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C-optimal threat decision n-tuples in collective bargaining

by

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In two-person game with bargaining, C-optimal threat decision pairs are defined. The definition is compared with that of optimal threat decision pairs in the sense of Nash. In the case of differential games, a sufficiency condition for C-optimality of a threat strategy pair is given and illustrated by examples of collective bargaining.
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1. Introduction

We shall be interested in the behavior of a set of "persons", called the *players*, each of whom strives to modify the state of a system or, as we shall say, the state of the *game*, in a most efficacious manner according to his own criterion. We shall first consider the case of games in which the rules assign to each player a *payoff function* of all the players' *decisions*; that is, the *rules of the game* prescribe mappings

$$W_i \ : \ \prod_{i=1}^{N} S_i \rightarrow \Omega_i, \quad i = 1, 2, \dots N, \tag{1}$$

where W_i and S_i is the payoff function and decision set, respectively, for player i in the set $J = \{J_1, J_2, \dots J_N\}$, and the Ω_i, $i = 1, 2, \dots N$, are linear spaces.

We wish to consider games for which a negotiated solution is envisaged. In general, cooperation entails bargaining for the reason that, in most cases where a cooperative mood of play exists, it is not unique. A decision N-tuple in the set of cooperative equilibria may be more desirable than another for some player. Accordingly, this player will try to convince the other players to choose that cooperation point. In practice, it appears that the efficiency of his argument will depend on his "strength", that is, on the efficiency of the *threats* he can put forward.

This problem was considered by Nash (1953)., and extended to differential games by Liu (1973). Here we shall use another approach based on the concept of C-optimality introduced in Blaquière and Ray (1981).

Cooperative and Competitive Games

In the case where $\Omega_i = R^1$, $i = 1, 2, \ldots N$, we suppose, loosely speaking, that each player desires to attain the greatest possible payoff to himself. A large part of the literature on games is concerned with two moods of play, one *cooperative* and the other *competitive*. These are due to economists Pareto (1909) and Nash (1951), respectively.

According to Pareto, a decision N-tuple is considered optimal if and only if one of two situations occurs: Adopting another decision N-tuple either results in no change in any of the payoffs (and hence there is no reason for adopting another decision) or it results in a payoff decrease to at least one player (which is undesirable in view of the cooperative mood of play). In other words, we have

Definition 1 For prescribed mappings $V_i : \prod_{i=1}^{N} S_i \to R^1$, $i = 1, 2, \ldots N$, a decision N-tuple $s^* \in \prod_{i=1}^{N} S_i$ is *Pareto-optimal* (or a *Pareto-equilibrium*) if and only if for every $s \in \prod_{i=1}^{N} S_i$ either

$$V_i(s) = V_i(s^*) \qquad i = 1, 2, \ldots N,$$

or there is at least one $i \in \{i = 1, 2, \ldots N\}$ such that

$$V_i(s) < V_i(s^*).$$

Later, we shall make use of the following lemma which embodies sufficiency conditions for Pareto-optimality: see Leitmann (1974)

Lemma 1 *Decision N-tuple $s^* \in \prod_{i=1}^{N} S_i$ is Pareto-optimal if there exist strictly positive numbers α_i, $i = 1, 2, \ldots N$, such that*

$$V_i(s) \leq V_i(s^*) \qquad \textit{for all} \;\; s \in \prod_{i=1}^{N} S_i,$$

where $V(s) = \sum_{i=1}^{N} \alpha_i V_i(s)$.

If the players do not cooperate, that is if they are in strict competition, a *rational* behavior for each player is to strive to attain the maximum of his own payoff regardless of the consequences to the other players, under the assumption that each of his opponents is rational. This leads to

Definition 2 A decision N-tuple $s^* \in \prod_{i=1}^N S_i$ is a *Nash-equilibrium* if and only if for all $i \in \{i = 1, 2, \dots N\}$

$$V_i(s^*) \geq V_i(s_1^*, \dots s_{i-1}^*, s_i, s_{i+1}^*, \dots s_N^*)$$

for all $s_i \in S_i$

2. C-optimality

Now, in the more general case where the rules of the game prescribe mappings of the type (1), we have introduced in Blaquière (1974), and further discussed in Blaquière (1975), Blaquière (1976a, b) the concept of C-optimality. This was motivated by the fact that, the concept of optimality being tied with the ones of preference and comparison, a *preference relation* and *comparison relation* need be associated with each player. Here, in general, the preference can not be defined by the natural ordering on the real line as in cases of the above paragraph.

