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A noisy duel with two kinds of weapons

by

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Part I

Noisy duel is considered in which Player I has two kinds of weapons: a gun with m bullets and a weapon which he can use when he meets the opponent. Player II has a gun with n bullets. Cases solved are: $m \in N$, $n = 1$; $m = 0$, $n \in N$, $N = \{1, 2, \ldots\}$; and $m \leq 20$, $n \leq 5$.

In this part of the paper the cases $m \in N$, $n = 1$ and $m = 0$, $n \in N$ are solved.

Keywords: noisy duel, game of timing, zero-sum game.

1. Definitions and assumptions

Consider the game which will be called the game (m, n). Two players: Player I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$.

Player I has two kinds of weapons: a gun with n bullets and a weapon which he can use when the distance between him and the opponent is zero. Player II has only a gun with n bullets.

At the beginning of the duel the players are at distance 1 from each other. Let $P(s)$ be probability of succeeding (destroying the opponent) by Player I (II) when the distance between them is $1 - s$. The function $P(s)$ will be called accuracy function. It is assumed that it is increasing and continuous in $[0, 1]$, has continuous second derivative in $(0, 1)$ and that $P(s) = 0$ for $s \leq 0$, $P(1) = 1$.

It is assumed also that at $s = 1$, Player I succeeds surely by his short distance weapon.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero sum game.

It is assumed that duel is noisy – each player hears every shot of his opponent.

As it will be seen from the sequel, we can suppose without loss of generality that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

For definitions and results in the theory of games of timing see Karlin S. (1959), Kimeldorf G. (1983), Restrepo R. (1957), Styszyński A. (1974), Yanovskaya E.B. (1969).

2. Duel $(0,1)$, $< a >$

Let us suppose that Player I has no bullets and Player II has one bullet. Let the distance between players at the beginning of the duel be $1-a$. Denote by $K(\hat{\xi},\hat{\eta})$ the expected gain for Player I if he applies strategy $\hat{\xi}$ and Player II applies strategy $\hat{\eta}$. We will consider two cases.

Case 1

Let $P(a) \le 1/2$. We define strategies ξ and η of Players I and II.
Strategy of Player I: Do not go beside the point s_0 such that $P(s_0) = 1/2$.
Strategy of Player II: If Player I reaches the point s_0 fire at $< s_0 >$. Otherwise do not fire.

The notation $< s >$ means the first time when Player I reaches the point s.

In the sequel we denote by $\hat{\xi}$ and $\hat{\eta}$ nonrandom strategies of Players I, II used against the strategies ξ and η, respectively.

Suppose that Player I applies the strategy ξ and that Player II fires at a' (strategy $\hat{\eta}$). Since if Player II fires and misses Player I surely succeeds we obtain

$$K(\xi,\hat{\eta}) \ge -P(a') + 1 - P(a') \ge 0$$

if $P(a') \le 1/2$ what always holds when Player I applies the strategy ξ.

If Player II does not fire we obtain

$$K(\xi,\hat{\eta}) = 0.$$

Then

$$K(\xi,\check{\eta}) \ge 0 \tag{1}$$

for any strategy $\check{\eta}$ of Player II.

On the other hand, suppose that Player I does not reach the point s_0. For such a strategy $\hat{\xi}$ we obtain

$$K(\hat{\xi},\eta) = 0.$$

If Player I attains point s_0 we have

$$K(\hat{\xi}, \eta) \leq -P(s_0) + 1 - P(s_0) = 0.$$

Then

$$K(\check{\xi}, \eta) \leq 0 \tag{2}$$

for any strategy $\check{\xi}$ of Player I.

From inequalities (1) and (2) it follows that strategies ξ and η are optimal (i.e. ξ is maximin and η is minimax) and the value of the game is

$$v_{01}(a) = 0. \tag{3}$$

Case 2

Let $P(a) \geq 1/2$. We define the strategies ξ and η of Players I and II.
Strategy of Player I: Escape to 0. If Player II has fired reach the point 1 and succeed surely.
Strategy of Player II: Fire at $< a >$.

We prove that strategies ξ and η are optimal.

Suppose that Player II fires at a'. We obtain

$$K(\xi, \hat{\eta}) \geq -P(a') + 1 - P(a') \geq 1 - 2P(a).$$

Suppose that Player II does not fire. For such a strategy $\hat{\eta}$

$$K(\xi, \hat{\eta}) = 0 \geq 1 - 2P(a)$$

if $P(a) \geq 1/2$.

On the other hand, for any strategy $\hat{\xi}$ of Player I

$$K(\hat{\xi}, \eta) \leq -P(a) + 1 - P(a) = 1 - 2P(a).$$

Then, strategies ξ and η are optimal and the value of the game is

$$v_{01}(a) = 1 - 2P(a). \tag{4}$$

3. Further definitions and assumptions

Suppose that the duel (m, n) begins when the distance between Players is $1 - a$. This duel will be denoted by $(m, n), < a >$.

In further part of the paper (and in the forthcoming paper Trybuła (1995)) we shall assume that between successive shots of the same player the time $\hat{\varepsilon}$ has to pass.

We assume that each strategy $\xi(\hat{\varepsilon})$ of Player I is defined for any $\hat{\varepsilon}, \hat{\varepsilon}_0 \geq \hat{\varepsilon} > 0$.

We say that Player I assures in the limit the value u_1 if for each $\hat{\varepsilon}, \hat{\varepsilon} \geq \hat{\varepsilon} > 0$, he has a strategy $\xi(\hat{\varepsilon})$ such that

$$K(\xi(\hat{\varepsilon}), (\hat{\eta})) \geq u_1 - k_1(\hat{\varepsilon}) \tag{5}$$

for any strategy $\eta(\hat{\varepsilon})$ of Player II, where $k_1(\hat{\varepsilon})$ is a function tending to 0 if $\hat{\varepsilon} \to 0$.

Similarly, Player II assures in the limit the value u_2 if for each $\hat{\varepsilon}$, $\hat{\varepsilon}_0 \geq \hat{\varepsilon} > 0$ he has a strategy $\eta(\hat{\varepsilon})$ such that

$$K(\hat{\xi}, (\hat{\varepsilon}), \eta(\hat{\varepsilon})) \leq u_2 - k_2(\hat{\varepsilon}) \tag{6}$$

for any strategy $\xi(\hat{\varepsilon})$ of Player I, where $k_2(\hat{\varepsilon}) \to 0$ if $\hat{\varepsilon} \to 0$.

If strategies $\xi(\hat{\varepsilon}), \eta(\hat{\varepsilon})$ assure in the limit the same value $v_{mn}(a)$ then $\xi(\hat{\varepsilon}), \eta(\hat{\varepsilon})$ are called optimal in the limit and the value $v_{mn}(a)$ is called the limit value of the game.

Assume that Player I assures in the limit the value $v_{mn}(a) - \varepsilon_1$ and Player II assures in the limit the value $v_{mn}(a) + \varepsilon_2$ for any $\varepsilon_1 > 0, \varepsilon_2 > 0$ and some $v_{mn}(a)$. The number $v_{mn}(a)$ will be called the limit ε–value of the game $(m, n), < a >$.

The strategy $\xi_\varepsilon(\hat{\varepsilon})$ is called ε–optimal in the limit if

$$K(\xi_\varepsilon(\hat{\varepsilon}), \hat{\eta}(\hat{\varepsilon})) \geq v_{mn}(a) - k_1(\hat{\varepsilon}) - \varepsilon$$

for any $\hat{\varepsilon}, \hat{\varepsilon}_0 \geq \hat{\varepsilon} > 0$ and any strategy $\eta(\hat{\varepsilon})$ of Player II, where $k_1(\hat{\varepsilon}) \to 0$ if $\hat{\varepsilon} \to 0$.

Similarly we define the ε–optimal in the limit strategy of Player II.

We shall consider a family of strategies such that for each $\varepsilon > 0$ there is a strategy $\xi_\varepsilon(\hat{\varepsilon})$ belonging to this family, optimal in the limit. If Player I has at his disposal such family of strategies he has a strategy $\xi(\hat{\varepsilon})$ optimal in the limit.

Similar corollary is true also for Player II.

4. Duel $(0, n), < a >$

Let us consider the duel in which Player I has no bullets, Player II has n bullets, $n \geq 2$, and the game is beginning when Player I is at the point a.

Let $> t <$ denote the point at which Player I is at time t.

We define the strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player I has fired – play optimally the resulting duel.

Strategy of Player II: Fire at $< a >$ and play optimally the resulting duel.

"Play optimally" means apply a strategy which is optimal in the limit.

For above strategies ξ and η we have

$$K(\xi, \eta) = -P(a) + Q(a) v_{0,n-1}(>< a > + \hat{\varepsilon} <)$$

where $Q(a) = 1 - P(a)$.

We prove that strategies ξ and η are optimal in the limit and that the limit value of the game $(0, n), < a >$ is

$$v_{0,n}(a) = \begin{cases} -1 + Q^{n-1}(a) & \text{if } P(a) \leq 1/2, \\ -1 + 2Q^n(a) & \text{if } P(a) \geq 1/2. \end{cases} \tag{7}$$

From (3) and (4) it follows that the formula (7) holds for $n = 1$. Suppose that it holds for $n = 1, \ldots, k-1$. Suppose at the moment that strategies ξ and η are optimal in the limit. Then

$$v_{0k}(a) = -P(a) + Q(a)v_{0,k-1}(a)$$

and the formula (7) holds for $n = k$.

Then it is sufficient to prove that strategies ξ and η are optimal in the limit. Suppose that Player II fires at a'. We obtain

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + Q(a')v_{0,n-1}(a') - k(\hat{\varepsilon}) \\ &= v_{0n}(a') - k(\hat{\varepsilon}) \quad \geq \quad v_{0n}(a) - k(\hat{\varepsilon}) \\ &= \begin{cases} -1 + Q^{n-1}(a) - k(\hat{\varepsilon}) & \text{if } P(a) \leq 1/2, \\ -1 + 2Q^{n}(a) - k(\hat{\varepsilon}) & \text{if } P(a) \geq 1/2. \end{cases} \end{aligned}$$

Suppose that Player II does not fire. For such strategy $\hat{\eta}$

$$K(\xi, \hat{\eta}) = 0 \geq v_{0n}(a).$$

Then, Player I, by applying ξ, assures in the limit the value $v_{0n}(a)$ given by (7).

