Control and Cybernetics

vol. **23** (1994) No. 1/2

Regularity of solutions in stability analysis of optimization and optimal control problems

by

Kazimierz Malanowski*)

Systems Research Institute
Polish Academy of Sciences
Warsaw
Poland

Stability of solutions to cone–constrained optimization problems in Banach spaces is investigated in the situation where the so called norm discrepancy takes place. A two–norm approach is proposed, in which an additional information on regularity of solutions is used.

The obtained abstract stability results are applied to control and state constrained optimal control problems for nonlinear o.d.e..

1. Introduction

Stability of solutions to finite dimensional mathematical programs has been studied for many years and the obtained results are fairly complete (cf. e.g., Auslender,Cominetti,1990; Bonnans,1992; Gauvin, Janin,1988; Gollan,1981; Jittorntrum,1984; Kojima,1980; Robinson,1987).

In recent years there appeared also several papers devoted to stability analysis of cone–constrained optimization problems in Banach space (cf. Alt,1990A; Alt,1990B; Dontchev,Hager,1993; Dontchev at al.,1994; Ioffe,1991; Ito,Kunisch, 1992; Malanowski 1992; Malanowski,1993; Malanowski,1994; Shapiro,1992; Shapiro,1994; Shapiro,Bonnans,1992; Tröltzsch,1991). Some of these papers present applications to optimal control.

However, there are serious difficulties in a direct application of stability results for abstract mathematical programs to nonlinear optimal control problems. These difficulties are due to the phenomenon of the so called *norm discrepancy*, which is intrinsicly connected with nonlinear optimal control problems

*) Supported by grant No. 3 P403 002 05 from Komitet Badań Naukowych [State Committee for Scientific Research].

(cf. Ioffe,1979; Maurer,1981). Namely, the differentiability properties as well as constraint qualifications used in stability analysis are satisfied in a stronger norm (of L^∞–type), whereas the second order sufficient optimality condition, which is also needed, holds in a weaker norm (of L^2–type).

A method to overcome these difficulties was proposed in Malanowski (1993), Malanowski (1994), where a modification of Robinson's generalized implicit function theorem, Robinson (1980), was used. This method is called *two–norm approach* since it uses both norms at the same time. The crucial role is played here by the regularity of optimal solutions, which has to be additionally studied. Therefore, the method is quite complicated technically.

The purpose of this paper is to present the very essence of the two–norm approach, which is rather simple and to emphasize the relations to finite dimensional mathematical programs.

Accordingly, only the general outlines of the proofs are given, and for technicalities the reader is refered to the original papers. Most of the presented results are contained in the papers by the author himself Malanowski (1992), Malanowski (1993), Malanowski (1994). The material related to second order sufficient optimality conditions can be found in papers by Dontchev,Hager (1993) and Dontchev et al. (1994).

The organization of the paper is the following. In Section 2 stability of solutions to linear–quadratic problems with respect to additive perturbations is analysed. In Section 3 the same class of problems is considered but in the situation of norm discrepancy.

Section 4 is devoted to nonlinear problems. It is recalled how Robinson's implicit function theorem for generalized equations, Robinson (1980), allows to reduce local stability analysis of nonlinear sytems to such an analysis for linear–quadratic problems with respect to additive perturbations, which was discussed in Section 2.

Nonlinear optimization problems in case of the norm discrepancy are considered in Section 5. It is shown that, if an additional information on regularity of the solutions is available, then a modification of Robinson's theorem together with the results of Section 3 can be used in stability analysis.

Finally in Section 6 the abstract stability results obtained in Section 5 are applied to control and state constrained optimal control problems for nonlinear ordinary differential equations.

Some used notation:
$X, Y, Z, \ldots$ denotes Banach spaces and $\widehat{X}, \widehat{Y}, \widehat{Z}, \ldots$ Hilbert spaces. Asterisks denote dual spaces. $L(X,Y)$ is the Banach space of linear continuous operators from X into Y.
The norms in Banach and Hilbert spaces are denoted by $[\!]\cdot[\!]$ and $\|\cdot\|$, respectively, with a subscript refering to the space.
$(\cdot,\cdot)_X$ denotes the duality pairing between X and X^*.
$\mathbf{R}^n$ is the n–dimensional Euclidean space with the inner product denoted by $\langle x, y\rangle$ and the norm $|x| = \langle x, x\rangle^{\frac{1}{2}}$.