Let the preference relation of J_i, $i = 1, 2, \dots N$ (reflexive, not necessarily transitive) be denoted by $(\geq)_i$, $\Omega_i^2 \supset (\geq)_i$; and

let the comparison relation of J_i, $i = 1, 2, \dots N$ (reflexive and symmetric) be denoted by C_i, $(\prod_{i=1}^N S_i)^2 \supset C_i$.

Then we have

Definition 3 A decision N-tuple $s^* \in \prod_{i=1}^N S_i$ is C-*optimal for player* J_i if and only if

$$W_i(s^*)\,(\geq)_i\,W_i(s) \qquad \text{for all} \quad s\,C_i\,s^*,$$

Definition 4 A decision N-tuple $s^* \in \prod_{i=1}^N S_i$ is C-*optimal* if and only if it is optimal for all the players, that is if and only if

$$W_i(s^*)\,(\geq)_i\,W_i(s) \qquad \text{for all} \quad s\,C_i\,s^*, \quad i = 1, 2, \dots N$$

Example 1

In order to illustrate the definition above, let us consider the case of a static game with the two players J_1 and J_2. Let the game be represented by

		J_2	
	4, 6	2, 5	6, 3
J_1	4, 3	1, 4	7, 2
	3, 2	6, 5	7, 4

J_1 chooses a line s_1, and J_2 chooses column s_2. The corresponding payoff for each player lies at the intersection of s_1 and s_2: the first number is the payoff of J_1, and the second one is the payoff of J_2. Define C_1 and C_2 by

$$(s_1, s_2)\,C_1\,(s_1', s_2') \iff s_2 = s_2',$$

$$(s_1, s_2)\, C_2\, (s_1', s_2') \iff s_1 = s_1'.$$

Let $\Omega_1 = \Omega_2 = R^1$ and $(\geq)_1 = (\geq)_2 = (\geq)$, where $\geq$ is natural ordering in R^1. Then the C-optimal decision pairs (s_1, s_2) are those corresponding to the payoffs (4,6) and (6,5). We see that, in that case, the definition of a C-optimal decision pair coincides with one of a *Nash equilibrium.*

Example 2
Let us modify Example 1 by letting $\Omega_1 = \Omega_2 = R^2$ and $(\geq)_1 = (\geq)_2 = (\geq)$, where the preference relation $(\geq)$ is defined by

$$[(x, y) \in R^2,\ (x', y') \in R^2 (x, y)\,(\geq)\,(x', y')] \Leftrightarrow$$

$$[(x > x' \text{ and/or } y > y') \text{ or } (x = x' \text{ and } y = y')]$$

Note that this preference relation is neither transitive nor antisymetric.

Define C_1 and C_2 such that any (s_1, s_2) is comparable with any (s_1', s_2'), for J_1 and J_2; here $C_1 = C_2$.

Then, there three C-optimal decision pairs (s_1, s_2), namely those corresponding to the payoffs (4,6), (6,5) and (7,4). We see that, in that case, the definition of a C-optimal decision pair coincides with the one a *Pareto equilibrium*. Note that the decision pairs corresponding to (4,6) and (6,5) are at once Nash and Pareto.

We will see another illustrative example in the next paragraph. In Blaquière (1976, b) Definition 4 is used in the study of *coalitions* and for introducing the concept of *diplomacy.*

3. C-optimal Threat Decision Pair

From now on, we shall consider two-person games for which a negotiated solution is envisaged. Before such a settlement can be arrived at, we will suppose that the players exchange threats in an attempt to influence the final outcome of the game. Whether negotiations take place and what are the results of such negotiations will depend on the threats made. The problem of bargaining has been considered by Nash (1953), and extended to the differential games by Liu (1973). Our approach, reported in Ray and Blaquière (1981) is different in that we define optimal threats independently of any negotiated stages, through the concept of C-optimality.

