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The system — assumed-model — reduced-model relationship: the optimal projection equations with input error

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A parameter-optimization problem for obtaining a reduced-model of an unknown parameter system with the use of a quadratic input-error criterion is formulated. An auxiliary optimization problem using an assumed model is also formulated to obtain a solution of the earlier one. The parameters of the system and its optimal reduced-model are found in terms of the assumed-model parameters in the optimal projection equations. Three optimal projectors are found, coupling each other by their factors.

1. Introduction

The parameters of a system (S) are usually estimated with the use of criterion imposed on an error, Nath and Nguyen (1991a), Soderstrom and Stoica (1989), Unbehauen and Rao (1987), Young (1981), in a cost function to be minimized. The minimization of the cost function w.r.t. the parameter vector of a model gives rise to a (linear) transformation on the model parameters into the space of the S parameters. The obtained S parameter vector is seen to belong to the set (linear manifold, Israel and Greville (1974), Rao and Mitra (1971)) resulting from the above transformation. Similarly, there exist various criteria in the form of functional cost for model reduction, Lastman and Sinha (1985), Nath and San (1991b), Sinha and Lastman (1989). The minimization of the cost function w.r.t. the parameters of a reduced-order model ($\boldsymbol{RM}$), Haddad and Bernstein (1988a,b), Hyland and Bernstein (1985), also gives rise to a transformation between the $\boldsymbol{RM}$ parameter space and that of the S parameters. Another manifold thus appears. As the two manifolds may have nonempty intersection, it is expected that the parameters of the S and $\boldsymbol{RM}$ are expressible in terms of the model parameters.

For a controllable and observable S, there exists a set of controllable and observable $\boldsymbol{RM}$s of different orders. The optimal parameters of the $\boldsymbol{RM}$ at a particular order have to satisfy a rank condition and two coupled modified Lyapunov equations in the optimal projection equations, Hyland and Bernstein (1985). This coupling effect has been interpreted to be an additionally constrained condition, Hyland and Bernstein (1985), to the L_2–optimization model reduction problem. Later a sufficient condition in the context of a global optimum for the model reduction has been obtained by putting the L_2–optimization problem under a preassigned H_∞ limit, Haddad and Bernstein (1989). However, it is found that apart from that, due to reduction, the error arising on the output side cannot be accepted in many practical situations, the obtained so far optimal projection equations demand the knowledge of the S configuration, further, that S is to be controllable and observable. The parameters corresponding to the mentioned part of the system have to be estimated first, which, in turn, requires the S input to be persistently exciting, Astrom and Bohlin (1965,66). This resriction can hardly be overcome in the real–time estimation by employing error other then the input one. The input-error approach to the S parameter estimation has been developed, Nath and San (1991a), to cope with the same, independent of the form of the S input. The input–error was either directly obtained by using a model–inverse, Nath and San (1991a), a suitable transformation of the state, Francis (1987), or indirectly obtained by using convolution operator, Haddad and Bernstein (1989), Hyland and Bernstein (1985), Nath and San (1991b), Wilson (1985,1989). Mreover, one can observe that with the use of input and output data, obtaining the $\boldsymbol{RM}$ is but a misorder case of the S parameter estimation, Nath and San (1991b). The result of the parameter estimation (S and $\boldsymbol{RM}$) problem depends on the measured data relating to the input and output of the S but not on the model. Then, in the development of optimal projection equations for reducing the order of an unknown parameter S, one may have right to think of using an assumed model ($\boldsymbol{AM}$) for escaping troublesome parameter estimation in the S. The relationship between the S, $\boldsymbol{AM}$ and $\boldsymbol{RM}$ in the optimal projection equation form has to be established.

In the present paper, a parameter–optimization problem for model reduction of an nth order, unknown parameter S, is formulated with the use of an error defined on an input side of the model reference technique for ensuring a total match at the output side. Through an auxiliary optimization problem formulated with an nth order $\boldsymbol{AM}$ having known parameters, an attempt is made to express the parameters of the S and its rth order optimal $\boldsymbol{RM}$ as functions of that of the $\boldsymbol{AM}$. Solving the auxiliary problem, an optimal projector of order $(n+r)$ appears. The parameters of the S and $\boldsymbol{RM}$ are either obtained in an augmented form or individually obtained in terms of that of the $\boldsymbol{AM}$. It is found that with a particular choice of generalized inverse of the optimal projector, the augmented parameters have to satisfy two modified Lyapunov equations which are then transferred into the standard–like ones. The augmented S having the determined parameters is shown to be stabilizable and detectable independent of the input and output of the S. That is, subject to any bounded input, the S is well replaced by the $\boldsymbol{RM}$ to deliver any desired output. Further, the augmented S is shown to be settled at the equilibrium point, which gives rises to two other optimal projectors and all the three are coupled to each other by their factors. This permits one to pick two out of the S, $\boldsymbol{RM}$ and $\boldsymbol{AM}$ to form an augmented S, whose parameters are related with those of the rest in optimal projection equations. Hence, optimization is generally permitted to be carried out w.r.t. any one amongst them.

The use of an $\boldsymbol{AM}$ in the present optimal projection equations is found applicable for estimating the S parameters and for reducing an S whose data are either parameters or are accessible by means of input and output terminals. The suitable choice of the generalized inverse of the optimal projector makes the modified Lyapunov equations become the standard–like ones, leading to a simple computation to be adopted. The optimal error that is referred to on the input side may be made available by a linear feedback for a total match at the output.

2. Notations, lemmas and definitions

Throughout the paper, the subscripts s, m, r stand for the S, $\boldsymbol{AM}$ and $\boldsymbol{RM}$ respectively.

$\mathbb{R}, \mathbb{R}^{r\times n}$	real number, $(r \times n)$ valued real matrix
$\varrho(\cdot); \mathrm{tr}(\cdot); (\cdot)^T$	rank; trace; transpose of a matrix
$(\cdot)^{-1}; (\cdot)^g; (\cdot)^+$	inverse; generalized inverse; pseudo inverse of a matrix
stable matrix	matrix having all eigenvalues on L.H.S of the S–complex plane
nonnegative definite matrix	symmetric matrix with nonnegative eigenvalues
positive definite matrix	symmetric matrix with positive eigenvalues
nonnegative semisimple matrix	matrix similar to a nonnegative definite matrix
positive semisimple matrix	matrix similar to a positive definite matrix
input	either a deterministic signal or a white noise with nonnegative definite intensity
$u_s(t); u_m(t); u_r(t)$	$\mathbb{R}^{p\times 1}$ input vector
$y_s(t); y_m(t); y_r(t)$	$\mathbb{R}^{q\times 1}$ output vector
$H_s(t); H_m(t); H_r(t)$	$\mathbb{R}^{q\times p}$ impulse reponse matrices
$A_s, A_m; B_s, B_m; C_s, C_m$	$\mathbb{R}^{n\times n}; \mathbb{R}^{n\times p}; \mathbb{R}^{q\times n}$
$A_r; B_r; C_r$	$\mathbb{R}^{r\times r}; \mathbb{R}^{r\times p}; \mathbb{R}^{q\times r}$

$x_{ij}(t)$	$\int_0^t e^{A_i(t-\tau)}B_iu_i(\tau)d\tau$
$\bar{X}(t); \tilde{X}(t)$	$[x^T_{sr}(t)\|x^T_{rr}(t)]^T; [x^T_{sm}(t)\|x^T_{mm}(t)\|x^T_{rm}(t)]^T$
Q	$\lim_{t\to\infty} \bar{X}(t)\bar{X}^T(t)$ or $\lim_{t\to\infty} \mathbb{E}[\bar{X}(t)\bar{X}^T(t)]$ corresponding to a deterministic or white noise input $u_r(t)$
$\tilde{Q}$	$\lim_{t\to\infty} \tilde{X}(t)\tilde{X}^T(t)$ or $\lim_{t\to\infty} \mathbb{E}[\tilde{X}(t)\tilde{X}^T(t)]$ corresponding to a deterministic or white noise input $u_m(t)$
$\bar{A}; \bar{B}; \bar{C}$	$\begin{bmatrix} A_s & 0 \\ 0 & A_r \end{bmatrix}; \begin{bmatrix} B_s \\ B_r \end{bmatrix}; \begin{bmatrix} C_s & -C_r \end{bmatrix}$
$\tilde{A}; \tilde{B}; \tilde{C}$	$\begin{bmatrix} A_s & 0 & 0 \\ 0 & A_m & 0 \\ 0 & 0 & A_r \end{bmatrix}; \begin{bmatrix} B_s \\ B_m \\ B_r \end{bmatrix}; \begin{bmatrix} -C_s & C_m & 0 \\ 0 & -C_m & C_r \end{bmatrix}$
V_r	I_p or $\lim_{t\to\infty} \mathbb{E}[u_r(t)u_r^T(t)] \in \mathbb{R}^{p\times p}$ corresponding to input $u_r(t)$
V_m	I_p or $\lim_{t\to\infty} \mathbb{E}[u_m(t)u_m^T(t)] \in \mathbb{R}^{p\times p}$ corresponding to input $u_m(t)$
R	$\mathbb{R}^{p\times p}$ positive definite matrix
$\alpha_s; \alpha_r$	$\int_0^\infty \|\|H_s^+(t-\tau)\|\|d\tau; \int_0^\infty \|\|H_r^+(t-\tau)\|\|d\tau;$
$R_s; R_{sr}, R_{rs}, R^T_{sr}; R_r$	$\alpha_s R\alpha_s; \alpha_s R\alpha_r; \alpha_r R\alpha_r;$
$\mathcal{L}_Z(\cdot)$	partial derivative of $\mathcal{L}(\cdot)$ w.r.t. Z