On the other hand, since Player II fires at $< a >$ we have for any strategy $\hat{\xi}$ of Player I

$$K(\hat{\xi}, \eta) \leq -P(a) + Q(a)v_{0,n-1}(a) + k(\hat{\xi}) = v_{0n}(a) + k(\hat{\varepsilon}),$$

where $v_{0n}(a)$ is given by (7). Then Player II, applying η, also assures in the limit the value $v_{0n}(a)$ which ends the proof of the assertion.

5. Duel $(1,1), < a >$

Let us consider the duel in which the players have one bullet each and the game is beginning when Player I is at the point a.

Case 1

Let $P(a) \leq 1/2$. We define strategies ξ and η of Players I and II.

Strategy of Player I: Go ahead and if Player I did not fire before fire at $< a_{11} >$, $P(a_{11}) = 1/3$, and if Player II did not fire still, do not go beside the point $s_0, P(s_0) = 1/2$. If Player II has fired (at any point) go to the point 1 to succeed surely.

Strategy of Player II: If Player I reaches the point a_{11} not having fired, fire with an absolutely continuous probability distribution (ACPD) in the interval $(< a_{11} >, < a_{11} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If Player I has fired, fire only when Player I reaches the point s_0. If Player I did not reach the point a_{11} and did not fire, do not fire neither.

The ACPD is chosen to make the strategy η ε-optimal in the limit.

Let $v_{11} = 1/3$. Suppose that Player II fires at point $a' < a_{11}$. We obtain

$$K(\xi, \hat{\eta}) \geq -P(a') + 1 - P(a') \geq 1 - 2P(a_{11}) = v_{11}.$$

Suppose that Player II fires at $a' = a_{11}$ together with Player I. We have

$$K(\xi, \hat{\eta}) \geq (1 - P(a_{11}))^2 \geq v_{11}.$$

Suppose that Player II does not fire, when Player I is before or at the point a_{11}. We have for such a strategy $\hat{\eta}$ of Player II

$$K(\xi, \hat{\eta}) \geq P(a_{11}) - (1 - P(a_{11}))v_{01}(a_{11}) = P(a_{11}) = v_{11}.$$

Then

$$K(\xi, \hat{\eta}) \geq v_{11} \tag{8}$$

for any strategy $\hat{\eta}$ of Player II.

On the other hand, suppose that Player I had fired at point a' before he reached the point a_{11}. We have

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq P(a') + Q(a')v_{01}(a') + k(\hat{\varepsilon}) = P(a') + k(\hat{\varepsilon}) \\ &\leq P(a_{11}) + k(\hat{\varepsilon}) = v_{11} + k(\hat{\varepsilon}). \end{aligned}$$

Suppose that Player I does not fire before $< a_{11} > +\alpha(\varepsilon)$. For such a strategy $\hat{\xi}$

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq -P(a_{11}) + Q(a_{11})v_{10}(a_{11}) + \varepsilon + k_1(\hat{\varepsilon}) \\ &= 1 - 2P(a_{11}) + k(\hat{\varepsilon}) = v_{11} + k(\hat{\varepsilon}) \end{aligned}$$

for properly chosen ACPD of Player II.

Then

$$K(\hat{\xi}, \eta) \leq v_{11} + k(\hat{\varepsilon}) \tag{9}$$

for any strategy $\hat{\xi}$ of Player I.

From (8) and (9) it follows that strategies ξ and η are optimal in the limit and

$$v_{11}(a) = v_{11} = 1/3$$

is the limit value of the game for $P(a) \leq 1/3$.

Case 2

Let $1/3 \leq P(a) \leq \frac{1}{2}(3-\sqrt{5}) \cong 0.38197$. We define strategies ξ and η of Players I and II.
Strategy of Player I: Fire at $< a >$ and play optimally the resulting duel.
Strategy of Player II: If Player I did not fire before fire with an ACPD in the interval $(< a >, < a > +\alpha(\varepsilon))$. If Player I has fired play optimally the resulting duel.

Suppose that Player II fires at $< a >$. For such strategy $\hat{\eta}$

$$K(\xi, \hat{\eta}) \geq (1 - P(a)))^2 \geq P(a)$$

when

$$P(a) \leq \frac{1}{2}(3 - \sqrt{5}).$$

If Player II does not fire at $< a >$

$$K(\xi, \hat{\eta}) \geq P(a) + Q(a)v_{01}(a) - k(\hat{\varepsilon}) = P(a) - k(\hat{\xi})$$

for $P(a) \leq 1/2$.

On the other hand, if Player I fires at $< a >$

$$K(\hat{\xi}, \eta) \leq P(a) + Q(a)v_{01}(a) + k(\hat{\varepsilon}) = P(a) + k(\hat{\varepsilon})$$

if $P(a) \leq 1/2$.

If Player I does not fire before $< a > +\alpha(\varepsilon)$

$$K(\hat{\xi}, \eta) \leq -P(a) + 1 - P(a) + k(\hat{\varepsilon}) \leq P(a) + k(\hat{\varepsilon})$$

if $P(a) \geq 1/3$.

Then for

$$1/3 \leq P(a) \leq \frac{1}{2}(3 - \sqrt{5})$$

strategies ξ and η are optimal in limit and the limit value of the game is

$$v_{11}(a) = P(a).$$

Case 3

Let $P(a) \geq \frac{1}{2}(3-\sqrt{5})$. We define strategies ξ and η of Players I and II.
Strategy of Player I: Fire at $< a >$ and play optimally the resulting duel.
Strategy of Player II: Fire at $< a >$ and play optimally the resulting duel.

Strategies ξ and η are optimal in limit and the limit value of the game is

$$v_{11}(a) = Q^2(a).$$

The proof is omitted.

6. Duel $(m,1)$

At the end we consider the duel in which Player I has m bullets, $m = 2, 3, \ldots$, Player II has one bullet and the game is beginning when Player I is before the point a_{m1} (see (13)). Such a game we shall denote by $(m,1)$. We define the strategies ξ and η of Players I and II.
Strategy of Player I: Go ahead and if Player I did not fire before, fire at $< a_{m1} >$ and play optimally the resulting duel. If Player II has fired go to the point 1 and succeed surely.
Strategy of Player II: If Player I has reached the point a_{m1} and did not fire before, fire with an ACPD in the interval $(< a_{m1} >, < a_{m1} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If Player I had fired before he reached the point a_{m1} play optimally the duel $(m-1,1)$. If he did not reach the point a_{m1} and did not fire, do not fire neither.

Let the numbers v_{m1}, a_{m1} be the solution of the equations

$$v_{m1} = P(a_{m1}) + 1 - P(a_{m1})v_{m-1,1} = 1 - 2P(a_{m1}),$$

$P(a_{11}) = 1/3, v_{11} = 1/3$. We obtain

$$P(a_{m1}) = \frac{1 - v_{m-1,1}}{3 - v_{m-1,1}}, \tag{10}$$

$$v_{m1} = \frac{1 + v_{m-1,1}}{3 - v_{m-1,1}}. \tag{11}$$

The solution of this equation is

$$v_{m1} = \frac{m}{m+2}, \tag{12}$$

$$P(a_{m1}) = \frac{1}{m+2}. \tag{13}$$

We prove that for these a_{m1} strategies ξ and η are optimal in the limit in the duel $(m,1)$ and the limit value of the game is given by (12).

Suppose that Player II fired at $a' < a_{m1}$. We have

$$K(\xi, \hat{\eta}) \geq -P(a') + 1 - P(a') \geq 1 - 2P(a_{m1}) = v_{m1}.$$

Suppose that Player II fired at $< a_{m1} >$ together with Player I. We obtain for such a strategy $\hat{\eta}$

$$K(\xi, \hat{\eta}) \geq (1 - P(a_{m1}))^2 > 1 - 2P(a_{m1}) = v_{m1}.$$

Suppose that Player II does not fire before or at $< a_{m1} >$ if Player I did not fire up to this time. We have

$$K(\xi, \hat{\eta}) \geq P(a_{m1}) + (1 - P(a_{m1}))v_{m-1,1} - k(\hat{\varepsilon}) = v_{m1} - k(\hat{\varepsilon}).$$

On the other hand, if Player I had fired at a' before he reached the point a_{m1}

$$\begin{aligned} K(\hat{\xi},\eta) &\le P(a') + (1-P(a'))v_{m-1,1} + k(\hat{\varepsilon}) \\ &\le P(a_{m1}) + (1-P(a_{m1}))v_{m-1,1} + k(\hat{\varepsilon}) \\ &= v_{m1} + k(\hat{\varepsilon}). \end{aligned}$$

At the end, if Player I does not fire before $< a_{11} > +\alpha(\varepsilon)$ we have

$$K(\hat{\xi},\eta) \le -P(a_{11}) + 1 - P(a_{11}) + k(\hat{\varepsilon}) = v_{m1} + k(\hat{\varepsilon})$$

for such a strategy $\hat{\xi}$.

This ends the proof of limit optimality of strategies ξ and η.

7. Results for the solved duels

$$v_{0n}(a) = \begin{cases} -1 + Q^{n-1}(a) & \text{if } P(a) \le 1/2, \\ -1 + 2Q^n(a) & \text{if } P(a) \ge 1/2 \end{cases}$$

$(n = 1, 2, \ldots)$.

$$v_{11}(a) = \begin{cases} 1/3 & \text{if } P(a) \le 1/3, \\ P(a) & \text{if } 1/3 \le P(a) \le \frac{1}{2}(3-\sqrt{5}), \\ Q^2(a) & \text{if } P(a) \ge \frac{1}{2}(3-\sqrt{5}). \end{cases}$$

$$v_{m1}(a) = \frac{m}{m+2} \text{ if } P(a) \le \tfrac{1}{m+2}$$

$(m = 1, 2, \ldots)$.

For the duels with retreat after firing his shots see Trybuła S. (1990).

For the duels with arbitrary moving see Trybuła S. (to be published).

For noisy duels see Fox M., Kimeldorf G. (1969), Radzik T. (1988), Teraoka Y. (1976), Trybuła S. (to be published), Vorob'ev N.N. (1984).

For other duels see Cegielski A. (1986a), Cegielski A. (1986b), Orłowski K., Radzik T. (1985a), Orłowski K., Radzik T. (1985b), Radzik T. (1988), Teraoka Y. (1979), Vorob'ev N.N. (1984).