Roughly speaking, we can think of a threat decision as a decision designed to inflict the greatest damage possible to the opponent. In so doing, each player will have to consider the possible reaction of his opponent. If the opponent behaves in the same way, then both players run to risk of having considerable losses. Thus, in choosing a threat decision, each player needs to consider the effect that it will have on the other player and also the risk to himself associated with it. In order to make this idea more precise, let us start with the mappings $V_i : S_1 \otimes S_2 \to R^1$, $i = 1, 2$, and with the following facts:

The selection of a decision $\underline{s}_1 \in S_1$ by player J_1 has two consequences: it will put an upper bound on the (scalar) payoff of his opponent, namely

$$V_2(\underline{s}_1, s_2') = \sup_{s_2 \in S_2} [V_2(\underline{s}_1, s_2)],$$

and a lower bound on his own (scalar) payoff, namely

$$V_1(\underline{s}_1, s_2'') = \inf_{s_2 \in S_2} [V_1(\underline{s}_1, s_2)].$$

Since $s_2' \neq s_2''$ in most cases, it will generally be necessary for a player J_2 to find a compromise between defending his own payoff and attacking his opponent. A similar consideration holds for player J_1.

The fact that each player is interested in *threat-risk* pair leads us to considering the mappings

$$W_i : S_1 \otimes S_2 \to R^2, \qquad i = 1, 2,$$

where $W_1(s_1, s_2) = W_2(s_1, s_2) := (V_1(s_1, s_2), V_2(s_1, s_2))$ for $(s_1, s_2) \in S_1 \otimes S_2$. Then, the framework of C-optimality provides us with a way for defining C-optimality of a threat decision pair: that is, we use Definition 4 with $(\geq)_i$ and C_i, $i = 1, 2$ defined by

$(x, y)\ (\geq)_1\ (x', y') \Leftrightarrow \{x > x' \text{ and/or } y < y'\} \text{ or } \{x = x' \text{ and } y = y'\}$,
$(x, y)\ (\geq)_2\ (x', y') \Leftrightarrow \{x < x' \text{ and/or } y > y'\} \text{ or } \{x = x' \text{ and } y = y'\}$,

$(s_1, s_2) C_1 (s_1', s_2') \Leftrightarrow s_2 = s_2'$,
$(s_1, s_2) C_2 (s_1', s_2') \Leftrightarrow s_1 = s_2'$.

In other words, a response decision $\underline{s}_2$ for player J_2 against $\underline{s}_1$ is optimal for J_2 if and only if

$$(V_1(\underline{s}_1, \underline{s}_2), V_2(\underline{s}_1, \underline{s}_2))\,(\geq)_2\,(V_1(\underline{s}_1, s_2), V_2(\underline{s}_1, s_2)) \qquad \text{for all } s_2 \in S_2.$$

This makes sense, because, if player J_2 selects any other decision $s_2 \in S_2$, then either his own scalar payoff is reduced, or the (scalar) payoff of player J_1 is increased, or both situations occur, unless there is no change in any of the payoffs. A similar consideration holds for player J_1; that is, response decision $\underline{s}_1$ for player J_1 against $\underline{s}_2$ is optimal for player J_1 if and only if

$$(V_1(\underline{s}_1, \underline{s}_2), V_2(\underline{s}_1, \underline{s}_2))\,(\geq)_1\,(V_1(s_1, \underline{s}_2), V_2(s_1, \underline{s}_2)) \qquad \text{for all } s_1 \in S_1.$$

Then a threat decision pair (s_1^*, s_2^*) is C-optimal if and only if it is optimal for both J_1 and J_2.

Noting that $(x, y)\,(\geq)_1\,(x', y') \Leftrightarrow (x', y')\,(\geq)_2\,(x, y)$, we see that a threat decision pair (s_1^*, s_2^*) is C-optimal if and only if

$$(V_1(s_1^*, s_2), V_2(s_1^*, s_2))\ (\geq)\ (V_1(s_1^*, s_2^*), V_2(s_1^*, s_2^*))\ (\geq)\ (V_1(s_1, s_2^*), V_2(s_1, s_2^*))$$

for all $s_1 \in S_1$, and for all $s_2 \in S_2$, where $(\geq)$ is written in place of $(\geq)_1$.