LEMMA 2.1 *Let the full column $G^T \in \mathbb{R}^{n\times r}$ and full row $\Gamma \in \mathbb{R}^{r\times n}$ rank matrices $(r < n)$ be given with $\Gamma G^T = I_r$, a projector σ of order n is obtained. Further, G^T and Γ are expressible in full rank factorization and out of the G^T's and Γ's factors a positive semisimple $M \in \mathbb{R}^{r\times r}$ can be formed. Then non-negative definite matrices $\hat{Q}, \hat{P} \in \mathbb{R}^{n\times n}$ can be found such that their product is a factorization of $(G^T M\Gamma)$.*

PROOF: If $\Gamma G^T = I_r$, then

$$(G^T\Gamma)^2 = G^T\Gamma G^T\Gamma = G^T\Gamma \in \mathbb{R}^{n\times n} \tag{2.1}$$

$G^T\Gamma = \sigma$ is a projector of order n. Conversely, as G^T, Γ are the matrices of full column and rank, respectively, the $G^\omega G^T = \Gamma\Gamma^\eta = I_r$ holds, where G^ω and Γ^η are the respective left and right generalized inverses of G^T and Γ, then multiplication on (2.1) on the left by G^ω and on the right by Γ^η gives $\Gamma G^T = I_r$.

It follows from theorem 5.6, Israel, Greville (1974), that there exists a symmetric matrix $\Phi \in \mathbb{R}^{r\times r}$ and a matrix $\Sigma \in \mathbb{R}^{r\times n}$ both of the rank r such that G^T and Γ can be expressed in full rank factorization as $G^T = \Sigma_G^T\Phi_G$ and $\Gamma = \Phi_\Gamma\Sigma_\Gamma$.

Since Φ_G and Φ_Γ are invertible, $\Gamma G^T = I_r$ implies $\Sigma_G \Sigma_\Gamma^T = \Phi_G^{-1} \Phi_\Gamma^{-1} \in \mathbb{R}^{r \times r}$ and thus a semisimple matrix M can be formed as $M = \Phi_G^{-1}\Phi_\Gamma^{-1}$. With $G^T M \Gamma = \Sigma_G^T \Phi_G \Sigma_G \Sigma_\Gamma^T \Phi_\Gamma \Sigma_\Gamma$, one can assign $\Sigma_G^T \Phi_G \Sigma_G = \hat{Q}$ and $\Sigma_\Gamma^T \Phi_\Gamma \Sigma_\Gamma = \hat{P}$. It can be seen that $\hat{Q}$, $\hat{P} \in \mathbb{R}^{n \times n}$ but $\varrho(\hat{Q}) = \varrho(\hat{P}) = r < n$ hence $\varrho(\hat{Q}\hat{P}) = r$ and $\hat{Q}$, $\hat{P}$ are nononegative definite matrices.

Conversely, suppose $\hat{Q}$, $\hat{P} \in \mathbb{R}^{n \times n}$ are nonnegative definite - then $\hat{Q}\hat{P}$ is nonnegative semisimple. In addition, if $\varrho(\hat{Q}\hat{P}) = r < n$, then there exists $G, \Gamma \in \mathbb{R}^{r \times n}$ and positive semisimple $M \in \mathbb{R}^{r \times r}$ such that $\hat{Q}\hat{P} = G^T M \Gamma$ and $\Gamma G^T = I_r$. The converse follows from the lemma 2.1, Hyland, Bernstein (1985).

REMARKS 2.1
1. Whenever $r = n$, $\hat{Q}$ and $\hat{P}$ are not only nononegative but also positive definite matrices.
2. As one will see in the proof of the main result (appendix) two matrices G^T and Γ with $\Gamma G^T = I_r$ are first obtained from the partial derivative w.r.t. A_m of a Lagrangian form and then M can be formed and $\hat{Q}$ and $\hat{P}$ follow. The Lemma 2.1 is, hence, a direct consequence and the same lemma in Hyland, Bernstein (1985) should be conversely used.

PROPOSITION 2.1 *Let the full column G^T and row Γ rank be given and nonnegative definite matrices $\hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$ be assigned as stated in lemma 2.1. Then the following identities can be obtained*

$$\begin{aligned} &\Sigma_G = \Gamma \hat{Q}, \quad \Sigma_\Gamma = G\hat{P}, \quad \Phi_G^{-1} = \Gamma \hat{Q} \Gamma^T, \quad \Phi_\Gamma^{-1} = G \hat{P} G^T, \\ &\hat{Q} = \sigma \hat{Q} = \hat{Q} \sigma^T, \quad \hat{P} = \sigma^T \hat{P} = \hat{P} \sigma \end{aligned} \tag{2.2}$$

PROOF: Substituting expressions for G, Γ, Σ_G, Σ_Γ, Φ_G, Φ_Γ and $\hat{P}$, $\hat{Q}$, from lemma 2.1 into (2.2), the identities are verified.

DEFINITION 2.1 *The projector obtained from the necessary conditions for a function to be an extremum is an optimal one and the corresponding nonnegative definite matrices $\hat{Q}$, $\hat{P}$ are also optimal if a factorization of $\hat{Q}\hat{P}$ is formed out of the optimal projector components in the sense of lemma 2.1.*

3. Formulation of the problem

3.1. Idea of the input–error in the model reference technique

The output–error approach to the solution of the S parameter estimation problem requires the S input to be persistently exciting, Unbehauen, Rao (1987), Sonderstrom, Stoica (1989), Astrom, Bohlin (1965,66), which is often not the case in a normal operating records (real–time). Moreover, the error inherently arising due to a reduction that is on the output side, Hyland, Bernstein (1985), Haddad, Bernstein (1989), cannot be accepted in a problem like the one of projective controls where a perfect matching at the output side may be desired and for this a direct adjustment on the tracking side is not at all possible. In such cases, one may think of making use of the term of input–error in the model reference technique.

An ***RM*** is considered in parallel with a S in the model reference technique. Let the response $y_s(t)$ of the S to an input $u_s(t)$ be given. For ensuring the output $y_r(t)$ of the ***RM*** to be the same as that of the S, the ***RM*** should have an input $u_r(t)$ other than $u_s(t)$. An error given by $u_s(t) - u_r(t)$ at the input side (input–error), thus appears reflecting the mismatch between the parameters as well as the orders of the two. It can be seen that if the parameters of the ***RM*** are changed so that the input-error reaches the minimum value, corresponding to which a set of optimal parameters of the ***RM*** is then obtained. That is, a quadratically weighted input–error to be minimized can be defined as

$$J = \int_0^\infty [u_s(\tau) - u_r(\tau)]^T R[u_s(\tau) - u_r(\tau)]d\tau = \int_0^\infty \|[u_s(\tau) - u_r(\tau)]\|_R^2 \quad (3.1)$$

where subscript R of the norm sign indicates the norm weighted.

The functional cost J defined in (3.1) represents the total difference in the optimal energies for realising the match at the output if the S and ***RM*** have respectively a strictly proper transfer function and the optimal energy criterion is considered for obtaining each input from the knowledge of a desired output.

3.2. Statement of the problem

Given an nth order causal, linear time–invariant S by its bounded response to a bounded input, determine an rth order ($q \leq r < n$) controllable and observable ***RM*** that minimizes (3.1).