Part II

A noisy duel is considered in which Player I has two kinds of weapons: a gun with m bullets and a weapon which he can use when he meets the oponent. Player II has a gun with n bullets. Players I and II can move as they want. Player I has greater maximal speed. Solved cases are: $n = 1$ for any $m, m = 0$ for any n, and $m \leq 20, n \leq 5$.

In this part the case of $n = 2, 3, m \leq n$ is solved.

1. Assumptions

Consider the game which will be called the game (m, n). It is assumed that Player I may use and succeeds surely by his second weapon only when the distance between him and his opponent is 0.

Player I gains 1 if he only succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero–sum game.

It is assumed that duel is noisy – each player hears every shot of his opponent.

As it will be seen from the sequel, we can assume without loss of generality that the maximal speed of Player I is 1 and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

Assume now that the duel (m, n) begins when the distance between players is $1 - a$. This duel will be denoted by $(m, n), < a >$.

As before between successive shots of the same player the time $\hat{\varepsilon} > 0$ has to pass.

Let (m, n), $< a \wedge c, a >$, $0 < c \leq \hat{\varepsilon}$, be the duel in which Player I has m bullets, Player II has n bullets, Player I is at the beginning of the duel at the point a, Player II at 1, but if $c < \hat{\varepsilon}$ Player II can fire his bullets beginning from the moment $< a >$ and Player I from the moment $< a > +c$; $< a >$ is the first time when Player I is at a. If $c = \hat{\varepsilon}$ the rule is the same with the only difference that Player II is not allowed to fire at $< a >$.

Similarly we define the duel (m, n), $< a, a \wedge c >$.

All other definitions and assumptions made for the duel (m, n) hold for the above two duels.

If in the duel (m, n),$< a >$ Player I fires as the first the bullet at the point a' and misses and Player II does not fire at the same time, the duel (m, n), $< a \wedge c, a >$ reduces to the game $(m - 1, n), < a' \wedge \hat{\varepsilon}, a' >$.

Moreover, if in the duel (m, n), $< a \wedge c, a >$ Player I fires as the first at the time $t \geq < a > +c$ and misses, $t = \dot{a}'$, and Player II did not fire at this moment then the duel (m, n), $< a \wedge c, a >$ reduces to the game $(m - 1, n)$, $< a' \wedge \hat{\varepsilon}, a' >$.

The sign $\dot{a}'$ denotes the time when Player I is at the point a' (not necessarily the earliest one which is denoted by $< a' >$).

If in the duel $(m, n), < a \wedge c, a >$ Player II fires first and misses, then: If in the duel $(m, n), < a \wedge c, a >$ Player II fires first and misses, then:

i) If he fires at $t < <a> +c$, $t = \dot{a}'$, the duel considered reduces to the game $(m, n-1)$, $< a', a' \wedge c_1 >$ for some $c_1, c_1 < \hat{\varepsilon}$.

ii) If he fires at $t \geq < a > +c$, the duel considered reduces to the game $(m, n-1)$,$< a', a' \wedge \hat{\varepsilon} >$ where $\dot{a}' = t$.

Similar situations arise when players fire at the same time.

Then each of the duels $(m,n), < a >; (m,n), < a \wedge c, a >$; $(m,n), < a, a \wedge c >$ reduces after a shot of a player to the duel of the above three kinds.

The same conclusion holds when players fire simultaneously.

For definitions and notions concerning duels see Berzin E.A. (1983), Karlin S. (1959), Restrepo R. (1957).

2. Duel $(1,2)$

Let us consider the duel in which Player I has 1 bullet and Player II has 2 bullets. The duel begins when Player I is at the point 0 and Player II is at the point 1. We define the following strategies ξ and η of Players I and II.
Strategy of Player I: Go ahead and if Player II did not fire before, fire with an absolute continuous probability distribution (ACPD) in the interval $(< a_{12} >, < a_{12} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If Player II has fired before play optimally the resulting duel $(1,1)$.
Strategy of Player II: If Player I reaches the point a_{12} and did not fire before, fire at $< a_{12} >$ and play optimally the duel $(1,1)$ or $(0,1)$. If he has fired before play optimally the resulting duel.

$$P(a_{12}) = \frac{\sqrt{7}-2}{3} = 0.21525. \tag{14}$$

"Play optimally" means: apply a strategy optimal in the limit.

Let v_{12} be a number satisfying the equation

$$v_{12} = P^2(a_{12}) = -P(a_{12}) + (1 - P(a_{12}))v_{11}$$

where (see part I) $v_{11} = 1/3$. Solving these equations we obtain (14) and

$$v_{12} = \frac{11 - 4\sqrt{7}}{9} = 0.04633. \tag{15}$$

Suppose that Player II has fired at $a' < a_{12}$. We obtain for such a strategy $\hat{\eta}$

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + (1 - P(a'))v_{11} - k(\hat{\varepsilon}) \\ &\geq -P(a_{12}) + (1 - P(a_{12}))v_{11} - k(\hat{\varepsilon}) \\ &= v_{12} - k(\hat{\varepsilon}) \end{aligned}$$

where $k(\hat{\varepsilon}) \to 0$ if $\hat{\varepsilon} \to 0$.

When $< a' > > < a_{12} > +\alpha(\varepsilon)$ we obtain for properly chosen $\alpha(\varepsilon)$

$$\begin{aligned} K(\xi, \tilde{\eta}) &\geq P(a_{12}) + (1 - P(a_{12}))v_{02}(a_{12}) - k_1(\hat{\varepsilon}) - \varepsilon \\ &= P(a_{12}) - (1 - P(a_{12}))P(a_{12}) - k(\hat{\varepsilon}) \\ &= P^2(a_{12}) - k(\hat{\varepsilon}) \quad = \quad v_{12} - k(\hat{\varepsilon}). \end{aligned}$$

For $v_{02}(a_{12})$ see part I, Section 6. Here also $k(\hat{\varepsilon}) \to 0$, $k_1(\hat{\varepsilon}) \to 0$ if $\hat{\varepsilon} \to 0$.
Then

$$K(\xi, \check{\eta}) \geq v_{12} - k(\hat{\varepsilon}) \tag{16}$$

for any strategy $\check{\eta}$ of Player II.
On the other hand, if Player I has fired at $a' < a_{12}$ (strategy $\hat{\xi}$) we have

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq P(a') - (1 - P(a'))v_{02}(a') + k(\hat{\varepsilon}) \\ &\leq P(a') + (1 - P(a'))P(a') + k(\hat{\varepsilon}) \\ &= P^2(a') + k(\hat{\varepsilon}) \quad \leq \quad P^2(a_{12}) + k(\hat{\varepsilon}) \\ &= v_{12} + k(\hat{\varepsilon}). \end{aligned}$$

If Player I fires at $< a_{12} >$ together with Player II we obtain

$$K(\hat{\xi}, \eta) \leq (1 - P(a_{12}))^2 v_{01}(a_{12}) + k(\hat{\varepsilon}) = k(\hat{\varepsilon}) < v_{12} + k(\hat{\varepsilon})$$

for such a strategy $\hat{\xi}$, since $v_{01}(a) = 0$ for $P(a) \leq 1/2$.
When Player I does not fire before or at $< a_{12} >$

$$K(\hat{\xi}, \eta) \leq -P(a_{12}) + (1 - P(a_{12}))v_{11} + k(\hat{\varepsilon}) = v_{12} + k(\hat{\varepsilon}).$$

Then

$$K(\check{\xi}, \eta) \leq v_{12} + k(\hat{\varepsilon}) \tag{17}$$

for any strategy $\check{\xi}$ of Player I.
From equations (16) and (17) it follows that strategies ξ and η are optimal in the limit and that limit value of the game is given by (15).

3. Duel $(2, 2)$

We define the strategies ξ and η of Players I and II.
Strategy of Player I: Go ahead and if Player II did not fire before, fire with an ACPD in the interval $(< a_{22} >, < a_{22} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If Player II has fired, play optimally the duel $(2, 1)$.
Strategy of Player II: If Player I reached the point a_{22} and did not fire, fire at $< a_{22} >$ and play optimally the resulting duel $(2, 1)$ or $(1, 1)$, if he fired – play optimally the duel $(1, 2)$.

The number a_{22} and the limit value of the game v_{22} is obtained from the equations

$$\begin{aligned} v_{22} &= P(a_{22}) + (1 - P(a_{22}))v_{12} \\ &= -P(a_{22}) + (1 - P(a_{22}))v_{21}. \end{aligned}$$

We have then

$$\begin{aligned} P(a_{22}) &= \frac{v_{21} - v_{12}}{2 + v_{21} - v_{12}}, \\ v_{22} &= (1 - v_{12})P(a_{22}) + v_{12}. \end{aligned} \tag{18}$$

Since $v_{21} = 1/2$ (see part I), v_{12} is given by (15), we obtain

$$P(a_{22}) \cong 0.18489, \quad v_{22} \cong 0.22266. \tag{19}$$

Proof of limit optimality of strategies ξ and η is omitted.

4. Duels $(1,3)$

Duel $(1,3), < a >$.

Case 1. $Q(a) \geq Q(\hat{a}_{13}) \cong 0.95572$.

We define the strategies ξ and η of Players I and II.
Strategy of Player I: Escape. If Player II has fired play optimally the duel $(1,2)$.
Strategy of Player II: If Player I escapes and does not fire, do not fire neither. If Player I approaches Player II and does not fire, fire when he is at $\hat{a}_{13}, Q(\hat{a}_{13}) = 1/(1 + v_{12}) \cong 0.95572$. If Player I has fired at $a', a' << \hat{a}_{13} >$ play optimally the resulting duel.