As a direct consequence of the definitions of $(\geq)$ and of a Pareto-equilibrium, we have

Lemma 2 *$(s_1^*, s_2^*) \in S_1 \otimes S_2$ is a C-optimal threat decision pair if and only if*

(a) s_1^* is a Pareto-equilibrium of $(V_1(s_1, s_2^*), -V_2(s_1, s_2^*))$; and

(b) s_2^* is a Pareto-equilibrium of $(-V_1(s_1^*, s_2), V_2(s_1^*, s_2))$.

Lemma 1 together with Lemma 2 result in Lemma 3 and Corollary 1 which embody sufficiency conditions for C-optimality of a threat decision pair.

Lemma 3 *Decision pair $(s_1^*, s_2^*) \in S_1 \otimes S_2$ is a C-optimal threat decision pair if there exists strictly positive numbers α_1, α_2, such that*
$V_1(s_1^*, s_2^*) - \alpha_1 V_2(s_1^*, s_2^*)) \geq V_1(s_1, s_2^*) - \alpha_1 V_2(s_1, s_2^*))$, *and*
$V_1(s_1^*, s_2) - \alpha_2 V_2(s_1^*, s_2)) \geq V_1(s_1^*, s_2^*) - \alpha_2 V_2(s_1^*, s_2^*))$,
for all $(s_1, s_2) \in S_1 \otimes S_2$.

Corollary 1 *Decision pair $(s_1^*, s_2^*) \in S_1 \otimes S_2$ is a C-optimal threat decision pair if there exists strictly positive number α such that the saddle-point condition*

$$V_1(s_1^*, s_2) - \alpha V_2(s_1^*, s_2) \geq V_1(s_1^*, s_2^*) - \alpha V_2(s_1^*, s_2^*) \geq V_1(s_1, s_2^*) - \alpha V_2(s_1, s_2^*)$$

is satisfied for all $(s_1, s_2) \in S_1 \otimes S_2$.

4. Nash-Optimal Threat Decision Pair

Again, consider the mappings

$$W_i : S_1 \otimes S_2 \rightarrow R^2, \qquad i = 1, 2,$$

where $W_1(s_1, s_2) = W_2(s_1, s_2) = W(s_1, s_2) := (V_1(s_1, s_2),\ V_2(s_1, s_2))$
for $(s_1, s_2) \in S_1 \otimes S_2$.
Let us denote by $\sum(s_1, s_2)$ the set

$$\{(r_1, r_2) \in S_1 \otimes S_2 : \quad [V_1(r_1, r_2) - V_1(s_1, s_2)] > 0,\ [V_2(r_1, r_2) - V_2(s_1, s_2)] > 0\}.$$

Then we have

Definition 5 *A Nash bargaining solution associated with $(s_1, s_2) \in S_1 \otimes S_2$,* whenever it exists, is a pair $(\sigma_1, \sigma_2) \in \sum(s_1, s_2)$ such that, either

(i) $[V_1(r_1, r_2) - V_1(s_1, s_2)][V_2(r_1, r_2) - V_2(s_1, s_2)] \leq$
$[V_1(\sigma_1, \sigma_2) - V_1(s_1, s_2)][V_2(\sigma_1, \sigma_2) - V_2(s_1, s_2)]$
for all $(r_1, r_2) \in \sum(s_1, s_2)$,

or

(ii) $(\sigma_1, \sigma_2) = (s_1, s_2)$ if (s_1, s_2) is a Pareto equilibrium $(V_1(r_1, r_2), V_2(r_1, r_2))$, in which case there is no (σ_1, σ_2) satisfying condition (i).