Although the knowledge of internal structure of the S is not available but the access to the S is by means of input and output terminals. This emphasizes the fact that with the given input and output data, the parameters constituting the controllable and observable part of the S in the state variable description have to be first estimated if the optimal projection equations established in Hyland, Bernstein (1985) are to be used. The error referred on the output side, however, is non-uniquely transferred to the input side as many different errors on the input side can give the small effect, for the trajectory between two outputs is not well specified. Hence, some more conditions may have to be used in addition to that established in Hyland, Bernstein (1985).

Further, although both $u_s(t)$ and $u_r(t)$ are the inputs, the minimization of (3.1) can be carried out if the input-error is expressed in terms of variables (the parameters of the ***RM***). The following steps are used for expressing (3.1) as a variable function.

3.3. Impartition of the problem

The convolution integral describing the input–output behaviour of an nth order causal, linear, time–invariant S is

$$y_s(t) = \int_0^t H_s(t-\tau)u_s(\tau)d\tau = \int_0^t C_s e^{A_s(t-\tau)} B_s u_s(\tau)d\tau \tag{3.2}$$

where $y_s(\cdot) \in \mathbb{R}^{q \times l}$ and $u_s(\cdot) \in \mathbb{R}^{p \times l}$ are the output and input vector, $H_s(\cdot) \in \mathbb{R}^{q \times p}$ is the transfer function matrix, $A_s \in \mathbb{R}^{n \times n}$, $B_s \in \mathbb{R}^{n \times p}$ and $C_s \in \mathbb{R}^{q \times n}$ are the parameters in the state representation.

As the S is causal, $C_s e^{A_s(t-\tau)} B_s = 0$ for $(t-\tau) \leq 0$, (3.2) can be rewritten as

$$y_s(t) = \int_0^\infty H_s(t-\tau)u_s(\tau)d\tau = \int_0^\infty C_s e^{A_s(t-\tau)} B_s u_s(\tau)d\tau \tag{3.3}$$

Equation (3.3) is valid when both $u_s(\cdot)$ and $y_s(\cdot)$ are vectors of integrable functions in L^2, Wilson (1985), Wilson (1989), Francis (1987), and implies that only the controllable and observable part of the S is of interest.

A similar expression is written for causal, linear, time–invariant, controllable and observable ***RM*** of order r with subscript r as

$$y_r(t) = \int_0^\infty H_r(t-\tau)u_r(\tau)d\tau = \int_0^\infty C_r e^{A_r(t-\tau)} B_r u_r(\tau)d\tau \tag{3.4}$$

where $y_r(\cdot) \in \mathbb{R}^{q \times l}$, $u_r(\cdot) \in \mathbb{R}^{p \times l}$ and $H_r(\cdot) \in \mathbb{R}^{q \times p}$ are the output, input vector and the transfer function matrix respectively. $A_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times p}$ and $C_r \in \mathbb{R}^{q \times r}$ are the parameters. The expression is also valid when both $u_r(\cdot)$ and $y_r(\cdot)$ are integrable functions in L^2.

Let the output of the ***RM*** be matched with that of the S. From (3.3) and (3.4) a convolution integral is obtained for tha said matching condition. Since the distribution property holds in a convolution integral, $H_s(\cdot)$ is the transfer function of a bounded input-bounded output S and $u_r(\cdot)$ is integrable function in L^2, $H_s(\cdot)u_r(\cdot)$ can be considered in addition to the integral obtained earlier. The expression related to the input–error is obtained as

$$\begin{aligned} &\int_0^\infty [u_s(\tau) - u_r(\tau)]d\tau \\ &= \int_0^\infty H_s^+(\cdot)[C_s e^{A_s(t-\tau)} B_s - C_r e^{A_r(t-\tau)} B_r] u_r(\tau)d\tau \end{aligned} \tag{3.5}$$

where the left generalized inverse of $H_s(\cdot)$ actually appeared, which, under the assumption made on the S (the bounded input-bounded output condition is equivalent to specified rank one, Israel, Greville (1974), Rao, Mistra (1971), is the same as the pseudoinverse $H_s^+(\cdot)$.

It is clearly seen from (3.5) that the input–error under integral sign on the L.H.S. accounting for the mismatch in parameters as well as in orders between the S and $\boldsymbol{RM}$ on the R.H.S. has a minimum value whenever the parameters of the $\boldsymbol{RM}$ are optimised w.r.t. that of the S.

With reference to the Holder's inequality for the integral, (3.5) is further written as

$$\int_0^\infty \|[u_s(\tau) - u_r(\tau)]\| d\tau$$
$$\leq \int_0^\infty \|H_s^+(\cdot)\| d\tau \cdot \|[C_s - C_r]\| \int_0^\infty \left\| \begin{bmatrix} e^{A_s(t-\tau)}B_s \\ e^{A_r(t-\tau)}B_r \end{bmatrix} u_r(\tau) \right\| d\tau \quad (3.6)$$

With the view of the Hankel operator norms, the first integral on the R.H.S. of (3.6) is positive and its value depends on the norm used. Let this value be α_s. If the Eucledean norm is used, α_s presents the maximum eigenvalue of the controllability and observability gramians of the S, Wilson (1989).

If the state of an augmented system is defined as

$$\bar{X}(t) = \int_0^t \begin{bmatrix} e^{A_s(t-\tau)}B_s \\ e^{A_r(t-\tau)}B_r \end{bmatrix} u_r(\tau) d\tau = [x_{sr}^T(t) \quad x_{rr}^T(t)]^T \quad (3.7)$$

where $x_{sr}(\cdot) \in \mathbb{R}^{n\times l}$ and $x_{rr}(\cdot) \in \mathbb{R}^{r\times l}$ are the respective state vectors of the S and $\boldsymbol{RM}$ corresponding to $u_r(\cdot)$, then by derivating (3.7) the dynamics of the said augmented system is obtained

$$\dot{\bar{X}}(t) = \bar{A}\bar{X}(t) + \bar{B}u_r(t) \quad (3.8)$$

and the output equation is

$$\bar{y}(t) = \bar{C}\bar{X}(t) \quad (3.9)$$

where

$$\bar{A} = \begin{bmatrix} A_s & 0 \\ 0 & A_r \end{bmatrix} \in \mathbb{R}^{(n+r)\times(n+r)}; \quad \bar{B} = \begin{bmatrix} B_s \\ B_r \end{bmatrix} \in \mathbb{R}^{(n+r)\times p}; \quad (3.10)$$

$$\bar{y}(t) = y_s(t) - y_r(t) \quad \text{and} \quad \bar{C} = [C_s \quad -C_r] \in \mathbb{R}^{q\times(n+r)} \quad (3.11)$$

Referring to (3.6), (3.7), (3.9) and (3.11), (3.1) becomes either

$$J \leq \lim_{t\to\infty} \bar{X}^T(t)\bar{C}^T \alpha_s R \alpha_s \bar{C}\bar{X}(t) \quad (3.12)$$

or

$$J \leq \lim_{t\to\infty} \mathbb{E}[\bar{X}^T(t)\bar{C}^T \alpha_s R \alpha_s \bar{C}\bar{X}(t)] \quad (3.13)$$

corresponding to the respective cases where $u_r(\cdot)$ is either a vector of deterministic signals or white noises with nonnegative definite intensity V_r. When $t \to \infty$, (3.12) and (3.13) are equivalent, and then (3.1) is

$$J \leq \operatorname{tr} Q \cdot R \quad (3.14)$$

where $R = \bar{C}^T \alpha_s R \alpha_s \bar{C} \in \mathbb{R}^{(n+r)\times(n+r)}$ and Q is either $\lim_{t\to\infty} \bar{X}(t)\bar{X}(t)^T$ or $\lim_{t\to\infty} \mathbb{E}[\bar{X}(t)\bar{X}(t)^T] \in \mathbb{R}^{(n+r)\times(n+r)}$ corresponding to each case of $u_r(\cdot)$.

As A_s is a stable matrix (S is bounded input, bounded output) and A_r is also stable (***RM*** is to be controllable and observable) hence, from (3.10), $\bar{A}$ is stable. Further, $(\bar{A}, \bar{B})$ and $(\bar{A}, \bar{C})$ are respectively controllable and observable, corresponding to which Q is the unique solution of Lyapunov equation

$$\bar{A}Q + Q\bar{A}^T + \bar{B}V_r\bar{B}^T = 0 \tag{3.15}$$

where V_r is either I_p for deterministic $u_r(t)$ or $\lim_{t\to\infty} \mathbb{E}[u_r(t)u_r^T(t)] \in \mathbb{R}^{p\times p}$ for white noise $u_r(t)$, respectively.

A finite value of J will be obtained independently of the initial conditions of (3.8) if (3.14) is minimized subject to (3.15).