We have

$$K(\xi, \eta) = 0.$$

Suppose that Player II fires at $a' \leq \hat{a}_{13}$. We have

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + (1 - P(a'))v_{12} - k(\hat{\varepsilon}) \\ &\geq -P(\hat{a}_{13}) + (1 - P(\hat{a}_{13}))v_{12} - k(\hat{\varepsilon}) \\ &= -k(\hat{\varepsilon}). \end{aligned}$$

If Player II does not fire

$$K(\xi, \hat{\eta}) = 0.$$

On the other hand, if Player I does not reach the point $\hat{a}_{13}$ and does not fire

$$K(\hat{\xi}, \eta) = 0.$$

If Player I does not fire and reaches the point $\hat{a}_{13}$

$$K(\hat{\xi}, \eta) \leq -P(\hat{a}_{13}) + (1 - P(\hat{a}_{13}))v_{12} + k(\hat{\varepsilon}) = k(\hat{\varepsilon}).$$

If Player I fires at $a' < \hat{a}_{13}$

$$\begin{aligned} K(\hat{\xi},\eta) &\le P(a') - (1-P(a'))(1-(1-P(a'))^2) + k(\hat{\varepsilon}) \\ &= 1 - 2Q(a') + Q^3(a') + k(\hat{\varepsilon}) \\ &\le k(\hat{\varepsilon}) \end{aligned}$$

for $Q(a') = 1 - P(a'), Q(a') \ge \frac{\sqrt{5}-1}{2}$.

If Player I fires at $< \hat{a}_{13} >$

$$K(\hat{\xi},\eta) \le -(1-P(\hat{a}_{13}))^2 P(\hat{a}_{13}) + k(\hat{\varepsilon}) \le k(\hat{\varepsilon}).$$

Then if $a \le \hat{a}_{13}$ strategies ξ and η are optimal in limit and $v_{13}(a) = 0$ is the limit value of the game.

Case 2. $0.85391 \cong Q(\check{a}_{13}) \le Q(a) \le Q(\hat{a}_{13}) \cong 0.95572$.

We define the strategies ξ and η of Players I and II.
Strategy of Player I: Escape. If Player II fires, play optimally the duel $(1,2)$.
Strategy of Player II: Fire at $< a >$ and play optimally the duel $(1,2)$. If Player I fired also at $< a >$ play optimally the resulting duel $(0,2)$.

Now

$$K(\xi,\eta) = -P(a) + (1-P(a))v_{12} + k(\hat{\varepsilon}).$$

Suppose that Player II fires at a'. In this case

$$\begin{aligned} K(\xi,\hat{\eta}) &\ge -P(a') + (1-P(a'))v_{12} - k(\hat{\varepsilon}) \\ &\ge -P(a) + (1-P(a))v_{12} - k(\hat{\varepsilon}). \end{aligned}$$

Suppose that Player II does not fire. For such a strategy $\hat{\eta}$

$$K(\xi,\hat{\eta}) = 0 \ge -P(a) + (1-P(a))v_{12}$$

if

$$Q(a) \le \frac{1}{1+v_{12}} = Q(\hat{a}_{13}).$$

On the other side, if Player I fires at $< a >$

$$\begin{aligned} K(\hat{\xi},\eta) &\le (1-P(a))^2 v_{02}(a) + k(\hat{\varepsilon}) \\ &= (1-P(a))^2 P(a) + k(\hat{\varepsilon}) \\ &\le -P(a) + (1-P(a))v_{12} + k(\hat{\varepsilon}) \end{aligned}$$

if

$$Q^3(a) - Q^2(a) + (1+v_{12})Q(a) + 1 \le 0 \tag{20}$$

which yields

$$Q(a) \geq Q(\breve{a}_{13}) \cong 0.85391$$

where $Q(\breve{a}_{13})$ is the root of multinomial (20).

If Player I does not fire at $< a >$ we have

$$K(\hat{\xi}, \eta) \leq -P(a) + (1 - P(a))v_{12} + k(\hat{\varepsilon}).$$

Then the strategies ξ and η are optimal in the limit and the limit value of the game is

$$v_{13}(a) = -P(a) + (1 - P(a))v_{12}.$$

Case 2a. $0.85391 \cong Q(\breve{a}_{13}) \leq Q(a) \leq Q(\bar{a}_{13}) \cong 0.88097$.

We define the strategies ξ and η of Players I and II.
Strategy of Player I: If Player II did not fire before, fire with an ACPD in the interval $(< a >, < a > + \alpha(\varepsilon))$ and escape. If he fired – play optimally the duel $(1, 2)$.
Strategy of Player II: Fire at $< a >$ and if Player I did not fire play optimally the duel $(1, 2)$. If he fired – play optimally the resulting duel.

Let $Q(\bar{a}_{13})$ be the root of the equation

$$Q^3(a) - (3 + v_{12})Q(a) + 2 = 0$$

($v_{12} \cong 0.04633$) and let $Q(\breve{a}_{13})$ be the root of the equation

$$Q^3(a) - Q^2(a) - (1 + v_{12})Q(a) + 1 = 0.$$

We have

$$Q(\bar{a}_{13}) \cong 0.88097, \quad Q(\breve{a}_{13}) \cong 0.85391.$$

If $\bar{a}_{13} \leq a \leq \breve{a}_{13}$ strategies ξ and η are optimal in the limit and the limit value of the game $(1, 3)$, $< a >$ is

$$v_{13}(a) = -P(a) + (1 - P(a))v_{12}.$$

The proof is omitted

Strategies presented in the cases 2 and 2a prove that Player I has sometimes more than one strategy optimal in the limit in the duel (m, n), $< a >$, when $m < n$.

Case 3. $0.78475 \cong Q(a_{12}) \leq Q(a) \leq Q(\breve{a}_{13}) \cong 0.85391$.

We define strategies ξ and η of Players I and II.
Strategy of Player I: Fire at $< a >$ and play optimally the resulting duel.
Strategy of Player II: Fire at $< a >$ and play optimally the resulting duel.
Now

$$K(\xi, \eta) = -(1 - P(a))^2 P(a) + k(\hat{\varepsilon}).$$

Suppose that Player II fires after $< a >$ or does not fire. For such a strategy $\hat{\eta}$ we obtain

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq P(a) - (1 - P(a))(1 - (1 - P(a))^2) - k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^3(a) - k(\hat{\varepsilon}) \\ &\geq -Q^2(a)(1 - Q(a)) - k(\hat{\varepsilon}). \end{aligned}$$

On the other hand, if Player I fires after $< a >$ or does not fire

$$K(\hat{\xi}, \eta) \leq -P(a) + (1 - P(a))v_{12} + k(\hat{\varepsilon}) \leq -Q^2(a)(1 - Q(a)) + k(\hat{\varepsilon})$$

if

$$Q^3(a) - Q^2(a) - (1 + v_{12})Q(a) + 1 \geq 0$$

i.e. if $Q(a) \leq Q(\breve{a}_{13})$. Taking into account the solution of the duel $(1, 2)$, we obtain that if $\breve{a}_{13} \leq a \leq a_{12}$ then strategies ξ and η are optimal in the limit and the limit value of the game is

$$v_{13}(a) = -Q^2(a)(1 - Q(a)).$$

Duel $(1, 3)$,$< a \wedge c, a >$.

Case 1. $Q(a) \geq Q(\hat{a}_{13}) \cong 0.95572$.

We define strategies ξ and η.
Strategy of Player I: Escape. If Player II has fired – play optimally the duel $(1, 2)$.
Strategy of Player II: If Player I escapes and does not fire, do not fire also. If Player I approaches Player II and does not fire, fire when he is at $\hat{a}_{13}$. If Player I has fired at a', $\dot{a}' << \hat{a}_{13} >$, play optimally the resulting duel.

The proof of limit optimality of strategies ξ and η is the same as in duel $(1, 3), < a >$, Case 1. The limit value of the game is

$$v_{13}(1, a) = 0.$$

Case 2. $0.78475 \cong Q(a_{12}) \leq Q(a) \leq Q(\hat{a}_{13}) \cong 0.95572$.

We define ξ and η.
Strategy of Player I: Escape. If Player II has fired play optimally the duel $(1,2)$.
Strategy of Player II: Fire at a' such that $< a > \leq \dot{a}' < \; < a > +c$ and play optimally the resulting duel.

Now

$$K(\xi, \eta) = -P(a) + (1 - P(a))v_{12} + k(\hat{\varepsilon}).$$

Suppose that Player II fires at a'. We have

$$\begin{aligned} K(\dot{\xi}, \hat{\eta}) &\geq -P(a') + (1 - P(a'))v_{12} - k(\hat{\varepsilon}) \\ &\geq -P(a) + (1 - P(a))v_{12} - k(\hat{\varepsilon}) \end{aligned}$$

On the other hand, for any strategy $\hat{\xi}$ of Player I

$$K(\hat{\xi}, \eta) \leq -P(a) + (1 - P(a))v_{12} + k(\hat{\varepsilon}).$$

Then strategies ξ and η are optimal in limit and

$$v_{13}(1, a) = -P(a) + (1 - P(a))v_{12}.$$

is the limit value of the game.

Duel $(1,3), < a, a \wedge c >$.

Case 1. $Q(a) \geq Q(\hat{a}_{13}) \cong 0.95572$.

For these a the strategies optimal in the limit are those defined in the duel $(1,3)$, $< a >$, Case 1, and

$$v_{13}(2, a) = 0.$$

Case 2. $0.88097 \cong Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}) \cong 0.95572$.

We define ξ and η.
Strategy of Player I: Escape. If Player II has fired – play optimally the duel $(1,2)$.
Strategy of Player II: If Player I did not fire before, fire at $< a > +c$ and play optimally the resulting duel.

We prove that for given a

$$v_{13}(2, a) = -P(a) + (1 - P(a))v_{12}.$$

The proof that Player I assures in the limit the value $-P(a) + (1 - P(a))v_{12}$ is the same as in the duel $(1,3), < a >$. On the other hand, if Player I fires at $< a > +c$ or after $< a > +c$ also the proof is the same. Assume then that Player I fires before $< a > +c$, at a'. We have

$$\begin{aligned} K(\hat{\xi},\eta) &\leq P(a) - Q(a)(1-Q^2(a)) + k(\hat{\varepsilon}) \\ &\leq -P(a) + Q(a)v_{12} + k(\hat{\varepsilon}) \end{aligned}$$

if

$$S(Q) = Q^3(a) - (3+v_{12})Q(a) + 2 \leq 0.$$

The function $S(Q)$ is decreasing and $S(Q) = 0$ for $Q = Q(\bar{a}_{13}) \cong 0.88097$ (see Case 2a). Then for $\hat{a}_{13} \leq a \leq \bar{a}_{13}$ strategies ξ and η are optimal in the limit and

$$v_{13}(2,a) = -P(a) + (1-P(a))v_{12}$$

is the limit value of the game.