Denote by $N(s_1, s_2)$ the set of all Nash bargaining solutions associated with $(s_1, s_2) \in S_1 \otimes S_2$. One can see easily that if $(\sigma_1, \sigma_2) \in N(s_1, s_2)$, then (σ_1, σ_2) is a Pareto equilibrium of $(V_1(r_1, r_2), V_2(r_1, r_2))$. In the sequel, we shall assume:

(A1) $\{(s_1, s_2): N(s_1, s_2) \neq \emptyset\} = S_1 \otimes S_2$, and

(A2) $W(S_1 \otimes S_2)$ is convex,

which ensure existence and uniqueness of a Nash bargaining solution associated with (s_1, s_2), for all $(s_1, s_2) \in S_1 \otimes S_2$.

From Definition 5 and elementary properties of convex sets, one obtains

Lemma 4 *Let (A1), (A2) hold. Then the following conditions are equivalent:*

(i) (σ_1, σ_2) *is the Nash bargaining solution associated with* $(s_1, s_2) \in S_1 \otimes S_2$*;*

(ii) there exists a unique $\mu > 0$*, such that*

(a) $[V_1(\sigma_1, \sigma_2) - V_1(s_1, s_2)] = \mu[V_2(\sigma_1, \sigma_2) - V_2(s_1, s_2)]$*; and*

(b) $V_1(\sigma_1, \sigma_2) + \mu V_1(\sigma_1, \sigma_2) \geq V_1(r_1, r_2) + \mu V_1(r_1, r_2)$,

for all $(r_1, r_2) \in S_1 \otimes S_2$.

Let (A1), (A2) hold. Let N denote the mapping which associates with each $(s_1, s_2) \in S_1 \otimes S_2$, the Nash bargaining solution (σ_1, σ_2). Let $Z = W \circ N$; that is $Z : S_1 \otimes S_2 \rightarrow R^2$, with $Z_i(s_1, s_2) = [V_1(N(s_1, s_2)), V_2(N(s_1, s_2))]$, $i = 1, 2$.

Definition 6 A Nash-optimal threat decision pair is a Nash-equilibrium of the game with payoff functions Z_1, Z_2.

From Lemma 4 and Definition 6, one can easily deduce the following:

Lemma 5 *Let (A1), (A2) hold. Then* (s_1^*, s_2^*) *is Nash-optimal threat decision pair if and only if the saddle-point condition*

$$V_1(s_1^*, s_2) - \mu V_2(s_1^*, s_2) \geq V_1(s_1^*, s_2^*) - \mu V_2(s_1^*, s_2^*) \geq V_1(s_1, s_2^*) - \mu V_2(s_1, s_2^*)$$

is satisfied for all $(s_1, s_2) \in S_1 \otimes S_2$.

From Corollary 1, we see that, *under* (A1) *and* (A2), *a Nash-optimal threat decision pair is special case of a C-optimal threat decision pair.*

5. C-Optimal Threat Strategy Pairs in Two-Player Differential Game

Consider now a two person-person differential game with state equations

$$dx(t)/dt = f(x(t), p^1(x(t), p^2(x(t))); \qquad (2)$$

where $x = (x_1, x_2, \ldots x_n) \in X$, $x_n \equiv t$, X is a domain in Euclidean space E^n, and f is Borel measurable on $X \otimes E^{d_1} \otimes E^{d_2}$. Let P^i denote the space of all Borel measurable functions from X into E^{d_i}, $i = 1, 2$. A *strategy* $p^i : X \to E^{d_i}$ is *admissible* if and only if $p^i \in P^i$ and

$$p^i(x) \in U^i(x) \qquad \text{for all } \ x \in X,$$

for given functions

$$U^i : X \to \text{set of all nonempty subsets of } \ E^{d_i}$$

We suppose that the target θ is a subset of ∂X.

A strategy pair $p = (p^1, p^2)$ is *playable* at x^o if it is admissible and generates at least one *terminating path* $x(.) : [t_o, t_f] \to X \cup \theta$, solution of (2), such that $x(t_o) = x^o$, $x(t_f) \in X$ for all $t \in [t_o, t_f)$, and $x(t_f) \in \theta$. Let $J(x^o)$ denote the set of all strategy pairs playable at x^o; we assume that $J(x^o)$ is nonempty. Let $I(x^o, p)$ denote the set of all terminating paths generated by p from x^o.