The problem is restated as : Given an nth order causal, linear, time-invariant S through its bounded response to a bounded input, determine an rth order $(q \leq r < n)$ ***RM*** such that the cost function (3.1) is minimized within the set $\mathcal{J} \triangleq \{S/\boldsymbol{RM}$: stabilisable & detectable $(\bar{A}, \bar{B})$: controllable, $(\bar{A}, \bar{C})$: observable$\}$ so that a finite value of J can be obtained.

If the parameters of the controllable and observable part of the S are known, then the problem can be easily solved and the optimal projection equations for computing the optimal parameters of the ***RM*** would be almost the same as that obtained in Hyland, Bernstein (1985). However, in view of (3.14) one can state that the maximum value of the present input–error would be the value of an input–error obtained by transferring from the output-error. This is due to α_s of which is but an additional condition put on the L_2–optimization problem.

The parameters of the S are not known, the technique adopted in Hyland, Bernstein (1985) cannot be employed in the present case, until the parameters of the mentioned part of the S are estimated. In such a problem of real–time estimation of the S parameters, different points arise. First, an S parameter estimation problem itself is an approximate one for which the solution is obtained depending upon the criterion used. Secondly, the input of the S is not as desired to be, persistently exciting, and the S like a bio–system or a chemical reaction process can not be excited by a pseudo random binary sequence. This makes the estimation of the S parameters a difficult task. Further, it is clear that the controllable and observable (constraint) conditions would be put twice on the S parameters, one in the estimation and other on the augmented S in the reduction. The double use of a constraint condition in the matrix form (like the Lyapunov equation) is a burden from the computational point of view.

A question thus arises, whether an ***AM*** having known parameters can be used to link the parameters of the S and those of the ***RM***in order to escape estimation of the S parameters. If it is the case, then what are the relationships between the parameters of the three? It is described in the next section how to bring an ***AM*** into the picture and the relationships are established in the section of main result. It will also be shown that inviting an ***AM*** will be useful not only for the problem of model reduction of an unknown parameter S but also for that of the S identification.

4. Formulation of the auxiliary problem for minimization

An nth order $\boldsymbol{AM}$ having known parameters is introduced in parallel with the S and $\boldsymbol{RM}$ in the model reference technique with an error on the input side. It can be observed that two errors appear; one is between the input of the S and that of the $\boldsymbol{AM}$ reflecting the mismatch between their parameters, and the other is between the inputs of the $\boldsymbol{AM}$ and the $\boldsymbol{RM}$ due to the differences in their orders and parameters.

If the first error is minimized, parameters of the S are obtained, expressed in terms of $\boldsymbol{AM}$. Likewise, the $\boldsymbol{RM}$ parameters are also derived if the second error is minimized. The outcome of the two minimization processes is that via the $\boldsymbol{AM}$ parameters it is possible to establish a relationship between the parameters of the S and the $\boldsymbol{RM}$ although none of them is known.

If two manifolds resulting from the said two minimization processes have nonempty intersection, then the only the process of minimization is required. For this, it needs a functional cost consisting of both errors, which is now described.

The cost function (3.1) is eqiuvalently written incorporating the $\boldsymbol{AM}$ input as

$$J = \int_0^\infty [u_s(\tau)-u_m(\tau)+u_m(\tau)-u_r(\tau)]^T R[u_s(\tau)-u_m(\tau)+u_m(\tau)-u_r(\tau)]d\tau$$

$$= \int_0^\infty [[u_m(\tau)-u_s(\tau)]^T[-u_m(\tau)+u_r(\tau)]^T] \begin{bmatrix} I_q \\ I_q \end{bmatrix} R \cdot$$

$$\cdot [I_q \quad I_q] \begin{bmatrix} u_m(\tau)-u_s(\tau) \\ -u_m(\tau)+u_r(\tau) \end{bmatrix} d\tau \tag{4.1}$$

Similarly to (3.3) for an nth order (causal, linear, time–invariant) controllable and observable $\boldsymbol{AM}$ with subscript m we can write

$$y_m(t) = \int_0^\infty H_M(t-\tau)u_m(\tau)d\tau = \int_0^\infty C_m e^{A_m(t-\tau)} B_m u_m(\tau)d\tau \tag{4.2}$$

where $y_m(\cdot) \in \mathbb{R}^{q\times l}$ and $u_m(\cdot) \in \mathbb{R}^{p\times l}$ are the output and input vectors, $H_m(\cdot) \in \mathbb{R}^{q\times p}$ is the transfer function matrix, $A_m \in \mathbb{R}^{n\times n}$, $B_m \in \mathbb{R}^{n\times p}$ and $C_m \in \mathbb{R}^{q\times n}$ are the parameters. The expression is also valid when $y_m(\cdot)$ and $u_m(\cdot)$ are the vectors of L^2 integrable.

Let $y_m(\cdot) = y_s(\cdot)$. Considering an additional term $H_s(\cdot)u_m(\cdot)$ in the integral obtained from (3.3) and (4.2), the relation related to the difference between the inputs of the S and $\boldsymbol{AM}$ is obtained as

$$\int_0^\infty \|[u_s(\tau)-u_m(\tau)]\| d\tau$$

$$\leq \int_0^\infty \|H_s^+(\cdot)\|d\tau \cdot \|[C_s - C_m]\| \int_0^\infty \left\| \begin{bmatrix} e^{A_s(t-\tau)}B_s \\ e^{A_m(t-\tau)}B_m \end{bmatrix} u_m(\tau) \right\| d\tau \quad (4.3)$$

Similarly, as $y_m(t) = y_r(\cdot)$ and considering one more term $H_r(\cdot)u_m(\cdot)$ one obtains from (3.4) and (4.2)

$$\int_0^\infty \|[u_r(\tau) - u_m(\tau)]\|d\tau$$

$$\leq \int_0^\infty \|H_r^+(\cdot)\|d\tau \cdot \|[C_r - C_m]\| \int_0^\infty \left\| \begin{bmatrix} e^{A_r(t-\tau)}B_r \\ e^{A_m(t-\tau)}B_m \end{bmatrix} u_m(\tau) \right\| d\tau \quad (4.4)$$

With reference to the Hankel operator norms, the first integral on the R.H.S. of (4.4) is a positive quantity denoted by α_r, whose value depends on the norm used.

Assume that the norm used in (4.4) is that of (4.3), one has

$$\int_0^\infty \{\|[u_s(\tau) - u_m(\tau)]\| + \|[u_r(\tau) - u_m(\tau)]\|\}d\tau$$

$$\leq [I_q \quad I_q] \left\| \begin{bmatrix} -\alpha_s C_s & \alpha_s C_m & 0 \\ 0 & -\alpha_r C_m & \alpha_r C_r \end{bmatrix} \right\| \int_0^\infty \left\| \begin{bmatrix} e^{A_s(t-\tau)}B_s \\ e^{A_m(t-\tau)}B_m \\ e^{A_r(t-\tau)}B_r \end{bmatrix} u_m(\tau) \right\| d\tau \quad (4.5)$$

Define

$$\tilde{X}(t) = \int_0^t \begin{bmatrix} e^{A_s(t-\tau)}B_s \\ e^{A_m(t-\tau)}B_m \\ e^{A_r(t-\tau)}B_r \end{bmatrix} u_m(\tau)d\tau = [x_{sm}^T(t) \quad x_{mm}^T(t) \quad x_{rm}^T(t)]^T \quad (4.6)$$

where $x_{sm}(\cdot), x_{mm}(\cdot) \in \mathbb{R}^{n\times l}$ and $x_{rm}(\cdot) \in \mathbb{R}^{r\times l}$ are state vectors of the S, $\boldsymbol{AM}$ and $\boldsymbol{RM}$ corresponding to $u_m(\cdot)$.

The dynamics of the augmented system consisting of all the three is obtained as

$$\dot{\tilde{X}}(t) = \tilde{A}\tilde{X}(t) + \tilde{B}u_m(t) \quad (4.7)$$

$$\tilde{y}(t) = \tilde{C}\tilde{X}(t) \quad (4.8)$$

where $\tilde{A} = \begin{bmatrix} A_s & 0 & 0 \\ 0 & A_m & 0 \\ 0 & 0 & A_r \end{bmatrix} \in \mathbb{R}^{(2n+r)\times(2n+r)}$; $\tilde{B} = \begin{bmatrix} B_s \\ B_m \\ B_r \end{bmatrix} \in \mathbb{R}^{(2n+r)\times p}$

and $\tilde{C} = \begin{bmatrix} -C_s & C_m & 0 \\ 0 & -C_m & C_r \end{bmatrix} \in \mathbb{R}^{q\times(2n+r)}$.