5. Results for the duels $(1,3)$.

$$v_{13}(1,a) = \begin{cases} 0 \quad \text{if} \quad Q(a) \geq Q(\hat{a}_{13}) \cong 0.95572, \\ -P(a) + (1-P(a))v_{12} \\ \qquad \text{if} \quad 0.78475 \cong Q(a_{12}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

$$v_{13}(a) = \begin{cases} 0 \quad \text{if} \quad Q(a) \geq Q(\hat{a}_{13}), \\ -P(a) + (1-P(a))v_{12} \\ \qquad \text{if} \quad 0.85391 \cong Q(\breve{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}), \\ -Q^2(a)(1-Q(a)) \\ \qquad \text{if} \quad Q(a_{12}) \leq Q(a) \leq Q(\breve{a}_{13}). \end{cases}$$

$$v_{13}(2,a) = \begin{cases} 0 \quad \text{if} \quad Q(a) \geq Q(\hat{a}_{13}), \\ -P(a) + (1-P(a))v_{12} \\ \qquad \text{if} \quad 0.88097 \cong Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

6. Duels $(2,3)$ and $(3,3)$.

Let us define strategies ξ and η in the duel $(2,3)$, $< a >$, $a \leq a_{23}$, $Q(a_{23})$ is the root of the equation (22), $Q(a_{23}) \cong 0.86167$.
Strategy of Player I: Go ahead. If Player II did not fire before, fire with an ACPD in the interval $(< a_{23} >, < a_{23} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If he has fired – play optimally the duel $(2,2)$.
Strategy of Player II: If Player I did not fire before, fire at $< a_{23} >$ and play

optimally the duel $(2,2)$ or $(1,2)$. If he has fired (say at a') play optimally the duel $(1,3), < a' \wedge \hat{\varepsilon}, a' >$.

Let

$$\begin{aligned} v_{23} &= P(a_{23}) + (1 - P(a_{23}))v_{13}(1, a_{23}) \\ &= -P(a_{23}) + (1 - P(a_{23}))v_{22}. \end{aligned} \tag{21}$$

If

$$0.78474 \cong Q(a_{12}) \le Q(a_{23}) \le Q(\hat{a}_{13}) \cong 0.95572$$

then

$$v_{13}(1, a_{23}) = -P(a_{23}) + (1 - P(a_{23}))v_{22}.$$

Substituting it in the equation (21) we obtain

$$(1 + v_{12})Q^2(a_{23}) - (3 + v_{22})Q(a_{23}) + 2 = 0 \tag{22}$$

what gives

$$Q(a_{23}) \cong 0.86167.$$

To prove that strategies ξ and η are optimal in the limit and that $v_{23}(a) = v_{23}$ for $a \le a_{23}$ suppose that Player II fires when Player I is at $a' \le a_{23}$. For such a strategy of Player II (denote it by $\hat{\eta}$) we obtain

$$\begin{aligned} K(\xi, \hat{\eta}) &\ge -P(a') + (1 - P(a'))v_{22} - k(\hat{\varepsilon}) \\ &\ge -P(a_{23}) + (1 - P(a_{23}))v_{22} - k(\hat{\varepsilon}) \\ &= v_{23} - k(\hat{\varepsilon}). \end{aligned}$$

If Player II fires after $< a_{23} > +\alpha(\varepsilon)$ or does not fire

$$K(\xi, \hat{\eta}) \ge P(a_{23}) + (1 - P(a_{23}))v_{13}(1, a_{23}) - k(\hat{\varepsilon}) = v_{23} - k(\hat{\varepsilon}).$$

On the other hand, if $a' < a_{23}$ (strategy $\hat{\xi}$) we have

$$\begin{aligned} K(\hat{\xi}, \eta) &\le P(a') + (1 - P(a'))v_{13}(1, a') + k(\hat{\varepsilon}) \\ &= \begin{cases} 1 - Q(a') + k(\hat{\varepsilon}) & \text{if } a' \le \hat{a}_{13}, \\ 1 - 2Q(a') + (1 + v_{12})Q^2(a') + k(\hat{\varepsilon}) & \text{if } \hat{a}_{13} \le a' \le a_{23}. \end{cases} \end{aligned}$$

The function $K(\hat{\xi}, \eta)$ defined in the above is increasing in a'. Then

$$\begin{aligned} K(\hat{\xi}, \eta) &\le 1 - 2Q(a_{23}) + (1 + v_{12})Q^2(a_{23}) + k(\hat{\varepsilon}) \\ &= P(a_{23}) + (1 - P(a_{23}))v_{13}(1, a_{23}) + k(\hat{\varepsilon}) \\ &= v_{23} + k(\hat{\varepsilon}). \end{aligned}$$

If Player I fires at $< a_{23} >$ together with Player II

$$\begin{aligned} K(\hat{\xi}, \eta) &= (1 - P(a_{23}))v_{12} + k((\hat{\varepsilon}) \\ &\cong 0.03440 + k(\hat{\varepsilon}) \\ &< 0.05354 + k(\hat{\varepsilon}) \\ &\cong v_{23} + k(\hat{\varepsilon}). \end{aligned}$$

At the end, if Player I fires after $< a_{23} >$ or does not fire at all

$$K(\hat{\xi}, \eta) \leq -P(a_{23}) + (1 - P(a_{23}))v_{22} + k(\hat{\varepsilon}) = v_{23} + k(\hat{\varepsilon}).$$

Then, strategies ξ and η are optimal in the limit and the limit value of the game is v_{23} defined in (21).

Let us consider the duel $(3,3)$. Let strategies ξ and η be now defined as follows.
Strategy of Player I: Go ahead. If Player II did not fire before, fire with an ACPD in the interval $(< a_{33} >, < a_{33} > +\alpha(\varepsilon))$, a_{33} is given by (23), and play optimally the duel $(2,3)$. If he has fired, play optimally the duel $(3,2)$.
Strategy of Player II: If Player I did not fire before, fire at $< a_{33} >$ (i.e. when Player I reached a_{33} for the first time) and play optimally the duel $(3,2)$. If he has fired play optimally the duel $(2,3)$.

Now

$$\begin{aligned} v_{33} &= P(a_{33}) + (1 - P(a_{33}))v_{23} \\ &= -P(a_{33}) + (1 - P(a_{33}))v_{32} \end{aligned}$$

yielding

$$\begin{aligned} Q(a_{33}) &= \frac{2}{2 + v_{32} - v_{23}} &\cong 0.87241, \\ v_{33} &= -1 + (1 + v_{32})Q(a_{33}) &\cong 0.17430 \end{aligned} \tag{23}$$

since

$$v_{23} \cong 0.05354, \quad v_{32} \cong 0.34604$$

(see Trybuła (1995), part V, and equations (19) and (22).

Proof that strategies ξ and η are optimal in the limit and $v_{33}(a) = v_{33}$ for $a \leq a_{33}$, is omitted.

7. Final remarks

The duel solved in Yanovskaya E.B. (1969), part I, and in this paper will be used to consider similar duels (m, n) for higher m and n, recursively.

For the duels with retreat after firing his shots see Trybuła S. (1990).

Duels with arbitrary moving are considered in Trybuła S. (to be published), Trybuła S. (1993), Trybuła S. (1995).

For noisy duels see Fox M., Kimeldorf G. (1969), Kimeldorf G. (1983), Trybuła S. (to be published), Trybuła S. (1995).

For other duels see Cegielski A. (1986a), Cegielski A. (1986b), Orłowski K., Radzik T. (1985a), Orłowski K., Radzik T. (1985b), Radzik T. (1988), Styszyński A. (1974), Teraoka Y. (1976), Teraoka Y. (1979), Vorob'ev N.N. (1984).

Part III

In this part the case of $n = 4$, $m \leq n$, is solved.

1. Definitions and assumptions

Given assumptions as before, let (m, n), $< a \wedge c, a >$; $0 < c \leq \hat{\varepsilon}$, be the duel in which Player I has m bullets, Player II has n bullets, Player I is at the beginning of the duel at a, Player II at 1, but if $c < \hat{\varepsilon}$ Player II can fire his bullets beginning from the time $< a >$ and Player II from the time $< a > +c$. If $c = \hat{\varepsilon}$ the rule is the same with the only exception that Player II is not allowed to fire at $< a >$.

Similarly we define the duel (m, n), $< a, a \wedge c >$.

All other definitions and suppositions made for the duel (m, n), hold also for the above duels.

For definitions and notions concerning duels see Karlin S. (1959), Trybuła S. (1995) and Vorob'ev N.N. (1984).

2. Duels $(1, 4)$

Duel $(1, 4), < a >$.

Let us consider the duel in which Player I has 1 bullet, Player II has 4 bullets and the game begins when Player I is at the point a.

Let $Q(a) \geq Q(\hat{a}_{14}) \cong 0.89814$. We define strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player II fires (say at a'), play optimally the duel $(1, 3), < a', a' \wedge \hat{\varepsilon} >$.

Strategy of Player II: Fire at $< a >$ and play optimally the duel $(1, 3), < a, a \wedge \hat{\varepsilon} >$ (or $(0, 3), < a > +\hat{\varepsilon}$).

The sign $< a >$ denotes the first time when Player I is at the point a.

"Play optimally" means: apply a strategy optimal in the limit.

Let $K(\hat{\xi}, \hat{\eta})$ be the expected gain of Player I if he applies strategy $\hat{\xi}$ and Player II applies the strategy $\hat{\eta}$. We have

$$\begin{aligned} K(\xi, \eta) &= -P(a) + Q(a) v_{13}(2, a) + k(\hat{\varepsilon}) \\ &= -P(a) + Q(a) \begin{cases} k_1(\hat{\varepsilon}) & \text{if} \quad Q(a) \geq Q(\hat{a}_{13}), \\ -P(a) + (1 - P(a)) v_{12} + k(\hat{\varepsilon}) & \\ & \text{if} \quad Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases} \\ &= \begin{cases} -1 + Q(a) + k(\hat{\varepsilon}) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12}) Q^2(a) + k(\hat{\varepsilon}) & \text{if } Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases} \end{aligned}$$

$v_{13}(2, a)$ is the limit value of the game in the duel $(1, 3)$, $< a, a \wedge \hat{\varepsilon} >$ (see Trybuła S. (1995)), $k(\hat{\varepsilon})$ is a functions tending to 0 if $\hat{\varepsilon} \to 0$, $Q(s) = 1 - P(s)$,

$Q(\bar{a}_{13}) \cong 0.88097$, $Q(\hat{a}_{13} \cong 0.95572$. For precise definition of $\bar{a}_{13}$ and $\hat{a}_{13}$ see Trybuła S. (1995). The number $v_{12} \cong 0.04633$ is the limit value of the game $(1,2)$.