The payoffs corresponding to path $x(.) : [t_o, t_1] \to X \cup \theta$, generated by a pair $p \in J(x^o)$ from x^o, are given by

$$V_i(x^o, p, x(.)) := \int_{t_o}^{t_1} h_i(x(t), p(x(t)))\, dt, \qquad i = 1, 2,$$

where $h_i : X \otimes E^{d_1} \otimes E^{d_2} \to R^1$ are given real valued Borel measurable functions satisfying a polynomial growth condition.

Definition 7 A pair $p^* = (p^{1*}, p^{2*})$ is a C-*optimal threat strategy pair at* x^o if and only if

(i) $p^* \in J(x^o)$, and

(ii) $V_i(x^o, p^*, x^*(.)) = V_i(x^o, p^*, x^{**}(.)) := V_i^*(x^o, p^*)$, $i = 1, 2$, for all $x^{**}(.) \in I(x^o, p^*)$,

(iii) $(V_1(x^o, p^{1*}, p^2, x(.)), V_2(x^o, p^{1*}, p^2, x(.)))\ (\geq)\ (V_1^*(x^o, p^*), V_2^*(x^o, p^*))$ for all $(p^{1*}, p^2) \in J(x^o)$, and $x(.) \in I(x^o, p^{1*}, p^2)$; and

(iv) $(V_1^*(x^o, p^*), V_2^*(x^o, p^*))\ (\geq)\ (V_1(x^o, p^1, p^{2*}, x(.)), V_2(x^o, p^1, p^{2*}, x(.)))$ for all $(p^1, p^{2*}) \in J(x^o)$, and all $x(.) \in I(x^o, p^1, p^{2*})$.

Below we state a sufficiency theorem. Before giving the theorem we need some definitions.

Definition 8 A *denumerable decomposition* D of a subset X of E^n is a denumerable collection of pairwise disjoint subset whose union is X. We shall write $D = \{X_k : k \in \tau\}$ where τ is a denumerable index set of pairwise disjoint subsets.

Definition 9 Let X be a subset of E^n and D a denumerable decomposition of X. A continuous $V : X \to R^1$ is continuously differentiable with respect to D if and only if there exists a collection $\{(D_k, V_k) : k \in \tau\}$ such that D_k is an open set containing X_k, $V_K : D_k \to R^1$ is continuously differentiable, and $V_k(x) = V(x)$ for $x \in X_k$.

Now we are ready to state

Theorem 1 *A strategy pair $p^* = (p^{1*}, p^{2*}) \in J(x^o)$ is a C-optimal threat strategy pair at x^o if there exists a denumerable decomposition D of X, two constants α_1, $\alpha_2 > 0$, and two continuous functions $V_i^* : X \cup \theta \to R^1$, $i = 1, 2$ which are continuously differentiable with respect to D, such that*

(i) $\int_{t_o}^{t_1^*} h_i(x^*(t), p^*(x^*(t)))\, dt = V_i^*(x^o) \qquad$ *for all* $x^*(.) \in I(x^o, p^*)$,
where t_1^* is the terminating time for $x^*(.)$;

(ii) $h_1(x, u, p^{2*}(x)) - \alpha_1 h_2(x, u, p^{2*}(x))$
$+grad(V_1^{*k} - \alpha_1 V_2^{*k})(x).f(x, u, p^{2*}(x)) \leq 0$
for all $x \in X_k$, $u \in U^1(x)$, $k \in \tau$;

(iii) $h_1(x, p^{1*}(x), v) - \alpha_2 h_2(x, p^{1*}(x), v)$
$+grad(V_1^{*k} - \alpha_2 V_2^{*k})(x).f(x, p^{1*}(x), v) \geq 0$
for all $x \in X_k$, $v \in U^2(x)$, $k \in \tau$;

(iv) $V_i^*(x) = 0$ *for all* $x \in \theta$, $i = 1, 2$;
where $\{(D_k, V_i^{*k}) : k \in \tau\}$ *is a collection associated with* V_i^* *and* $D = \{X_k : k \in \tau\}$ *for each* $i = 1, 2$.

That theorem is straightforward consequence of Theorem 1 of Stalford and Leitmann (1973), and Lemma 3.