For a similar argument as made for (3.14), (4.1) is written as

$$j \leq \operatorname{tr} \tilde{Q}\tilde{R} \quad (4.9)$$

where $\tilde{R} = \begin{bmatrix} -\alpha_s C_s & \alpha_s C_m & 0 \\ 0 & -\alpha_r C_m & \alpha_r C_r \end{bmatrix}^T \begin{bmatrix} I_q \\ I_q \end{bmatrix} R[I_q \quad I_q] \begin{bmatrix} -\alpha_s C_s & \alpha_s C_m & 0 \\ 0 & -\alpha_r C_m & \alpha_r C_r \end{bmatrix} \in \mathbb{R}^{(2n+r)\times(2n+r)}$ and $\tilde{Q}$ is either $\lim_{t\to\infty} \tilde{X}(t)\tilde{X}^T(t)$ or $\lim_{t\to\infty} \mathbb{E}[\tilde{X}(t)\tilde{X}^T(t)] \in \mathbb{R}^{(2n+r)\times(2n+r)}$ corresponding to $u_m(\cdot)$ a vector of either deterministic signals or white noises with nonnegative definite intensity V_m.

As $(\tilde{A}, \tilde{B})$ is controllable, the matrix $\tilde{Q}$ consisting of the auto– and cross–relations between the states corresponding to $u_m(\cdot)$ satisfies the Lyapunov equation

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}V_m\tilde{B}^T = 0 \tag{4.10}$$

in which V_m is either I_p or $lim_{t\to\infty}\mathbb{E}[u_m(t)u^T(t)]$ corresponding to a deterministic signal or white noise input respectively.

This ensures that the control law $U_m(\cdot)$ will be linear w.r.t. the defined state $\tilde{X}(t)$. Moreover, as $u_m(\cdot)$ is bounded, a finite value of J will be obtained, independent of the initial conditions of the augmented system described by (4.7).

The problem of reduction of the order of an unknown parameter S stated in (3.2) has been transferred into that minimizing (4.9) subject to the constraint (4.10). This means that the minimization of (4.1) is restricted to the set $\mathcal{J} \triangleq \{\ S/\boldsymbol{AM}/\boldsymbol{RM}$: stabilisable & detectable, $(\tilde{A}, \tilde{B})$: controllable, $(\tilde{A}, \tilde{C})$: observable $\}$.

5. The main results

Solving the auxiliary minimization problem, two theorems are stated as

THEOREM 5.1 *1. — For a controllable and observable S of order n there exists a set of controllable and observable $\boldsymbol{AM}$s of order n and a set of controllable and observable $\boldsymbol{RM}$s of order r $(r < n)$. With a chosen $\boldsymbol{AM}$, there exists an optimal projector $\sigma_2 = G_2^T\Gamma_2$ of order $(n+r)$, the optimal augmented parameters are expressible as functions of the $\boldsymbol{AM}$ parameters and the components of the projector as*

$$\bar{A} = \sigma_2^g G_2^T A_m \Gamma_2 \sigma_2^g \tag{5.1}$$

$$\bar{B} = -\sigma_2^g G_2^T B_m \tag{5.2}$$

$$\mathcal{R}\begin{bmatrix} C_s & 0 \\ 0 & C_r \end{bmatrix} = -\mathcal{R}\begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m \Gamma_2 \sigma_2^g \tag{5.3}$$

where $\mathcal{R} = [(R_{sr} - R_s) \quad (R_{sr} - R_r)]$.

2. — There exist two optimal positive definite matrices of order $(n+r)$ $\bar{Q}$ and $\bar{P}$ in the sense of the definition 2.1 such that with σ_2 as a prticular choice for σ_2^g, the conditions to be satisfied are

$$\varrho(\bar{Q}) = \varrho(\bar{P}) = \varrho(\bar{Q}\bar{P}) = (n+r) \tag{5.4}$$

$$G_2^T A_m \Gamma_2 \bar{Q} + \bar{Q}\Gamma_2^T A_m^T G_2 + G_2^T B_m V_m B_m^T G_2 = 0 \tag{5.5}$$

$$\Gamma_2^T A_m^T G_2 \bar{P} + \bar{P} G_2^T A_m \Gamma_2 + \Gamma_2^T C_m^T [I_q \quad I_q] \begin{bmatrix} R_s & -R_{sr} \\ -R_{sr} & R_r \end{bmatrix} \begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m \Gamma_2 = 0 \tag{5.6}$$

PROOF: The proof of the theorem is given in the appendix.

REMARKS 5.1

1. — For a matching condition between the outputs of the $\boldsymbol{AM}$ and S, the $\boldsymbol{AM}$ input is seen to be a persitently exciting one independent of the form of the S input due to the involvement of the state transition matrices of the S and $\boldsymbol{AM}$. The matrix R is clearly seen to be nonzero. With these conditions, the optimal augmented parameters are obtained in (5.1)–(5.3). Hence (5.5) and (5.6) follow, although in these equations V_m and $\mathcal{R}$ can be zero.

2. — (5.5), (5.6) are referred to as the modified Lyapunov equations. In these equations, the matrices $\bar{Q}$ and $\bar{P}$ play the analogous roles as that played by the controllability and observability gramians of the augmented systems but they are not these gramians (appendix), hence $\bar{Q}$ and $\bar{P}$ are termed the controllability and observability anagramians.

3. — It is noted that the factors G_2 and Γ_2 of the optimal projector are dimensionless. However, with σ_2 chosen for σ_2^g, each amongst $\sigma_2^g G_2^T$ and $\Gamma_2 \sigma_2^g$ is related with only a quantity presenting the controllability and observability gramian respectively and so are $\bar{B}$ in (5.2) and $\bar{C}$ in (5.3). This leads (5.5) and (5.6), the two modified Lyapunov equations, to be in the standard–like form without coupling. However, the effect of decoupling of two modified Lyapunov equations leads to a simple computation to be adopted.

4. — Pre– and post–multiplying of (5.5) by Γ_2 and Γ_2^T and of (5.6) by G_2 and G_2^T respectively, with $\Gamma_2 G_2^T = G_2 \Gamma_2^T = I_n$, the two modified Lyapunov equations become

$$A_m \Gamma_2 \bar{Q} \Gamma_2^T + \Gamma_2 \bar{Q} \Gamma_2^T A_m^T + B_m V_m B_m^T = 0 \tag{5.7}$$

$$A_m^T G_2 \bar{P} G_2^T + G_2 \bar{P} G_2^T A_m + C_m^T [I_q \quad I_q] \begin{bmatrix} R_s & -R_{sr} \\ -R_{sr} & R_r \end{bmatrix} \begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m = 0 \tag{5.8}$$

Two standard Lyapunov equations applicable to the $\boldsymbol{AM}$ are

$$A_m W_{mc} + W_{mc} A_m^T + B_m V_m B_m^T = 0 \tag{5.9}$$

$$A_m^T W_{mo} + W_{mo} A_m + C_m^T [I_q \quad I_q] \begin{bmatrix} R_s & -R_{sr} \\ -R_{sr} & R_r \end{bmatrix} \begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m = 0 \tag{5.10}$$

where W_{mc} and W_{mo} are the respective controllability and observability gramians of the $\boldsymbol{AM}$. (5.9) and (5.10) are easily computed yielding

$$\Gamma_2 \bar{Q} \Gamma_2^T = Q_m \tag{5.11}$$

$$G_2 \bar{P} G_2^T = P_m \tag{5.12}$$

Referring to proposition 2.1, $\bar{Q} = \sigma\bar{Q} = \bar{Q}\sigma^T$, $\bar{P} = \sigma^T\bar{P} = \bar{P}\sigma$, pre– and post–multiplying on (5.11) by G^T and G, and on (5.12) by Γ^T and Γ respectively gives

$$\bar{Q} = G_2^T W_{mc} G_2 \quad \text{and} \quad \bar{P} = \Gamma_2^T W_{mo} \Gamma_2 \tag{5.13}$$

Thus, instead of computing $\bar{Q}$ and $\bar{P}$ from respective (5.5) and (5.6), one can use (5.13) after computing W_{mc} and W_{mo} from the standard equations applicable to the $\boldsymbol{AM}$, i.e., (5.9) and (5.10).

5. — Several remarks and propositions regarding the change of bases stated in Hyland, Bernstein (1985), Haddad, Bernstein (1989), can be made in the present case with the change of the $\boldsymbol{AM}$ basis, with that of projector's basis, the Drazin group inverse, Israel, Greville (1974), Hyland, Bernstein (1985), i.e., different criteria adopted to obtain different forms of the presently obtained result, which may facilitate construction suitable algorithms, in the context of a global optimum, for computation.