Suppose that Player II fires when Player I is at a'. For such a strategy (denote it by $\hat{\eta}$) we obtain

$$\begin{aligned} K(\xi,\hat{\eta}) &\geq -P(a') + (1-P(a'))v_{13}(2,a') + k(\hat{\varepsilon}) \\ &= \begin{cases} -P(a') + k(\hat{\varepsilon}) \geq -P(a) + k(\hat{\varepsilon}) \\ \quad \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1+v_{12})Q^2(a') + k(\hat{\varepsilon}) \geq -1 + (1+v_{12})Q^2(a) + k(\hat{\varepsilon}) \\ \quad \text{if } Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases} \end{aligned}$$

Suppose that Player II does not fire at all. For such a strategy $\hat{\eta}$ we obtain

$$K(\xi,\hat{\eta}) = 0 \geq \begin{cases} -P(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1+v_{12})Q^2(a) & \text{if } Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

Then Player I applying strategy ξ assures in the limit the value $-P(a)$ if $a \leq \hat{a}_{13}$ and the value $-1 + (1+v_{12})Q^2(a)$ if $\hat{a}_{13} \leq a \leq \bar{a}_{13}$.

On the other hand, suppose that Player II applies the strategy η and Player I fires at $< a >$. We have for $a \leq \hat{a}_{13}$

$$\begin{aligned} K(\hat{\xi},\eta) &\leq -Q^2(a)v_{02}(a) + k(\hat{\varepsilon}) \\ &= -Q^2(a)(1-Q^2(a)) + k(\hat{\varepsilon}) \\ &\leq -P(a) + k(\hat{\varepsilon}) \end{aligned}$$

if

$$Q^2(a)(1-Q(a)) \leq 1$$

what always holds for $Q(a) \geq Q(\bar{a}_{13})$.

If $\hat{a}_{13} \leq a \leq \bar{a}_{13}$ we obtain

$$K(\hat{\xi},\eta) \leq -Q^2(a)(1-Q^2(a)) + k(\hat{\varepsilon}) \leq -1 + (1+v_{12})Q^2(a) + k(\hat{\varepsilon})$$

what leads to the inequality

$$S(Q) = Q^4(a) - (2+v_{12})Q^2(a) + 1 \leq 0.$$

This multinomial is a decreasing function of the variable Q and $S(Q) = 0$ for $Q = Q(\hat{a}_{14}) \cong 0.89814$. Then the inequality holds for $a \leq \hat{a}_{14}$.

Suppose that Player I fires after $< a >$ or does not fire at all. For such a strategy $\hat{\xi}$ we have

$$K(\hat{\xi},\eta) \leq -P(a) + (1-P(a))v_{13}(2,a) + k(\hat{\varepsilon})$$

for $Q(a) \geq Q(\breve{a}_{13})$.

Then for $Q(a) \geq Q(\hat{a}_{14})$ strategies ξ and η are optimal in the limit and the limit value of the game is

$$v_{14}(a) = \begin{cases} -1 + Q(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^2(a) & \text{if } Q(\hat{a}_{14}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

Duel $(1,4)$, $< a \wedge c, a >$. $Q(a) \geq Q(\bar{a}_{13}) \cong 0.88097$.

Let us consider the strategies ξ and η of Players I and II.
Strategy of Player I: Escape. If Player II fired (say at a'), play optimally the resulting duel $(1,3), < a', a' \wedge c_1 >$.
Strategy of Player II: Fire at a', $< a > \leq a' << a > +c$ and play optimally the resulting duel $(1,3)$, $< a', a' \wedge c_2 >$.

In the paper $\dot{s}$ denotes the time when Player I is at the point s (not necessarily the first time which is denoted by $< s >$).

The proof that strategies ξ and η are optimal in the limit and the limit value of the game is

$$v_{14}(1,a) = \begin{cases} -1 + Q(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^2(a) & \text{if } Q(\bar{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases}$$

is nearly the same as for the duel $(1,4), < a >$. (In the proof of limit optimality of the strategy η only this case is different when both players fire at the same time $< a >$).

2.1. Duel $(1,4)$, $< \ddot{a}, a \wedge c >$. $Q(a) \geq Q(\breve{a}_{14}) \cong 0.90920$

Define ξ and η.
Strategy of Player I: Escape. If Player II fired (say at a'), play optimally the resulting duel $(1,3)$,$< a', a' \wedge c_1 >$.
Strategy of Player II: Fire at $< a > +c$ and play optimally the resulting duel.

Here also the proof of limit optimality is nearly the same as for the duel $(1,4), < a >$. Now there comes in addition the case:
(a) Player I fires before $< a > +c$ (at a''). In this case

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq P(a) - Q(a)(1 - Q^3(a)) + k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^4(a) + k(\hat{\varepsilon}) \\ &\leq -1 + Q(a) + k(\hat{\varepsilon}) \end{aligned}$$

if

$$Q^4(a) - 3Q(a) + 2 \leq 0$$

which always holds for $Q \geq Q(\hat{a}_{13}) \cong 0.95572$.

Moreover

$$K(\hat{\xi}, \eta) \leq 1 - 2Q(a) + Q^4(a) + k(\hat{\varepsilon}) \leq -1 + (1 + v_{12})Q^2(a) + k(\hat{\varepsilon})$$

if

$$S(Q) = Q^4(a) - (1 + v_{12})Q^2(a) - 2Q(a) + 2 \le 0. \tag{24}$$

The function $S(Q)$ is decreasing in Q and $S(Q) = 0$ for

$$Q = Q(\hat{a}_{14}) \cong 0.90920$$

Then for $a \le \check{a}_{14}$ the inequality (24) holds.

From the above it follows that if $a \le \check{a}_{14}$ the strategies ξ and η are optimal in the limit and the value of the game is the same as in the duel $(1,4), < a >$.

3. Results for the duels $(1,4)$.

Let $v_{14}(1,a), v_{14}(a), v_{14}(2,a)$ be the limit values for the duels $(1,4), < a \wedge c, a >$; $(1,4), < a >$; $(1,4)$, $< a, a \wedge c >$, respectively. From previous section we obtain

$$v_{14}(1,a) = \begin{cases} -1 + Q(a) & \text{if} \quad Q(a) \ge Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^2(a) & \\ & \text{if} \quad 0.88097 \cong Q(\bar{a}_{13}) \le Q(a) \le Q(\hat{a}_{13}), \end{cases}$$

$$v_{14}(a) = \begin{cases} -1 + Q(a) & \text{if} \quad Q(a) \ge Q(a_{13}), \\ -1 + (1 + v_{12})Q^2(a) & \\ & \text{if} \quad 0.89815 \cong Q(\hat{a}_{14}) \le Q(a) \le Q(\hat{a}_{13}), \end{cases}$$

$$v_{14}(2,a) = \begin{cases} -1 + Q(a) & \text{if}\ Q(a) \ge Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^2(a) & \\ & \text{if} \quad 0.90920 \cong Q(\check{a}_{14}) \le Q(a) \le Q(\hat{a}_{13}). \end{cases}$$

4. Duels $(2,4)$

Duel $(2,4), < a >$

Case 1. $Q(a) \ge Q(\hat{a}_{24}) \cong 0.95105$.

We denote by ξ and η the strategies of Player I and II:
Strategy of Player I: If Player I did not fire before, reach the point $\hat{a}_{13}$, fire with an ACPD in the interval $(< a_{13} >, < a_{13} > + \alpha(\varepsilon))$ and play optimally the resulting duel.
Strategy of Player II: If Player I does not fire and does not reach the point $\hat{a}_{24}$,

$$Q(\hat{a}_{24}) = \frac{1 + P^2(\hat{a}_{13})}{1 + v_{23}} \cong 0.95105,$$

do not fire neither. If he reaches the point $\hat{a}_{24}$, fire at $< \hat{a}_{24} >$ and play optimally the resulting duel. If he fires (say at a') before he reaches the point $\hat{a}_{24}$ play optimally the duel $(1,4)$, $< a', a' \wedge \hat{\varepsilon} >$.

We prove that for $a \le \hat{a}_{24}$ the strategies ξ and η are optimal in the limit and

$$v_{24}(a) = P(\hat{a}_{13}) + Q(\hat{a}_{13})v_{14}(1, \hat{a}_{13}) = P^2(\hat{a}_{13}).$$

Suppose that Player II fires when Player I is at the point a' before he (possibly) reaches $\hat{a}_{13}$. We have

$$K(\xi, \hat{\eta}) \ge -P(a') + Q(a')v_{23} - k(\hat{\varepsilon}) \ge P^2(\hat{a}_{13}) - k(\hat{\varepsilon})$$

if

$$Q(a') \ge \frac{1 + P^2(\hat{a}_{13})}{1 + v_{13}} = Q(\hat{a}_{24})$$

then always for $Q(a') \ge Q(\hat{a}_{13})$.

Suppose that Player II does not fire before $< \hat{a}_{13} > +\alpha(\varepsilon)$. For such a strategy $\hat{\eta}$ we obtain

$$K(\xi, \hat{\eta}) \ge P(\hat{a}_{13}) + Q(\hat{a}_{13})v_{14}(1, \hat{a}_{13}) - k(\hat{\varepsilon}) = P^2(\hat{a}_{13}) - k(\hat{\varepsilon}).$$

To prove that Player II applying η assures in the limit the value $P^2(\hat{a}_{13})$ assume that

a) Player I reaches $\hat{a}_{24}$ and does not fire. We have
$$K(\hat{\xi}, \eta) \le -P(\hat{a}_{24}) + Q(\hat{a}_{24})v_{23} + k(\hat{\varepsilon}) = P^2(\hat{a}_{13}) + k(\hat{\varepsilon});$$

b) Player I does not reach the point $\hat{a}_{24}$ and does not fire. We have
$$K(\hat{\xi}, \eta) = 0 < P^2(\hat{a}_{13});$$

c) Player I fires at $a' < \hat{a}_{24}$,
$$\begin{aligned} K(\hat{\xi}, \eta) &\le P(a') + (1 - P(a'))v_{14}(1, a') + k(\hat{\varepsilon}) \\ &= \begin{cases} P^2(a') \quad \text{if} \quad Q(a') \ge Q(\hat{a}_{13}), \\ 1 - 2Q(a') + (1 + v_{12})Q^3(a') \\ \qquad\qquad \text{if} \quad 0.88097 \cong Q(\bar{a}_{13}) \le Q(a') \le Q(\bar{a}_1) \end{cases} \\ &\le P(\hat{a}_{13}) + (1 - P(\hat{a}_{13}))v_{14}(1, \hat{a}_{13}) + k(\hat{\varepsilon}) \\ &= P^2(\hat{a}_{13}) + k(\hat{\varepsilon}); \end{aligned}$$

d) Player I fires at $< \hat{a}_{24} >$ together with Player II,
$$\begin{aligned} K(\hat{\xi}, \eta) &\le Q^2(\hat{a}_{24})v_{13}(\hat{a}_{24}) + k(\hat{\varepsilon}) \\ &= -Q^2(\hat{a}_{24}) + (1 + v_{12})Q^3(\hat{a}_{24}) + k(\hat{\varepsilon}) \\ &= -0.00616 + k(\hat{\varepsilon}) \\ &\le P^2(\hat{a}_{13}) + k(\hat{\varepsilon}). \end{aligned}$$

Case 2. $0.91636 \cong Q(\breve{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}) \cong 0.95105$.