6. Examples of C-Optimal Threat Strategies in Collective Bargaining

Example 1

Theorem 1 can be be easily applied to the differential game considered by Liu (1973). The state equations are

$$dx_1(t)/dt = \boldsymbol{u}_1(t) + \boldsymbol{u}_2(t), \qquad dx_2(t)/dt = 1, \qquad t \in [t_o, t_1],$$

where $u_1(t) = p^1(x(t))$, $u_2(t) = p^2(x(t))$, $x(t) = (x_1(t), x_2(t))$,

$$u_1(t) \in [-1, 0], \qquad u_2(t) \in [-1, 0],$$

$$X = \{(x_1, x_2) : x_1 > 0\}, \quad \theta = \{(x_1, x_2) : x_1 = 0\}.$$

The payoff functions are

$$h_1 = -1 + ax_1, \qquad h_2 = -1 - ax_1,$$

where $a > 0$ is a constant.

Theorem 1 gives sufficiency conditions for Nash-optimality of strategy pair $p^* = (p^{1*}, p^{2*}) \in J(x^o)$ with respect to the pair of value functions

$$(V_1(x^o, p, x(.)) - \alpha_1 V_2(x^o, p, x(.)),\, V_1(x^o, p, x(.)) - \alpha_2 V_2(x^o, p, x(.))).$$

Necessary conditions for Nash-optimality of that strategy pair are given by Theorem 1 of Stalford and Leitmann (1973). These necessary conditions will provide us with *candidates* to Nash-optimality; then, *if we can verify that these candidates are indeed Nash-optimal*, Lemma 3 tells us that p^* is C-optimal. This invites us to first use the necessary conditions

$$H_1(x, u_1, \lambda') = -1 + ax_1 + \alpha_1(1 + ax_1) + \lambda'(u_1 + p^{2*}(x)) \le 0, \tag{3}$$
$$H_2(x, u_1, \lambda'') = -1 + ax_1 + \alpha_2(1 + ax_1) + \lambda''(p^{1*}(x) + u_2) \ge 0, \tag{3'}$$

where $x = x^*(t)$, $\lambda' = \lambda'(t)$, $\lambda'' = \lambda''(t)$, $t \in [t_o, t_1]$, from which follows that:

$$\begin{aligned} u_1^*(t) &= 0 \quad \text{if } \lambda'(t) > 0 \\ u_1^*(t) &= -1 \quad \text{if } \lambda'(t) < 0 \\ u_2^*(t) &= -1 \quad \text{if } \lambda''(t) > 0 \\ u_2^*(t) &= 0 \quad \text{if } \lambda''(t) < 0, \end{aligned} \tag{4}$$

where $u_1^*(t) = p^{1*}(x^*(t))$, $u_2^*(t) = p^{2*}(x^*(t))$.
Disregarding the case $(u_1^*(t) = 0,\ u_2^*(t) = 0)$, for which the corresponding strategy pair is not playable, one can readily deduce from (3), (3′) and (4) that

$$\begin{aligned} p^{1*}(x) &= 0 \quad \text{if } ax_1 > b, \\ p^{1*}(x) &= -1 \quad \text{if } ax_1 < b, \\ p^{2*}(x) &= -1 \quad \text{if } ax_1 > c, \\ p^{2*}(x) &= 0 \quad \text{if } ax_1 < c, \end{aligned}$$

where

$$b = \frac{1 - \alpha_1}{1 + \alpha_1}, \quad c = \frac{1 - \alpha_2}{1 + \alpha_2}. \tag{5}$$

In the following, we shall assume that $c < b$, which ensures that $p^* = (p^{1*}, p^{2*})$ is playable for each initial point. If $b < c$, playability is ensured for initial points x^o such that $ax_1^o < b$, only.

To verify that p^* is indeed Nash-optimal, and accordingly C-optimal, we apply Theorem 1. We define the decomposition

$$D = \{X_1, X_2, X_3\}$$

where

$X_1 = \{(x_1, x_2) : x_1 > b/a\}$,
$X_2 = \{(x_1, x_2) : c/a < x_1 < b/a\}$,
$X_3 = \{(x_1, x_2) : x_1 < c/a\}$.