The parameters of the augmented S, S and $\boldsymbol{RM}$ are obtained in the following corollaries.

COROLLARY 5.1 *Assume that relations (5.1) through (5.3) hold, then the augmented parameters are given by*

$$\bar{A} = G_2^T A_m \Gamma_2, \quad \bar{B} = -G_2^T B_m, \quad \begin{bmatrix} C_s & 0 \\ 0 & -C_r \end{bmatrix} = -\begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m \Gamma_2 \tag{5.14}$$

PROOF: Following the property (e) [Boullion, Odell (1971), p.7], σ_2 is chosen for σ_2^g. With reference to the proposition 2.1 and to the choice for σ_2^g, the first and second identities are readily obtained from (5.1) and (5.2) respectively. The third one is obtained from (5.3) with a condition that the matrix $\mathcal{R}$ be nonzero.

The third result stated in the above corollary is equivalently obtained considering two input–errors, which were defined as the difference between the inputs of the $\boldsymbol{AM}$ and S and that between the $\boldsymbol{AM}$ and $\boldsymbol{RM}$ inputs, to be uncorrelated. In the correlated input–errors case, the parameters of the S and $\boldsymbol{RM}$ are separately stated in the following corollaries.

COROLLARY 5.2 *Assume that the augmented parameters are expressed in (5.1)–(5.3), then the parameters of the S are given by*

$$A_s = [I_n \quad 0_{n\times r}] G_2^T A_m \Gamma_2 \begin{bmatrix} I_n \\ 0_{r\times n} \end{bmatrix} \tag{5.15}$$

$$B_s = -[I_n \quad 0_{n\times r}]G_2^T B_m \tag{5.16}$$

$$C_s = -[I_q \quad (R_{sr}-R_s)^{-1}(R_{sr}-R_r)]\begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m \Gamma_2 \begin{bmatrix} I_n \\ 0_{r\times n} \end{bmatrix} \tag{5.17}$$

PROOF: It is readily seen that $A_s = [I_n \quad 0_{n\times r}]\bar{A}\begin{bmatrix} I_n \\ 0_{r\times n} \end{bmatrix}$ and $B_s = -[I_n \quad 0_{n\times r}]\bar{B}$. Using $\bar{A}$ and $\bar{B}$ as given in (5.1) and (5.2), (5.15) and (5.16) are then obtained. It is noted that with the choice of σ_2^g mentioned earlier, (5.3) is written in another form as

$$[(R_{sr}-R_s)C_s \quad (R_{sr}-R_r)C_r] = [(R_{sr}-R_s) \quad (R_{sr}-R_r)]\begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m \Gamma_2$$

Hence, (5.17) is derived.

COROLLARY 5.3 *Assume that the augmented parameters are expressed in (5.1)–(5.3), then the parameters of the **RM** are given by*

$$A_r = [0_{r\times n} \quad I_r]G_2^T A_m \Gamma_2 \begin{bmatrix} 0_{n\times r} \\ I_r \end{bmatrix} \tag{5.18}$$

$$B_r = -[0_{r\times n} \quad I_r]G_2^T B_m \tag{5.19}$$

$$C_r = -[(R_{sr}-R_r)^{-1}(R_{sr}-R_s) \quad I_q]\begin{bmatrix} I_q \\ I_q \end{bmatrix} C_m \Gamma_2 \begin{bmatrix} 0_{n\times r} \\ I_r \end{bmatrix} \tag{5.20}$$

PROOF: Similar to corollary 5.2.

REMARKS 5.2

1. — In the case where only the estimation of the S parameters is of interest, the blocks related to the ***RM*** with subscript r of $\tilde{R}$ in (4.2) and 3 of $\tilde{Q}$ and $\tilde{P}$ in the appendix are considered to be absent. 6 out of 15 equations ((7.2)–(7.16)) disappear in addition to that the first three ((7.2)–(7.4)) consist only of the quantities related to the S and ***RM***. $P^* = P_{12}$, $Q^* = Q_{12}$, $\Gamma_2 = P_{22}^{-1}P_{12}^T$ and $G_2^T = Q_{12}Q_{22}^{-1}$ are $n \times n$ matrices. The optimal projector $\sigma_2 = Q_{12}Q_{22}^{-1}P_{22}^{-1}P_{12}^T$ is of order n having the rank n, hence σ_2^g becomes σ_2^{-1} The anagramians $\bar{Q} = Q_{12}Q_{22}^{-1}Q_{12}^T$ and $\bar{P} = P_{12}P_{22}^{-1}P_{12}^T$ consist only of quantities related to the respective gramians of the augmented S and so on. The theorem is dealing with the optimal projector equations for parameters which have the responsibility for the controllable and observable part of the S. This part of the S, although is not uniquely described by the state–variable representation, is expected to be obtained using a similarity transformation on the ***AM*** parameters. However, the result of the S parameter estimation problem does not depend on parameters of the ***AM*** because the data used for the purpose are those related with the controllability and observability gramians of the S

in Q_{12} and P_{12}, respectively. For the estimation purpose, the linear dynamical operator, Unbehauen, Rao (1987), Sonderstrom, Stoica (1989), supplying the required data has to be the type of the state–observer based.

2. — In the case where the parameters of the S to be reduced are known, the parameters of the $\boldsymbol{RM}$ only are of interest. The relationship between the parameters of the S and $\boldsymbol{RM}$ can be found out by the use of the results stated in corollaries 5.2 and 5.3. Alternatively, since the S parameters are known, the S is chosen as an $\boldsymbol{AM}$, the blocks with subscripts s of $\tilde{R}$ in (4.2) and l of $\tilde{Q}$ and $\tilde{P}$ in the appendix are interpreted to be absent. The matrices $\tilde{R}$, $\tilde{Q}$ and $\tilde{P}$ are of the order $(n+r)$. Only 9 out of 15 equations ((7.2)–(7–16)) are present involving the quantities for the $\boldsymbol{AM}$ (presently S) and $\boldsymbol{RM}$. $P^* = P_{23}^T$, $Q^* = Q_{23}^T$ are $r \times n$ matrices, $\Gamma_2 = P_{22}^{-1}P_{23}$, $G_2 = Q_{22}^{-1}Q_{23}$ are $n \times r$ matrices. In this case σ_2^g becomes also σ_2^{-1} and the anagramians $\bar{Q} = Q_{23}^T Q_{22}^{-1} Q_{23}$ and $\bar{P} = P_{23}^T P_{22}^{-1} P_{23}$ are of the order r having rank r. The present result is the same as that obtained by Hyland and Bernstein in [Hyland, Bernstein (1985)] whereas the S was in the balanced optimal projection basis.

3. — In the context of a global minimum, the present result can be used along with other criteria which may be the principle of cost–ranking, Skelton, Yousuff (1983), the internally balanced condition, Mustafa, Glover (1991), etc.. However, a sufficient condition for the auxiliarly formulated problem can also be obtained if one puts the problem under a preassigned H_∞ limit.

4. — If a set of the parameters in the state variable description is defined as $\mathcal{M} := (A, B, C)$, then upon minimizing $\mathcal{L}(\cdot)$ w.r.t. the set of the $\boldsymbol{AM}$ there exist transformations $\mathcal{T}_s$ and $\mathcal{T}_r$

$$\mathcal{T}_s \mathcal{M}_m = \mathcal{M}_s, \quad \mathcal{T}_r \mathcal{M}_m = \mathcal{M}_r \tag{5.21}$$

such that,

$$(\mathcal{T}_r^+ \mathcal{M}_r - \mathcal{T}_s^+ \mathcal{M}_s) \in (\mathcal{N}(\mathcal{T}_s) + \mathcal{N}(\mathcal{T}_r)) \tag{5.22}$$

for the existing set of common solutions, with the view of corollary 3 [Israel, Greville (1974), p.209], which is one of the following equivalent manifolds

$$\mathcal{T}_s^+ \mathcal{M}_s + \mathcal{P}_{\mathcal{N}(\mathcal{T}_s)}[\mathcal{P}_{\mathcal{N}(\mathcal{T}_s)} + \mathcal{P}_{\mathcal{N}(\mathcal{T}_r)}]^+ [\mathcal{T}_r^+ \mathcal{M}_r - \mathcal{T}_s^+ \mathcal{M}_s] + [\mathcal{N}(\mathcal{T}_s) \cap \mathcal{N}(\mathcal{T}_r)] \tag{5.23}$$