$L^2(0,T;\mathbf{R}^n)$ is the Hilbert space of square integrable vector functions, with the inner product

$$(x,y) = \int_0^T \langle x(t), y(t)\rangle \, dt$$

and the norm

$$\|x\|_2 = (x,x)^{\frac{1}{2}}.$$

$M^2(0,T;\mathbf{R}^n)$ denotes the Hilbert space of square integrable vector functions that assume a finite value at $t=0$, with the inner product

$$(x,y)_M = \langle x(0), y(0)\rangle + (x,y).$$

$L^\infty(0,T;\mathbf{R}^n)$ is the Banach space of essentially bounded vector functions with the norm

$$\|x\|_\infty = \max_i \operatorname{ess} \sup_{t\in[0,T]} |x^i(t)|.$$

$C^0(0,T;\mathbf{R}^n)$ and $C^1(0,T;\mathbf{R}^n)$ are the spaces of continuous and continuously differentiable vector functions, respectively, equipped with the usual norms.

$$W^{1,p}(0,T;\mathbf{R}^n) = \{x \in L^p(0,T;\mathbf{R}^n) \mid \dot{x} \in L^p(0,T;\mathbf{R}^n)\}, \qquad p = 2, \infty,$$

denote the Sobolev spaces of absolutely continuous functions with the norms

$$\|x\|_{1,2} = \{|x(0)|^2 + \|\dot{x}\|_2^2\}^{\frac{1}{2}}$$

and

$$\|x\|_{1,\infty} = \max\{|x(0)|, \|\dot{x}\|_\infty\},$$

respectively.

2. Stability analysis of linear–quadratic problem

In this section we are going to analyse stability of solutions to cone–constrained linear–quadratic problems of optimization in Banach spaces subject to linear perturbations.

Let Z and Y be two Banach spaces, space of arguments and constraints, respectively. Moreover there are given Hilbert spaces $\widehat{Z}$ and $\widehat{Y}$. We assume

$$Z \subset \widehat{Z} = \widehat{Z}^* \subset Z^*, \qquad Y \subset \widehat{Y} = \widehat{Y}^* \subset Y^* \tag{2.1}$$

with all embeddings being dense and continuous.

In the space Y there is given a closed cone K with the vertex at the origin, which induces a partial order in Y. We denote

$$K^+ = \{\lambda \in Y^* \mid (\lambda, y)_Y \geq 0 \quad \text{for all} \quad y \in K\}.$$

Let us denote $X = Z \times Y^*$, $X^* = Z^* \times Y$, $\delta = (a, b) \in X^*$ and let $\mathcal{O}^\rho \subset X^*$ be the open ball of radius ρ around the origin.

There are given two linear continuous operators

$$Q \in L(Z, Z^*), \qquad D \in L(Z, Y),$$

and elements $q \in Z^*$, $e \in Y$.

We consider the family of the following optimization problems depending on the additive parameter $\delta \in X^*$:

$$(L_\delta) \qquad \begin{array}{l} \min_{w \in Z} \left\{ \frac{1}{2}(Qw, w)_Z + (q + a, w)_Z \right\} \\ \text{subject to} \\ Dw + e + b \in K. \end{array}$$

We are going to investigate local stability of solutions to (L_δ). More precisely, we are looking for sufficient conditions under which there exists $\rho > 0$ such that for each $\delta \in \mathcal{O}^\rho$ Problem (L_δ) has a unique solution w_δ, which is a Lipschitz continuous function of δ.

Let us assume for a moment that the following two conditions are satisfied:

$$DZ = Y, \text{ i.e. } D \in L(Z, Y) \text{ is surjective,} \tag{2.2}$$

$$(Qw, w)_Z \geq \alpha [\![w]\!]_Z^2, \qquad \alpha > 0 \tag{2.3}$$

for all $w \in \Phi := \{w \in Z \mid Dw \in K + (-K)\}$.

