most negative** term as the term for which the weight should be increased (this is similar to the criterion in LP for introducing a nonbasic variable as basic variable).

When there are more than one positive terms in (6), it may be difficult to decide which weights to reduce, except when $\underline{\bar{a}}_j$ can be replaced by one original $\underline{a}_{j_k}$ vector, in which case g_k^j is set to one and all the other weights in S_k are set to zero.

For simplicity, we will assume in the following that we reduce the weight of only one positive term (as in (7)), and let that term be the most positive. Without loss of generality we also assume that the variables are ordered so that

$$\bar{B}^{-1}\underline{\bar{a}}_j = \underline{e}_j$$

where $\underline{e}_j$ is a unit vector with 1 in the j–th position. We then have the following relations:

$$\bar{B}^{-1}\underline{\bar{a}}_j = \bar{B}^{-1}(\underline{a}_{j_1}g_1^j + \underline{a}_{j_2}g_2^j + \cdots + \underline{a}_{j_{n_j}}g_{n_j}^j) = \underline{e}_j \tag{8}$$

Now suppose that we increase g_i^j and decrease g_k^j as in (7). Then we get:

$$\underline{\hat{a}}_j = \underline{a}_{j_1} g_1^j + \cdots + \underline{a}_{j_i}(g_i^j + \alpha) + \cdots + \underline{a}_{j_k}(g_k^j - \alpha) + \cdots + \underline{a}_{j_{n_j}} g_{n_j}^j \tag{9}$$

and

$$\hat{c}_j = c_{j_1} g_1^j + \cdots + c_{j_i}(g_i^j + \alpha) + \cdots + c_{j_k}(g_k^j - \alpha) + \cdots + c_{j_{n_j}} g_{n_j}^j. \tag{10}$$

From (8) we then get:

$$\begin{aligned} \underline{\hat{a}}'_j &= \bar{B}^{-1}\underline{\hat{a}}_j = \underline{e}_j + \alpha \bar{B}^{-1}(\underline{a}_{j_i} - \underline{a}_{j_k}) \\ &= \underline{e}_j + \alpha \underline{a}'' \end{aligned}$$

$$\text{where } \underline{a}'' = \bar{B}^{-1}(\underline{a}_{j_i} - \underline{a}_{j_k}) \tag{11}$$

The elements of vector $\underline{\hat{a}}'_j$ are thus:

$$\left.\begin{aligned} \underline{\hat{a}}'_i &= \alpha a''_i, \quad i \neq j,\ i = 1, \ldots, m \\ \underline{\hat{a}}'_j &= 1 + \alpha a''_j \end{aligned}\right\} \tag{12}$$

The new solution $\underline{\hat{X}}$, which we get when $\underline{\bar{a}}_j$ is replaced by $\underline{\hat{a}}_j$, is then basic feasible if the following conditions are satisfied:

$$\hat{a}'_j = 1 + \alpha a''_j > 0 \tag{13}$$

$$\text{and } \frac{X_j}{\hat{a}'_j} \leq \frac{X_i}{\hat{a}'_i}, \quad \hat{a}'_i > 0,\ i = 1, \ldots, m,\ i \neq j. \tag{14}$$

The restriction on α imposed by (13) is simply that

$$0 < \alpha < \frac{1}{|a''_j|} \quad \text{if } a''_j < 0 \tag{15}$$

The restriction on α imposed by (14) may be found by substituting (12) into (13) and (14):

$$\begin{aligned} \frac{X_j}{1 + \alpha a''_j} &\leq \frac{X_i}{\alpha a''_i}, \quad a''_j > 0,\ i = 1, \ldots, m,\ i \neq j. \\ \frac{\alpha X_j}{1 + \alpha a''_j} &\leq \frac{X_i}{a''_i}. \end{aligned}$$

Now let

$$\frac{X_r}{a''_r} = \min_i \left\{ \frac{X_i}{a''_i},\ a''_i > 0 \right\}, \quad i = 1, \ldots, m,\ i \neq j.$$

Then we can determine α such that:

$$\frac{\alpha X_j}{1 + \alpha a''_j} = \frac{X_r}{a''_r}$$

giving

$$\alpha = \frac{X_r}{X_j a''_r - X_r a''_j}. \tag{16}$$

To summarize, in view of (9)–(10), (13), (14), (15) and (16) we then have proved the following:

THEOREM 2 *When the steplength α is limited by*

$$0<\alpha<\min\left[(1-g_i^j), g_k^j, \left(\frac{1}{|a''_j|}, a''_j<0\right), \left(\frac{X_r}{X_j a''_r - X_r a''_j}, a''_r>0\right)\right], \tag{17}$$

the new weights given by (7) yield an aggregated problem which has a basic feasible solution given by

$$\begin{aligned} \hat{X}_j &= X_i - \frac{\hat{a}_i}{\hat{a}_j}X_j, \quad i=1,\ldots,m, \quad i\neq j \\ \hat{X}_j &= \frac{X_j}{\hat{a}_j} \end{aligned} \tag{18}$$

and the corresponding objective value given by

$$\hat{z} = \bar{z} - (\underline{\bar{u}}\,\underline{\hat{a}}_j - \hat{c}_j)\hat{X}_j. \tag{19}$$

Consider again the numerical example in the preceding section, and let us now use (17) to determine the steplength.