Calculating the payoffs $V_j(x^o, p^*, x^*(.))$, $j = 1, 2$, we find

$$\begin{aligned}
V_1^*(x) &= -x_1 + ax_1^2/2 + (b-c)/2a - (b^2 - c^2)/4a, && \text{for } x \in X_1,\\
V_2^*(x) &= -x_1 + ax_1^2/2 + (b-c)/2a - (b^2 - c^2)/4a, && \text{for } x \in X_1,\\
V_1^*(x) &= -x_1/2 + ax_1^2/4 - c/2a + c^2/4a, && \text{for } x \in X_2,\\
V_2^*(x) &= -x_1/2 - ax_1^2/4 - c/2a - c^2/4a, && \text{for } x \in X_2,\\
V_1^*(x) &= -x_1 + ax_1^2/2, && \text{for } x \in X_3,\\
V_2^*(x) &= -x_1 - ax_1^2/2, && \text{for } x \in X_3.
\end{aligned}$$

One can readily verify that all requirements of Theorem 1 are satisfied, with α_1, α_2 given by (5).

If $b = c$, then we have $V_1^*(x) = -x_1 + ax_1^2/2$, $V_2^*(x) = -x_1 - ax_1^2/2$, for all $x \in X$, and requirements of Theorem 1 are met if

$$\alpha_1 = \alpha_2 = (1-b)/(1+b) = (1-c)/(1+c).$$

Thus, $\alpha_1 = \alpha_2$ gives one special case of optimal threat strategies. In that case, Theorem 1 reduces to sufficiency conditions for p^* to be a saddle-point of

$$V_1(x^o, p, x(.)) - \alpha V_2(x^o, p, x(.))$$

where $\alpha = \alpha_1 = \alpha_2$.

Example 2

Theorem 1 can be easily applied to a dynamic game model of labor-management negotiations during a period that may but need not include a strike.

Let $[0, T]$ denote the unspecified interval during which negotiations take place. At $t \in [0, T]$, let $o(t)$ denote the offer by management of total wages per unit time, $d(t)$the demand by labor for total wages per unit time, and $k =$const the gross profit of company per unit time. The evolution of the game is governed by differential equations

$$\begin{aligned}
do(t)/dt &= u(t), && u(t) \in [0,1],\\
dd(t)/dt &= -v(t), && v(t) \in [0,1].
\end{aligned}$$

Starting from given initial conditions, settlement is reached the first time the offer equals demand, that is, at time T such that $d(T) - o(T) = 0$.

Thus, management chooses the rate of change of the offer, and the union chooses the rate of change of the demand. In addition, the union has the option of calling, or not calling, a strike. We represent this by another control variable w for the union, where $w \in \{0,1\}$. We take $w = 1$ to correspond to a strike and $w = 0$ to the absence of strike.

The objective of management is to minimize the final offer $o(T)$ and the profit lost during strikes, assumed given by

$$\int_0^T \{w(x(t))[k - d(t)]\}dt.$$

The union, for its part, wishes to maximize the final offer $o(T)$ and minimize the wages lost during strikes, given by

$$\int_0^T \{w(x(t))o(t)\}dt.$$

We thus take the payoffs

$$V_1(x^o, u(.), v(.), w(.), x(.)) = -o(T) - a\int_0^T \{w(x(t))[k - d(t)]\}dt,$$

$$V_2(x^o, u(.), v(.), w(.), x(.)) = o(T) - b\int_0^T \{w(x(t))o(t)\}dt$$

for the management and union, respectively, where a, b are positive constants.

This example has been worked out by Ray (1981) from the point of view of C-optimal threat strategy pairs. The general conclusion is the following: (i) whether or not the union threatens to strike depends on whether the offer $o^*(t)$ is less or greater than a certain fraction of the potential profit $k - d^*(t)$; and (ii) if a strike is threatened, then the union will also threaten not to lower the demand as termination is approached. This example has been discussed earlier by Leitmann (1973) who characterizes rational behaviour by saddle-point condition. It follows that Leitmann's solution is Nash-optimal threat strategy solution.

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