If some additional conditions are satisfied (see Theorem II.10 in Brezis,1983), in particular if Z and Y are Hilbert spaces, then by (2.2) there exists a right invers $T \in L(Y, Z)$ of D. For the sake of simplicity we assume, in this section, that $Z = Z^*$ and $Y = Y^*$ are Hilbert spaces. Let us introduce the change of variables putting:

$$v = w + T(e + b). \tag{2.4}$$

In terms of v Problem (L_δ) takes on the form:

$$\begin{array}{l} \min_{v \in Z} \left\{ \frac{1}{2}(Qv, v)_Z + (q' + a', v)_Z \right\} \\ \text{subject to} \\ Dv \in K, \end{array} \tag{2.5}$$

where

$$q' = q - QTe, \qquad a' = a - QTb. \tag{2.6}$$

By (2.2) the feasible set of (2.5) is non empty and by (2.3) there exists a unique solution v_δ of (2.5), which is characterized by the following variational inequality:

$$(Qv_\delta + q' + a', v - v_\delta)_Z \geq 0 \qquad \text{for all feasible } v.$$

Let v_{δ_1}, v_{δ_2} be the solutions to (2.5) corresponding to the values δ_1 and δ_2 of the parameter. We have

$$\begin{aligned}(Qv_{\delta_1} + q' + a_1', v_{\delta_2} - v_{\delta_1})_Z \geq 0,\\ (Qv_{\delta_2} + q' + a_2', v_{\delta_1} - v_{\delta_2})_Z \geq 0.\end{aligned}$$

Adding these inequalities and using (2.3) we obtain

$$\begin{aligned}&\alpha\|v_{\delta_1} - v_{\delta_2}\|_Z \leq (Q(v_{\delta_1} - v_{\delta_2}), (v_{\delta_1} - v_{\delta_2}))_Z \leq\\ &\leq (a_2' - a_1', v_{\delta_1} - v_{\delta_2})_Z \leq \|a_1' - a_2'\|_{Z^*}\|v_{\delta_1} - v_{\delta_2}\|_Z,\end{aligned}$$

i.e., in view of (2.6).

$$\|v_{\delta_1} - v_{\delta_2}\|_Z \leq \beta\ \|\delta_1 - \delta_2\|_{X^*} \tag{2.7}$$

By (2.2) there exists a unique Lagrange multiplier μ_δ such that the following Kuhn–Tucker conditions hold:

$$\begin{aligned}&Qv_\delta + q' + a' - D^*\mu_\delta = 0,\\ &(\mu_\delta, D^*v_\delta)_Y = 0, \qquad \mu_\delta \in K^+.\end{aligned} \tag{2.8}$$

In view of (2.2), the first equation in (2.8) yields

$$\mu_{\delta_1} - \mu_{\delta_2} = T^*(Q(v_{\delta_1} - v_{\delta_2}) + (a_1' - a_2')),$$

i.e., by (2.6) and (2.7)

$$\|\mu_{\delta_1} - \mu_{\delta_2}\|_{Y^*} \leq \beta\|\delta_1 - \delta_2\|_{X^*}. \tag{2.9}$$

Thus we have obtained Lipschitz continuity of the solutions to (L_δ) and of the associated Lagrange multipliers.

The assumptions (2.2) and (2.3) are very strong and we would like to weaken them as much as possible, still preserving the above Lipschitz continuity properties.

Before doing that, let us recall some stability results for quadratic programming problems in finite dimension, i.e., let us assume for a moment that $Z = \mathbf{R}^n$, $Y = \mathbf{R}^m$ and $K \subset \mathbf{R}^m$ is the non-negative octant. Then problem (L_δ) takes on the form:

$$(MP_\delta) \qquad \begin{aligned}&\min_{w\in\mathbf{R}^n}\left\{\tfrac{1}{2}\langle Qw, w\rangle + \langle q + a, w\rangle\right\}\\ &\text{subject to}\\ &\langle d^i, w\rangle + e^i + b^i \geq 0, \qquad i = 1, 2, ..., m.\end{aligned}$$