Since $n_2 = 2$ we increase the weight of $\underline{a}_3$ and reduce the weight of $\underline{a}_4$ as before. We have:

$$\bar{B} = \begin{bmatrix} 4.5 & 8.5 \\ 1.5 & 1.5 \end{bmatrix}, \quad \bar{B}^{-1} = \begin{bmatrix} -1/4 & 17/12 \\ 1/4 & -3/4 \end{bmatrix}, \quad \underline{\bar{x}} = \begin{bmatrix} 2/3 \\ 6 \end{bmatrix}.$$

From (11) we find:

$$\underline{a}'' = \bar{B}^{-1}(\underline{a}_3 - a_4) = \bar{B}^{-1}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 0 \end{bmatrix}$$

Since $a''_1 < 0$ and $a''_2 = 0$, the last two terms in (17) are not relevant. Then, since $g_1^2 = g_2^2 = 1/2$, we get $\alpha = 1/2$ (which means that $\underline{\bar{a}}_2$ should be replaced by $\underline{a}_3$).

From (12) we find:

$$\begin{aligned} \hat{a}'_1 &= \frac{1}{2}\cdot(-2/3) = -\frac{1}{3} \\ \hat{a}'_2 &= 1 + \frac{1}{2}\cdot 0 = 1, \end{aligned}$$

and from (18) and (19) we find:

$$\hat{X}_1' = 2/3 - \frac{-1/3}{1} \cdot 6 = 2\ 2/3$$
$$\hat{X}_2' = \frac{6}{1} = 6,$$

giving the new objective value $\hat{Z} = 31\ 1/3$. The new basis matrix $\bar{B} = \begin{bmatrix} 4.5 & 7 \\ 1.5 & 1 \end{bmatrix}$ is optimal for the new aggregated problem.

The process may now be repeated if a better objective is wanted. The new dual variables are: $\underline{\hat{u}} = \begin{bmatrix} 0.541667 \\ 0.208333 \end{bmatrix}$, giving

$$\underline{\hat{u}}\,\underline{a}_1 - c_1 = -1/8, \quad \text{and} \quad \underline{\hat{u}}\,\underline{a}_2 - c_2 = 1/8,$$

indicating that the weight of $\underline{a}_1$ should be increased. Doing so, the same process then yields a new basis matrix which is the optimal basis matrix for the original problem.

In general, as long as the conditions of Theorem 2 are satisfied, repeated application of the process will yield a nondecreasing sequence of objective values.

5. Disscussion

We have developed a procedure for improving a given set of weights used to aggregate variables (columns) in linear programming models. The procedure is based on standard postoptimal analysis of the basis matrix. Only two weights are considered to be changed in each step. It is then rather straightforward to develop bounds on how much the weights may be changed.

If more than two weights are considered simultaneously, the calculation of such bounds becomes much more complicated.

We have selected the most negative term in (6) as the indicator of which weight to be increased (the most positive to be decreased). It would of course be desirable to change the weight which would contribute most to the increase in the objective value. However, the extra calculations necessary to find this weight are greater than our gain when using the simple rule (this is analogous to the standard simplex method where we use the most negative reduced cost to introduce a new basic variable, and avoid calculating which nonbasic variable would contribute most to the change in the objective value). The procedure is heuristic in the sense that it may not converge to the optimal set of weights. If one or more of the variables (columns) of the optimal solution of the original problem (1), (2) are not aggregated into any of the optimal variables (columns) of the first aggregated problem (3), (4) they will never appear later since only the weights corresponding to the vectors of the optimal basis matrix B will be changed by this procedure.

This may be avoided by considering **all** negative terms (not only those defined in (6)) when we decide which weight should be increased. Then the steplength again becomes difficult to determine. In a forthcomming paper we will study how this can be done.

The present procedure has been successfully used to generate good approximate solutions to large problems. Numerical results may be found in Jörnsten and Leisten (1992).

References

CHVATAL V., (1983) Linear Programming, W.H. Freeman & Co.

HALLEFJORD Å. AND STORØY S. (1990) Aggregation and Disaggregation in Integer Programming Problems, *Operations Research*, **38**, 619–623.

JÖRNSTEN K. AND LEISTEN R. (1992) Column Aggregation and Primal Decomposition in Linear Programming: Some Observations, paper to appear in *Optimization*.

JÖRNSTEN K. AND LEISTEN R. (1990) Aggregation and Decomposition for MultiDivisional Linear Programs, Discussion Paper, Universität Heidelberg.

KALLIO M. (1977) Computing Bounds for the Optimal Value in Linear Programming, *Naval Research Logistics Quarterly*, **24**, 301–308.

KNOLMAYER G. (1986) Computing a Tight A Posteriori Bound for Column Aggregated Linear Programs, *Methods of Operations Research*, **53**, 103–114.

MENDELSSOHN R. (1980) Improved Bounds for Aggregated Linear Programs, *Operations Research*, **28**, 1450–1453.

ROGERS D.,F., PLANTE R.,D., WONG R.,T., AND EVANS J.,R. (1991) Aggregation and Disaggregation Techniques and Methodology in Optimization, *Operations Research*, **39**, 553–582.

ZIPKIN P.H. (1980) Bounds on the Effect of Aggregating Variables in Linear Programs, *Operations Research*, **28**, 403–418.