$$\mathcal{T}_r^+ \mathcal{M}_r - \mathcal{P}_{\mathcal{N}(\mathcal{T}_r)}[\mathcal{P}_{\mathcal{N}(\mathcal{T}_s)} + \mathcal{P}_{\mathcal{N}(\mathcal{T}_r)}]^+ [\mathcal{T}_r^+ \mathcal{M}_r - \mathcal{T}_s^+ \mathcal{M}_s] + [\mathcal{N}(\mathcal{T}_s) \cap \mathcal{N}(\mathcal{T}_r)] \tag{5.24}$$

$$[\mathcal{T}_s^+ \mathcal{M}_s + \mathcal{T}_r^+ \mathcal{M}_r]^+ [\mathcal{T}_s^+ \mathcal{M}_s + \mathcal{T}_r^+ \mathcal{M}_r] + [\mathcal{N}(\mathcal{T}_s) \cap \mathcal{N}(\mathcal{T}_r)] \tag{5.25}$$

where $\mathcal{T}_r^+$, $\mathcal{T}_s^+$ stand for the generalized inverse of the triansformations $\mathcal{T}_r$, $\mathcal{T}_s$ respectively; $\mathcal{N}(\cdot)$ and $\mathcal{P}_{\mathcal{N}(\cdot)}$ are respectively denoted the null space of the transformation and the transformation into null space of the transformation and the symbol $\cap$ stands for the intersection of two null spaces.

The conditions to be satisfied by the parameters of the S and $\boldsymbol{RM}$ for ensuring the manifold of the common solutions are nonempty can, hence, be derived.

Further, it may be possible to show that $\mathcal{T}_s$, $\mathcal{T}_r$ are not only the transformations but also the projections and each is a partial isometry of the augmented transformation. In addition to that, $\mathcal{T}_r$ is a partial isometry of $\mathcal{T}_s$, which can be easily seen if the diagonalization of $\mathcal{T}_s$ and $\mathcal{T}_r$ is performed.

It has been shown that the optimal parameters of the S and ***RM*** are related to those of the ***AM*** by the optimal projection equations. It is found that the optimal parameters of the three are trangularly coupled each other via three optimal projectors. If one set of the parameters is known the other sets can be obtained. The following theorem states the manner of coupling of the optimal projectors instead of different analysis are to be carried out in connection with the theorem 5.1.

THEOREM 5.2 *Let the augmented parameters be given according to the theorem 5.1, then there exist two nonnegative definite matrices $\mathcal{Q}$ and $\mathcal{P}$ both of order $(n+r)$ such that in addition to the optimal projector two more projectors appear and all the projectors are coupled to each other via their factors.*

PROOF: Define $(n+r)\times(n+r)$ matrices $\tilde{\mathcal{Q}} = \begin{bmatrix} Q_{11} & Q_{13} \\ Q_{13}^T & Q_{33} \end{bmatrix}, \tilde{\mathcal{P}} = \begin{bmatrix} P_{11} & P_{13} \\ P_{13}^T & P_{33} \end{bmatrix}$, $\mathcal{Q} = \tilde{\mathcal{Q}} - \bar{\mathcal{Q}}$ and $\mathcal{P} = \tilde{\mathcal{P}} - \bar{P}$. It is readily seen that $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{P}}$ are nonnegative definite. Referring to the theorem 5.1, the system of matrix equations consisting of (7.14), (7.15), $(7.15)^T$ and (7.16) (Appendix) is written as $\bar{A}\tilde{\mathcal{Q}} + \tilde{\mathcal{Q}}\bar{A}^T + \bar{B}V_m\bar{B}^T = 0$. This equation, with reference to defined $\mathcal{Q}$, is also written as $\bar{A}\mathcal{Q} + \mathcal{Q}\bar{A}^T = 0$, which gives a unique solution. Since $\bar{A}$ is to be a stable matrix, the unique solution is zero matrix.

Similarly, referring to the theorem 5.1 and $\mathcal{P}$, the system of equations (7.8), (7.9), $(7.9)^T$ and (7.10) (Appendix) leads to equation $\bar{A}\mathcal{P} + \mathcal{P}\bar{A}^T = 0$, whose unique solution is also zero matrix.

Since $\mathcal{Q}$ and $\mathcal{P}$ are zero matrices and Q_{11}, Q_{22}, Q_{33}, P_{11}, P_{22}, P_{33} are invertible (Appendix), the identities $\Gamma_1 G_1^T = \Gamma_2 G_2^T = I_n$ and $\Gamma_3 G_3^T = I_r$ are easily obtained where $\Gamma_1 = P_{11}^{-1}[P_{12} \quad P_{13}]$, $G_1 = -Q_{11}^{-1}[Q_{12} \quad Q_{13}]$, $\Gamma_2 = P_{22}^{-1}[P_{12}^T \quad P_{23}]$, $G_2 = -Q_{22}^{-1}[Q_{12}^T \quad Q_{23}]$, $\Gamma_3 = P_{33}^{-1}[P_{13}^T \quad P_{23}^T]$ and $G_3 = -Q_{33}^{-1}$ $[Q_{13}^T \quad Q_{23}^T]$. It follows from lemma 2.1 that the optimal projectors are $\sigma_1 = G_1^T\Gamma_1$, $\sigma_2 = G_2^T\Gamma_2$ and $\sigma_3 = G_3^T\Gamma_3$. It can be seen that the three optimal projectors are coupled to each other by their factors.

REMARKS 5.3

1. — Due to the extremal conditions to be achieved w.r.t. the ***AM*** parameters in the problem, the optimal projector σ_2 appears and is used for obtaining the optimal parameters of the S and ***RM*** in the augmented form. The parameters of the S and ***RM*** are obtained individually with a choice of the generalized inverse of σ_2. The two other optimal projectors are made to appear and all the three couple to each other by their factors. The optimal projectors are called "jointly optimal projectors".

2. — It can be easily shown that the jointly optimal projectors are uniquely determined in the sense that knowing leads to deducing the other two. (If the optimization is carried out w.r.t. the S parameters, σ_1 would appear and the optimal parameters of the $\boldsymbol{AM}$ and $\boldsymbol{RM}$ would be related with that of the S in the same manner as stated in the theorem 5.1 with a suitable change in the quantities of its expressions). This implies that optimization can be carried out w.r.t. the parameters of anyone amongst the S, $\boldsymbol{AM}$ and $\boldsymbol{RM}$. Fact is that the transformation from one space to another due to the process of optimization is bi–directional within the generalized concept. The intersection space of two transformations can always be made non–empty due to a large number of possibilities for choosing a generalized inverse of the jointly optimal projectors.
3. — Futher, it is found that the optimal projectors also appear within the factor of each pair formed amongst the S, $\boldsymbol{AM}$ and $\boldsymbol{RM}$. This implies that to estimate the S parameters, a model of any order can be chosen and the S parameters are obtained by minimizing the error. The order of the model is then increased for finding the minimum of the minimized errors. If the estimation process is performed up to a certain order, an optimal reduced model corresponding to that order is then obtained.
4. — If a change of the basis of the $\boldsymbol{AM}$ is made, the bases of the S and $\boldsymbol{RM}$ are accordingly changed due to the coupled optimal projectors. It can be shown that if the $\boldsymbol{AM}$ is transferred into the balanced optimal projection basis, with respect to which the optimal projector σ_2 is unitary and two anagramians are diagonalized, then the other two σ_1, σ_3 are also unitary and two corresponing pairs of the anagramians are also diagonalized. This implies that the S and $\boldsymbol{RM}$ are also transferred into the balanced optimal projector basis.
5. — If two $(n+r) \times (n+r)$ matrices $\mathcal{Q}$, $\mathcal{P}$ in the proof of the theorem 5.2 were defined as $\mathcal{Q} = \tilde{\mathcal{Q}} + \bar{Q}$, $\mathcal{P} = \tilde{\mathcal{P}} + \bar{P}$, two modified matrix Riccati equations would appear. A close–loop treatments can then be carried out with the help of the said Riccati equations, Haddad, Bernstein (1989).
6. — It is clear that the theorem 5.1 has been analysed by theorem 5.2 because the optimality in the problem of S parameter estimation was reconfirmed to be achieved and so in the reduction.

6. Discussions and conclusions

In order to match the output of the model ($\boldsymbol{AM}$ and $\boldsymbol{RM}$) with that of the S, the model should have an input other than the S one, the input–error is thus brought in. In the case where $\boldsymbol{AM}$ is different from $\boldsymbol{RM}$, the augmented system consists of all of the three. An optimal projector appears and the other two are derived coupling with the earlier one. The parameter estimation ($\boldsymbol{RM}$ or S) can be carried out following the projection equations of which the proposed development hides behind an idea of bringing the reduction problem to a misorder case of the S parameter estimation.