Let w_0 be the solution to (MP_0). Denote by

$$I = \left\{i \in \{1, 2, ..., m\} \mid \langle d^i, w_0\rangle + e^i = 0\right\}$$

the set of indices of constraints active at v_0 and by D_I the matrix with rows d^i, $i \in I$. We say that the condition of *linear independence of the gradients of active constraints* (LIC) is satisfied if

$$D_I \text{ is of full rank.} \tag{2.10}$$

If (2.10) holds, then there exists a unique Lagrange multiplier

$$\mu_0 = (\mu_0^1, \mu_0^2, ..., \mu_0^m)$$

associated with v_0. Denote $J = \{i \in I \mid \mu_0^i > 0\}$. We say that the *strong second order sufficient optimality condition* (SOC) holds if

$$\begin{aligned} &\langle Qw, w\rangle \geq \alpha |w|^2 \\ &\text{for all } w \in \{w \in \mathbf{R}^n \mid \langle d^i, w\rangle = 0, \quad i \in J\}. \end{aligned} \tag{2.11}$$

Let us modify (MP_δ) as follows:

$$(\widetilde{MP}_\delta) \qquad \begin{aligned} &\min_{v \in \mathbf{R}^n} \left\{\tfrac{1}{2}\langle Qv, v\rangle + \langle q+a, v\rangle\right\} \\ &\text{subject to} \\ &\langle d^i, v\rangle + e^i + b^i = 0, \qquad i \in J, \\ &\langle d^i, v\rangle + e^i + b^i \geq 0, \qquad i \in I \setminus J\,. \end{aligned}$$

It ie easy to see that if (LIC) and (SOC) hold then for $(\widetilde{MP}_\delta)$ conditions (2.2) and (2.3) are satisfied, so there exists a unique solution v_δ of $(\widetilde{MP}_\delta)$ and a unique associated Lagrange multiplier ν_δ. Both v_δ and ν_δ are Lipschitz continuous functions of δ.

Certainly $v_0 = w_0$ and $\nu_0 = \mu_0$. Moreover, by Lipschitz continuity of v_δ and ν_δ, for $\delta > 0$ sufficiently small we have

$$\begin{aligned} &\langle d^i, v_\delta\rangle + e^i + b^i > 0 \qquad &&\text{for } i \notin I, \\ &\nu_\delta^i > 0 &&\text{for } i \in J. \end{aligned} \tag{2.12}$$

It implies that $v_\delta = w_\delta$ and $\nu_\delta = \mu_\delta$, i.e., w_δ and μ_δ are locally Lipschitz continuous functions of δ.

An idea similar to that presented above will be also used in stability analysis of solutions to infinite dimensional problems (L_δ).

Let us start with such a reformulation of the definitions of (LIC) and (SOC), which would fit to infinite dimensional situations.

For $y \in K$ and $\mu \in K^+$ define the subspaces:

$$\begin{aligned} M_y &= (K + [y]) \cap (-K + [y]) \subset Y, \\ N_\mu &= (K^+ + [\mu]) \cap (-K^+ + [\mu]) \subset Y^*, \end{aligned} \tag{2.13}$$

where $[y] = \{\kappa y \mid \kappa \in \mathbf{R}^1\}$, and denote $M_0 = M_{y_0}, N_0 = N_{\mu_0}$, where $y_0 = Dw_0 + e$, and $N_0^\perp = \{y \in Y \mid (\mu, y)_Y = 0 \text{ for all } \mu \in N_0\}$. It can be easily checked that in case of (MP_0), M_0 is the subspace generated by nonzero components y_0^i of the vector y_0, while $N_0^{\perp} = \{y \in Y \mid \langle \mu^i, y \rangle = 0,\ i \in J\}$. Hence conditions (2.10) and (2.11) can be rewritten in the form

$$\text{(LIC)} \qquad DZ + M_0 = Y,$$

$$\text{(SOC)} \qquad (Qw, w)_Z \geq \alpha [\![w]\!]_Z^2 \quad \text{for all } w \in \{w \in Z \mid Dw \in N_0^{\perp}\}.$$

In finite dimensional case conditions (LIC) and (SOC) are stable under small perturbations, in the sense that they are preserved if we substitute M_0 and N_0 by M_y and N_λ with $y \in K$ and $\mu \in K^+$ sufficiently close to y_0 and μ_0, respectively. This fact has been reflected in (2.12). Such a stability is not automatic in infinite dimensional spaces. To assure it we will strengthen conditions (LIC) and (SOC) introducing some "margin of freedome". Namely we assume:

(I 1) There exists a closed subspace $M \subset M_0$ and a linear continuous mapping $\Pi : Y \mapsto M$ such that

$$DZ + \Pi Y = Y \tag{2.14}$$

and moreover there exists a neighborhood $\mathcal{Y}_0$ of y_0 such that

$$M \subset M_y \qquad \text{for all } y \in \mathcal{Y}_0 \cap K. \tag{2.15}$$

(I 2) There exists a closed subspace $N \subset N_0$ and a constant $\alpha > 0$ such that

$$(Qw, w)_Z \geq \alpha [\![w]\!]_Z^2 \quad \text{for all } z \in \{z \in Z \mid Dz \in N^{\perp}\} \tag{2.16}$$

and moreover there exists a neighborhood Λ_0 of μ_0 such that

$$\Lambda_0 \cap (K^+ + N) \subset K^+. \tag{2.17}$$

The modified problem $(\widetilde{L}_\delta)$ analogous to $(\widetilde{MP}_\delta)$ and corresponding to (L_δ) is defined as follows:

$$(\widetilde{L}_\delta) \qquad \begin{array}{l} \min_{v \in Z} \left\{ \frac{1}{2}(Qv, v)_Z + (q + a, v)_Z \right\} \\ \text{subject to} \\ Dv + e + b \in \widetilde{K} := \overline{(K \cap N^{\perp}) + M}. \end{array}$$

In order to assure that condition (2.2) is satisfied, we enlarge the space of arguments to $Z \times Y$ and modify $(\widetilde{L}_\delta)$ as follows:

$$(\widetilde{L}'_\delta) \qquad \begin{array}{l} \min_{(v,u) \in Z \times Y} \left\{ \frac{1}{2}[(Qv, v)_Z + (u, u)_Y] + (q + a, v)_Z \right\} \\ \text{subject to} \\ Dv + \Pi u + e + b \in \widetilde{K}. \end{array}$$

Due to the additional term in the cost functional the coercivity condition (2.3) is satisfied. Hence by (2.7) and (2.9) the solutions (v_δ, u_δ) of $(\widetilde{L}'_\delta)$ and the associated Lagrange multiplers ν_δ are Lipschitz continuous functions of δ.

On the other hand, in view of the definitions of Π and $\widetilde{K}$ it can be shown (cf. Lemma 3.7 in Malanowski,1992 as well as Lemmata 4.3 and 4.4 in Malanowski,

1993) that a pair $(v, u) \in Z \times Y$ is feasible for $(\widetilde{L}'_\delta)$ if and only if $v \in Z$ is feasible for $(\widetilde{L}_\delta)$. Hence, it follows from the form of the cost functional in $(\widetilde{L}'_\delta)$ that for any $\delta \in X^*$ the solution to $(\widetilde{L}'_\delta)$ has the form $(v_\delta, 0)$, where v_δ is the solution to $(\widetilde{L}_\delta)$. Certainly also the Lagrange multiplier ν_δ is the same for $(\widetilde{L}'_\delta)$ and $(\widetilde{L}_\delta)$.

At the reference point $\delta = 0$ we have

$$v_0 = w_0, \qquad \nu_0 = \mu_0,$$

where w_0 is the solution to (L_0) and μ_0 — the associated Lagrange multiplier. It is shown in Malanowski (1993) (cf. Lemmata 4.3 and 4.4) that for δ sufficiently small this property is preserved, i.e., we have

$$v_\delta = w_\delta, \qquad \nu_\delta = w_\delta. \tag{2.18}$$

Hence we finally obtain:

THEOREM 2.1 *If Z and Y are Hilbert spaces and assumptions (I 1), (I 2) are satisfied, then there exists a constant $\rho > 0$ such that for all $\delta \in \mathcal{O}^\rho$ there is a unique solution w_δ of (L_δ) and a unique associated Lagrange multiplier μ_δ, and both w_δ and μ_δ are Lipschitz continuous functions of δ.* ■

In order to illustrate the nature of assumptions (I 1) and (I 2) as well as the reason why for δ sufficiently small the solutions to (L_δ) and $(\widetilde{L}_\delta)$ coincide, let us consider the following simple example.

EXAMPLE 2.2 Let $Z = Y = L^\infty(0,1)$, $K \subset Y$ be the cone of non-negative functions,

$$(Qw, w)_Z = \int_0^1 g(t) w^2(t) dt, \quad (Dw)(t) = d(t)w(t), \quad \text{where } g, d \in C^0(0,1).$$

Assume that w_0 and μ_0 are given by functions of class $C^0(0,1)$.