We denote by ξ and η the following strategies of Player I and II.
Strategy of Player I: If Player II did not fire before, escape to the the point $\hat{a}_{13}$, fire with an ACPD in the time interval $(< \hat{a}_{13} >, < \hat{a}_{13} > +\alpha(\varepsilon))$ and play

optimally the resulting duel.
Strategy of Player II: Fire at $< a >$ and play optimally the resulting duel.

We prove that for properly chosen $\alpha(\varepsilon)$ strategies ξ and η are optimal in the limit and

$$v_{24}(a) = -P(a) + (1 - P(a))v_{23}.$$

Suppose that Player II fires when Player I is at a' before he reaches $\hat{a}_{13}$. We have $\hat{a}_{13} \leq a' \leq a$ and

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + Q(a')v_{23} - k(\hat{\varepsilon}) \\ &\geq -P(a) + Q(a)v_{23} - k(\hat{\varepsilon}) \\ &= v_{24}(a) - k(\hat{\varepsilon}) \end{aligned}$$

for $Q(a) \leq Q(\hat{a}_{13})$.

Suppose that Player II does not fire before $< \hat{a}_{13} > +\alpha(\varepsilon)$. Then, for properly chosen $\alpha(\varepsilon)$

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq P(\hat{a}_{13}) + Q(\hat{a}_{13})v_{14}(1, \hat{a}_{13}) - \varepsilon - k_1(\hat{\varepsilon}) \\ &= P^2(\hat{a}_{13}) - k(\hat{\varepsilon}) \quad \geq \quad -P(a) + (1 - P(a))v_{23} - k(\hat{\varepsilon}) \end{aligned}$$

if

$$Q(a) \leq \frac{1 + P^2(\hat{a}_{13})}{1 + v_{23}} = Q(\hat{a}_{24}).$$

On the other hand, if Player I fires at $< a >$

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq Q^2(a)v_{13}(a) + k(\hat{\varepsilon}) \\ &= -Q^2(a) + (1 + v_{12})Q^3(a) + k(\hat{\varepsilon}) \\ &\leq -1 + (1 + v_{23})Q(a) + k(\hat{\varepsilon}) \end{aligned}$$

for $0.85391 \cong Q(\check{a}_{13}) \leq Q(a) \leq Q(\hat{a}_{13})$. Then we have

$$S(Q) = (1 + v_{12})Q^3(a) - Q^2(a) - (1 + v_{23})Q(a) + 1 \leq 0.$$

The only root of the multinomial $S(Q)$ is $Q = Q(\check{a}_{24}) \cong 0.91636$ and the function $S(Q)$ is decreasing for $Q \leq Q(\check{a}_{24})$. Then $S(Q) \leq 0$ for $Q(a) \geq Q(\check{a}_{24})$.

If Player I fires after $< a >$ or does not fire at all

$$K(\hat{\xi}, \eta) \leq -P(a) + (1 - P(a))v_{23} + k(\hat{\varepsilon}) = v_{24}(a) + k(\hat{\varepsilon})$$

which ends the proof of limit optimality of strategies ξ and η.

Case 2a. $0.91636 \cong Q(\breve{a}_{24}) \le Q(a) \le Q(\bar{a}_{24}) \cong 0.93571$.

We define strategies ξ and η of Players I and II.
Strategy of Player I: If Player II did not fire before, fire with an ACPD in the time interval $(< a >, < a > +\alpha(\varepsilon))$ and escape. If he fired, play optimally the duel $(2,3)$.
Strategy of Player II: Fire at the beginning of the duel and play optimally afterwards.

We prove that as before

$$v_{24}(a) = -P(a) + (1 - P(a))v_{23}. \tag{25}$$

Suppose that Player II fires at time $< a >$. We have

$$K(\xi, \hat{\eta}) \ge -P(a) + (1 - P(a))v_{23} - k(\hat{\varepsilon}).$$

Suppose that Player II fires at a', $\dot{a}' \ge < a > +\alpha(\varepsilon)$. For properly chosen $\alpha(\varepsilon)$

$$\begin{aligned} K(\xi, \hat{\eta}) &\ge P(a) + (1 - P(a))v_{14}(1, a) - k_1(\hat{\varepsilon}) - \varepsilon \\ &= 1 - 2Q(a) + (1 + v_{12})Q^3(a) - k(\hat{\varepsilon}) \end{aligned}$$

if

$$0.88097 \cong Q(\bar{a}_{13}) \le Q(a) \le Q(\hat{a}_{13}).$$

We have

$$K(\xi, \hat{\eta}) \ge v_{24}(a) - k(\hat{\varepsilon}) \tag{26}$$

if

$$(1 + v_{12})Q^3(a) - (3 + v_{23})Q(a) + 2 \ge 0$$

which is satisfied if

$$Q(a) \le Q(\bar{a}_{24}) \cong 0.93571.$$

Then the inequality (26) holds for $\bar{a}_{24} < a < \bar{a}_{13}$.

On the other hand, if Player I also fires at $< a >$

$$K(\hat{\xi}, \eta) \le Q^2(a)v_{13}(a) + k(\hat{\varepsilon}) = -Q^2(a) + (1 + v_{12})Q^3(a) + k(\hat{\varepsilon})$$

if

$$0.85391 \cong Q(\breve{a}_{13}) \le Q(a) \le Q(\hat{a}_{13}).$$

We obtain

$$K(\hat{\xi}, \eta) \le v_{24}(a) + k(\hat{\varepsilon})$$

if

$$(1 + v_{12})Q^3(a) - Q^2(a) - (1 + v_{23})Q(a) + 1 \leq 0$$

which, as it was proved in Case 2, is satisfied if $Q(a) \geq Q(\breve{a}_{24}) \cong 0.91636$.

Suppose that Player I does not fire at $< a >$. For such a strategy $\hat{\xi}$ we have

$$K(\hat{\xi}, \eta) \leq -P(a) + (1 - P(a))v_{23} + k(\hat{\varepsilon})$$

which ends the proof that strategies ξ and η are optimal in the limit and that the limit value of the game is given by (25).

From Case 2 and 2a it follows that sometimes Player I has more than one strategy optimal in the limit.

Case 3. $0.89815 \cong Q(\breve{a}_{14}) \leq Q(a) \leq Q(\breve{a}_{24}) \cong 0.91636$.

We define ξ and η.

Strategy of Player I: Fire at the beginning of the duel and play optimally afterwards.

Strategy of Player II: Fire at the beginning of the duel and play optimally afterwards.

We prove that now

$$v_{24}(a) = Q^2(a)\ddot{v}_{13}(a) = -Q^2(a) + (1 + v_{12})Q^3(a)$$

if $\breve{a}_{24} \leq a \leq \breve{a}_{14}$.

Suppose that Player II does not fire at $< a >$. We obtain for such a strategy $\hat{\eta}$:

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq P(a) + Q(a)v_{14}(1, a) - k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + (1 + v_{12})Q^3(a) - k(\hat{\varepsilon}) \,. \\ &\geq -Q^2(a) + (1 + v_{12})Q^3(a) - k(\hat{\varepsilon}) \end{aligned}$$

for any a if $v_{14}(1, a) = -1 + (1 + v_{12})Q(a)$ i.e. if $\hat{a}_{13} \leq a \leq \bar{a}_{13}$, $Q(\bar{a}_{13}) \cong 0.88097$.

On the other hand, if Player I does not fire at the time $< a >$

$$\begin{aligned} K(\hat{\xi}, \eta) &< -P(a) + Q(a)v_{23} + k(\hat{\varepsilon}) \\ &\leq -Q^2(a) + (1 + v_{12})Q^3(a) + k(\hat{\varepsilon}) \end{aligned}$$

if

$$(1 + v_{12})Q^3(a) - Q^2(a) - (1 + v_{23})Q(a) + 1 \geq 0$$

what holds, as we know, for $Q(a) \leq Q(\breve{a}_{24}) = 0.91636$.

This ends the proof of the assertion.

Duel $(2,4), < a \wedge c, a >$.

Case 1. $Q(a) \geq Q(\hat{a}_{24}) \cong 0.95105$.

We define ξ and η.
Strategy of Player I: If Player II did not fire before, reach the point $\hat{a}_{13}$, fire with an ACPD in the time interval $(< \hat{a}_{13} >, < \hat{a}_{13} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If Player II fires (say at a') play optimally the duel $(2,3)$.
Strategy of Player II: If Player I does not reach the point $\hat{a}_{24}$ and does not fire, do not fire neither. If he reaches $\hat{a}_{24}$ and has not fired, fire at $< \hat{a}_{24}$ and play optimally the resulting duel. If he fires before he reaches the point $\hat{a}_{24}$ (say at a') play optimally the duel $(1,4), < a', a' \wedge \hat{\varepsilon} >$.

Now

$$v_{24}(1,a) = P^2(\hat{a}_{13}) \tag{27}$$

for $a \leq \hat{a}_{24}$.

Case 2. $0.86167 \cong Q(a_{23}) \leq Q(a) \leq Q(\hat{a}_{24}) \cong 0.95105$.

We define ξ and η.
Strategy of Player I: The same as in Case 1.
Strategy of Player II: Fire at $a', \dot{a}' << a > +c$, and play optimally the obtained duel.