Instead of direct minimization of the input–error appearing in the auxiliary problem, the problem can be well tackled with the use of linear quadratic method, i.e., the Riccati–equation approach. The input–error can be made available by a linear feedback, which, in turn, can be thought of as a linear regulator problem. This permits to get the idea that the theory established for the linearly optimal regulator can be applied to reduced S in a closed–loop.

7. Appendix

As the uniqueness in state variable description does not obtain, a set of ***AM***s and that of ***RM***s can, hence, be obtained for a given S.

For minimizing (4.4) subject to (4.5), a Lagrangian function is formed as

$$\mathcal{L}(A_m, B_m, C_m, \tilde{Q}, \tilde{P}, \lambda) = \operatorname{tr}[\lambda \tilde{Q}\tilde{R} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{B}V_m\tilde{B}^T)\tilde{P}] \tag{7.1}$$

where multipliers $\lambda \geq 0$ and $\tilde{P} \in R^{(2n+r)\times(2n+r)}$ should not simultanously be zero. Moreover, to ensure the constraint conditions to be effective and the system of equations inside the trace to be linear independent, $\tilde{P}$ should be a positive definite matrix. The matrix $\tilde{Q}$ is symmetric, nonnegative definite. Perform a partition of $\tilde{Q}$ and $\tilde{P}$ as

$$\tilde{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \in \begin{bmatrix} (n\times n) & (n\times n) & (n\times r) \\ (n\times n) & (n\times n) & (n\times r) \\ (r\times n) & (r\times n) & (r\times r) \end{bmatrix}$$

The optimal augmented parameters are the parameters of the ***AM*** resulting from the first–order necessary conditions for (7.1) to be an extremum. Taking partial derivatives of $\mathcal{L}(\cdot)$ w.r.t. the unknowns [Athans (1968)] and equating the results to zero, one obtains

$$\mathcal{L}_{A_m}(\cdot) = P_{22}Q_{22} + P_{12}^TQ_{12} + P_{23}Q_{23}^T = 0 \tag{7.2}$$

$$\mathcal{L}_{B_m}(\cdot) = (P_{12}^TB_s + P_{23}B_r + P_{22}B_m)V_m = 0 \tag{7.3}$$

$$\begin{aligned}\mathcal{L}_{C_m}(\cdot) = \lambda\{&(R_{sr}^T - R_s)C_sQ_{12} + (R_{sr} - R_r)C_rQ_{23}^T \\ &-(R_{sr}^T - R_s + R_{sr} - R_r)C_mQ_{22}\} = 0\end{aligned} \tag{7.4}$$

$$\mathcal{L}_{Q_{22}}(\cdot) = A_m^TP_{22} + P_{22}A_m + \lambda C_m^T(R_s - R_{sr}^T + R_r - R_{sr})C_m = 0 \tag{7.5}$$

$$\mathcal{L}_{Q_{12}}(\cdot) = A_s^TP_{12} + P_{12}A_m - \lambda C_s^T(R_s - R_{sr}^T)C_m = 0 \tag{7.6}$$

$$\mathcal{L}_{Q_{23}}(\cdot) = A_m^TP_{23} + P_{23}A_r - \lambda C_m^T(R_r - R_{sr})C_r = 0 \tag{7.7}$$

$$\mathcal{L}_{Q_{11}}(\cdot) = A_s^TP_{11} + P_{11}A_s + \lambda C_s^TR_sC_s = 0 \tag{7.8}$$

$$\mathcal{L}_{Q_{13}}(\cdot) = A_s^TP_{13} + P_{13}A_r - \lambda C_s^TR_{sr}C_r = 0 \tag{7.9}$$

$$\mathcal{L}_{Q_{33}}(\cdot) = A_r^T P_{33} + P_{33} A_r - \lambda C_r^T R_r C_r = 0 \tag{7.10}$$

$$\mathcal{L}_{P_{22}}(\cdot) = A_m Q_{22} + Q_{22} A_m^T + B_m V_m B_m^T = 0 \tag{7.11}$$

$$\mathcal{L}_{P_{12}}(\cdot) = A_s Q_{12} + Q_{12} A_m^T + B_s V_m B_m^T = 0 \tag{7.12}$$

$$\mathcal{L}_{P_{23}}(\cdot) = A_m Q_{23} + Q_{23} A_r^T + B_m V_m B_r^T = 0 \tag{7.13}$$

$$\mathcal{L}_{P_{11}}(\cdot) = A_s Q_{11} + Q_{11} A_s^T + B_s V_m B_s^T = 0 \tag{7.14}$$

$$\mathcal{L}_{P_{13}}(\cdot) = A_s Q_{13} + Q_{13} A_r^T + B_s V_m B_r^T = 0 \tag{7.15}$$

$$\mathcal{L}_{P_{33}}(\cdot) = A_r Q_{33} + Q_{33} A_r^T + B_r V_m B_r^T = 0 \tag{7.16}$$

It is seen from (7.8), (7.5) and (7.10) that A_s, A_m and A_r are required to be stable matrices, if $\lambda = 0$, then P_{11}, P_{22} and P_{33} are also zero matrices. This, together with the Lagrange multipliers' rule and the argument made earlier ($\tilde{P}$ should be a positive definite) makes that λ cannot be assigned to zero. Let $\lambda = 1$. The positive definiteness of P_{11}, P_{22} and P_{33} can also be seen from the earlier mentioned Lyapunov equations, hence they are invertible. Referring to (7.14), (7.11) and (7.16), Q_{11}, Q_{22} and Q_{33} are also invertible.

Two $(n+r) \times n$ matrices $P^* = \begin{bmatrix} P_{12} \\ P_{23}^T \end{bmatrix}$ and $Q^* = \begin{bmatrix} Q_{12} \\ Q_{23}^T \end{bmatrix}$ are defined. It is seen from (7.2) that $\Gamma_2 G_2^T = I_n$, where $\Gamma_2 = P_{22}^{-1} P^{*T} \in \mathbb{R}^{n \times (n+r)}$ and $G_2^T = -Q^* Q_{22}^{-1} \in \mathbb{R}^{(n+r) \times n}$. It follows from the lemma 2.1 and the definition 2.1 that the optimal projector is $\sigma_2 = G_2^T \Gamma_2 \in \mathbb{R}^{(n+r)\times(n+r)}$. The equality $G_2^T M_2 \Gamma_2 = \bar{Q}\bar{P}$ is formed in accordance with the lemma 2.1, where the positive semisimple $M_2 = -Q^{*T} P^* = Q_{22} P_{22}$ and the two nonnegative definite matrices are assigned as $\bar{Q} = Q^* Q_{22}^{-1} Q^{*T}$, $\bar{P} = P^* P_{22}^{-1} P^{*T}$. Further, it follows from the standard Sylvester's inequality that $\varrho(\bar{Q}) = \varrho(Q_{23}) + \varrho(Q_{23}) = (n+r)$ and $\varrho(\bar{P}) = \varrho(P_{23}) + \varrho(P_{23}) = (n+r)$ which are their dimensions, hence $\bar{Q}$ and $\bar{P}$ are positive definite.

Referring to the defined matrices Q^*, P^*, proposition 2.1, (7.2) and (7.4) and using $(7.5) \times Q_{22} + (7.6) \times Q_{12} + (7.7) \times Q_{23}^T$, (5.1) is derived. (5.2) and (5.3) are obtained from (7.3) and (7.4) respectively. (5.1) is also derived using $P_{22} \times (7.11) + P_{12}^T \times (7.12) + P_{23} \times (7.11)^T$.

The first two rank conditions of (5.4) are already proved for $\bar{Q}$ and $\bar{P}$ to be positive definite. The third part follows also from the standard Sylvester's inequality.

Referring to corollary 5.1 and 5.2 and proposition 2.1 and using $((7.12) + (7.13)) \times G_2$, (5.5) is obtained. (5.5) is also derived from (7.11) with reference to proposition 2.1.

Similarly, with reference to proposition 2.1, (5.6) is obtained either using (7.4) or using $((7.6) + (7.7)) \times \Gamma_2$ with the corollaries 5.1 and 5.3.

It can be seen that (7.8), (7.9), (7.10), (7.14), (7.15) and (7.16) do not intervene in the proof of theorem 5.1, however, and are used for theorem 5.2, hence there is no overdetermination.

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