Now

$$v_{24}(1,a) = -P(a) + (1 - P(a))v_{23} \tag{28}$$

for

$$0.86167 \cong Q(a_{23}) \leq Q(a) \leq Q(\hat{a}_{24}).$$

The proofs are omitted.

Duel $(2,4), < a, a \wedge c >$.

Case 1. $Q(a) \geq Q(\hat{a}_{24}) \cong 0.95105$.

Strategy of Player I: If Player II did not fire before, reach the point $\hat{a}_{13}$, fire with an ACPD in the interval $(< \hat{a}_{13} >, < \hat{a}_{13} > +\alpha(\varepsilon))$ and play optimally the resulting duel. If Player II fires (say at a') play optimally the duel $(2,4)$.
Strategy of Player II: If Player I does not fire and does not reach the point $\hat{a}_{24}$, do not fire neither. If he has reached $\hat{a}_{24}$ and has not fired, fire at $< \hat{a}_{24} >$ and play optimally the resulting duel. If he had fired (say at a') before he reached the point $\hat{a}_{24}$, play optimally the resulting duel $(1,4), < a', a' \wedge c >$.

Now also

$$v_{24}(2,a) = P^2(\hat{a}_{13}). \tag{29}$$

Case 2. $0.93571 \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{24}) \cong 0.95105$.

Strategy of Player I: The same as in Case 1.
Strategy of Player II: If Player I does not fire before, fire at $< a > +c$ and play optimally the obtained duel. If he fired (say at a') play optimally the resulting duel $(1,4), < a' \wedge c, a' >$.

$$v_{24}(2,a) = -P(a) + Q(a)v_{23}$$

for $\hat{a}_{24} < a < \bar{a}_{24}$.

Case 3. $0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(\bar{a}_{24}) \cong 0.93571$.

Strategy of Player I: Fire before $< a > +c$ and play optimally the resulting duel.
Strategy of Player II: If Player I did not fire before, fire at $< a > +c$ and play optimally the obtained duel. If he has fired (say at a') play optimally the resulting duel $(1,4), < a' \wedge c, a' >$.

Now

$$v_{24}(2,a) = 1 - 2Q(a) + (1 + v_{12})Q^3(a) \tag{30}$$

for $\bar{a}_{24} \leq a \leq \breve{a}_{24}$.

The proofs that above strategies are optimal in limit are omitted.

5. Results for the duels $(2,4)$.

$$v_{24}(1,a) = \begin{cases} P^2(\hat{a}_{13}) \quad \text{if} \quad Q(a) \geq Q(\hat{a}_{24}) \cong 0.95105, \\ -P(a) + (1 - P(a))v_{23} \\ \qquad\qquad \text{if} \quad 0.86167 \cong Q(a_{23}) \leq Q(a) \leq Q(\hat{a}_{24}), \end{cases}$$

$$v_{24}(a) = \begin{cases} P^2(\hat{a}_{13}) \quad \text{if} \quad Q(a) \geq Q(\hat{a}_{24}), \\ -P(a) + (1 - P(a))v_{23} \\ \qquad\qquad \text{if} \quad 0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{24}), \\ -Q^2(a) + (1 + v_{12})Q^3(a) \\ \qquad\qquad \text{if} \quad 0.88097 \cong Q(\bar{a}_{13}) \leq Q(a) \leq Q(\breve{a}_{24}), \end{cases}$$

$$v_{24}(2,a) = \begin{cases} P^2(\hat{a}_{13}) \quad \text{if} \quad Q(a) \geq Q(\hat{a}_{24}), \\ -P(a) + (1 - P(a))v_{23} \\ \qquad\qquad \text{if} \quad 0.93571 \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{24}), \\ 1 - 2Q(a) + (1 + v_{12})Q^3(a) \\ \qquad\qquad \text{if} \quad 0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(\bar{a}_{24}). \end{cases}$$

6. Duel $(3,4)$.

Duel $(3,4), < a > .Q(a) \geq Q(a_{34}) \cong 0.89726$.

We define strategies ξ and η of Players I and II.
Strategy of Player I: If Player II did not fire before, fire with an ACPD in the interval $(< a_{34} >, < a_{34} > +\alpha(\varepsilon))$ and play optimally the resulting duel $(3,3)$.
Strategy of Player II: If Player I did not fire before, fire at $< a_{34} >$ and play optimally the duel $(3,3)$. If Player I did not reach the point a_{34} and did not fire, do not fire neither. If he fired (say at a') play optimally the duel $(2,4), < a' \wedge c, a' >$.

The number a_{34} is the solution of the equation (31).

Let the numbers v_{34} and a_{34} satisfy the equations

$$\begin{aligned} v_{34} &= P(a_{34}) + Q(a_{34})v_{24}(1, a_{34}) \\ &= -P(a_{34}) + Q(a_{34})v_{33}. \end{aligned}$$

We have

$$v_{24}(1, a_{34}) = -1 + (1 + v_{23})Q(a_{34})$$

for

$$Q(a_{23}) \leq Q(a_{34}) < Q(\hat{a}_{24}) \cong 0.95105.$$

Then we obtain

$$(1 + v_{23})Q^2(a_{34}) - (3 + v_{33})Q(a_{34}) + 2 = 0. \tag{31}$$

Since $v_{23} \cong 0.05354$, $v_{33} \cong 0.17430$, we obtain

$$Q(a_{34}) \cong 0.89726, \quad v_{34} \cong 0.05365.$$

Then $\hat{a}_{24} \leq a_{34} \leq a_{23}$ as it should be.

We prove that for $a \leq a_{34}$ the strategies ξ and η are optimal in the limit and the limit value of the game $v_{34}(a) = v_{34}$.

Suppose that Player II fires when Player I is at a', before he reaches the point a_{34}. We have

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + (1 - P(a'))v_{33} - k(\hat{\varepsilon}) \\ &\geq -P(a_{34}) + (1 - P(a_{34}))v_{33} - k(\hat{\varepsilon}) \\ &= v_{34} - k(\hat{\varepsilon}). \end{aligned}$$

Suppose that Player II fires after $< a_{34} > +\alpha(\varepsilon)$ or does not fire at all. For such a strategy $\hat{\eta}$ we have

$$K(\xi, \hat{\eta}) \geq P(a_{34}) + Q(a_{34})v_{24}(1, a_{34}) - k(\hat{\varepsilon}) = v_{34} - k(\hat{\varepsilon})$$

for properly chosen $\alpha(\varepsilon)$.

On the other hand, if Player I fires at $a' < a_{34}$

$$\begin{aligned}
K(\hat{\xi},\eta) &\le P(a') + (1-P(a'))v_{24}(1,a') + k(\hat{\varepsilon}) \\
&= \begin{cases} 1-(1-P^2(\hat{a}_{13}))Q(a') + k(\hat{\varepsilon}) \\ \qquad \text{if} \quad Q(a') \ge Q(\hat{a}_{24}), \\ 1-2Q(a') + (1+v_{23})Q^2(a') + k(\hat{\varepsilon}) \\ \qquad \text{if} \quad Q(a_{34}) \le Q(a') \le Q(\hat{a}_{24}) \end{cases} \\
&\le 0.05365 + k(\hat{\varepsilon}) \\
&\le v_{34} + k(\hat{\varepsilon}).
\end{aligned}$$

Suppose that Player I fires at $< a_{34} >$. In this case

$$K(\hat{\xi},\eta) \le Q^2(a_{34})v_{23} + k(\hat{\varepsilon}) \cong 0.04310 + k(\hat{\varepsilon}) < v_{34} + k(\hat{\varepsilon}).$$

Suppose, finally, that Player I does not fire before or at $< a_{34} >$. Then

$$K(\hat{\xi},\eta) \le -P(a_{34}) + Q(a_{34})v_{33} + k(\hat{\varepsilon}) = v_{34} + k(\hat{\varepsilon}).$$

The strategies ξ and η are now optimal in limit for $a \le a_{34}$.

It is easy to see that strategies ξ and η are also optimal in limit in the duels $(3,4), < a, a \wedge c >$ and $(3,4), < a \wedge c, a >$ if $a \le a_{34}$. We denote all these duels simply by $(3,4)$.

7. Duels $(4,4)$.

Duel $(4,4), < a >$. $Q(a) \ge Q(a_{44}) \cong 0.90240$.

Define ξ and η.
Strategy of Player I: If Player II did not fire before, go ahead, fire in the time interval $(< a_{44} >, < a_{44} > + \alpha(\varepsilon))$ and play optimally the duel $(3,4)$. If he fired play optimally the duel $(4,3)$.
Strategy of Player II: If Player I did not fire before, fire at $< a_{44} >$ and play optimally the duel $(4,3)$ or $(3,3)$. If he fired play optimally the duel $(3,4)$.

Now

$$\begin{aligned}
v_{44}(a) = v_{44} &= P(a_{44}) + Q(a_{44})v_{34} \\
&= -P(a_{44}) + Q(a_{44})v_{43}.
\end{aligned}$$

Since $v_{34} \cong 0.05365, v_{43} = 0.26997$ (see Trybuła S. (1995)), we obtain from the above

$$\begin{aligned}
Q(a_{44}) &= \frac{2}{2+v_{43}-v_{34}} &&\cong 0.90240, \\
v_{44} &= -1 + (1+v_{43})Q(a_{44}) &&\cong 0.14602.
\end{aligned}$$

Proof that strategies ξ and η are optimal in the limit is omitted.

Here it is easy to see that strategies ξ and η are optimal in the limit also in the duels $(4,4), < a \wedge c, a >$ and $(4,4), < a, a \wedge c >$. We denote these duels together with $(4,4), < a >$ simply by $(3,4)$.

For the duels with retreat after firing shots see Trybuła S. (1990).

For the duels with arbitrary movements see Trybuła S. (to be published), Trybuła S. (1993), Trybuła S. (1995).

For noisy duels see Fox M., Kimeldorf G. (1969), Karlin S. (1959), Teraoka Y. (1976), Trybuła S. (to be published).

For other duels see Cegielski A. (1986a), Cegielski A. (1986b), Kimeldorf G. (1983), Orłowski K., Radzik T. (1985a), Orłowski K., Radzik T. (1985b), Radzik T. (1988), Restrepo R. (1957), Styszyński A. (1974), Teraoka Y. (1976).

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