For $\epsilon \geq 0$ and $\eta \geq 0$ we define the sets (see Figure 1):

$$\Omega^\epsilon = \{t \in [0,1] \mid y_0(t) := Dw_0(t) + e(t) > \epsilon\},$$
$$\Xi^\eta = \{t \in [0,1] \mid \mu_0(t) > \eta\}.$$

Let us introduce the following subspaces:

$$M^\epsilon = \{y \in L^\infty(0,T) \mid y(t) = 0 \quad \text{for all } t \in [0,1] \setminus \Omega^\epsilon\},$$
$$N^\eta = \{\mu \in L^\infty(0,T) \mid \mu(t) = 0 \quad \text{for all } t \in [0,1] \setminus \Xi^\eta\}.$$

It is easy to see that for any $\epsilon > 0$ and $\eta > 0$ we have $M^\epsilon \subset M_0$ and $N^\eta \subset N_0$.

Let us choose any $\epsilon > 0$ and $\eta > 0$ and define $M = M^\epsilon$, $N = N^\eta$. Note that M and N are closed in $L^\infty(0,1)$, but N is not closed in the space $Y^* = (L^\infty(0,1))^*$ — this fact will be discussed later on.

We define

$$\Pi y(t) = \begin{cases} y(t) & \text{if } t \in M^\epsilon \\ 0 & \text{if } t \notin M^\epsilon \end{cases}$$

It is easy to see that (2.15) is satisfied if we choose

$$\mathcal{Y}_0 = \{y \in L^\infty(0,1) \mid |y(t) - y_0(t)| < \frac{\epsilon}{2} \quad \text{for all } t \in [0,1]\}$$

while (2.14) holds if $d(t) \neq 0$ for all $t \in [0,1] \setminus \Omega^{2\epsilon}$.

Coercivity condition (2.16) is never satisfied in the norm of the original space $L^\infty(0,1)$. However, it is satisfied in the weaker norm of the space $L^2(0,1)$, provided that $g(t) > 0$ for all $t \in [0,1] \setminus \Xi^\eta$.

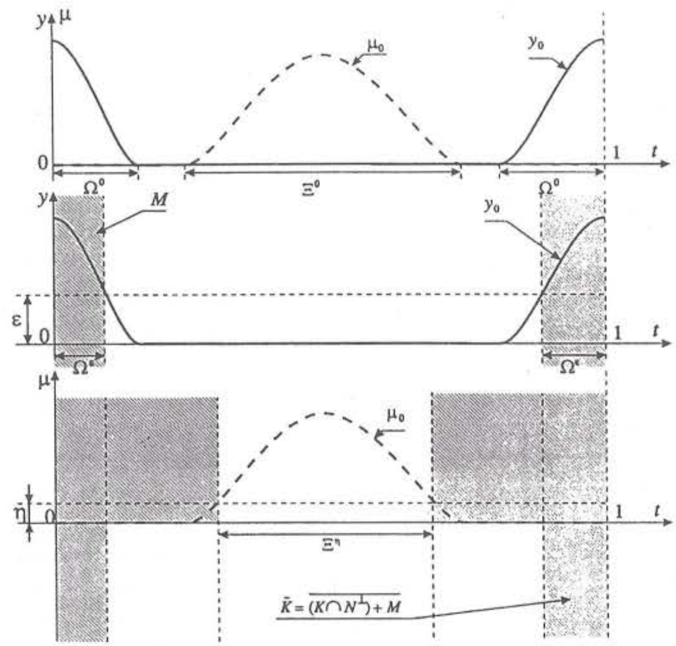

Figure 1

Constraints in $(\widetilde{L}_\delta)$ have the form:

$$d(t)v(t) + e(t) + b(t) \begin{cases} = 0 & \text{for} \quad t \in \Xi^\eta, \\ \geq 0 & \text{for} \quad t \in ([0,1] \setminus \Xi^\eta) \setminus \Omega^\epsilon \\ \text{arbitrary} & \text{for} \quad t \in \Omega^\epsilon. \end{cases} \tag{2.19}$$

To illustrate the reason why for δ sufficiently small the solutions to (L_δ) and $(\widetilde{L}_\delta)$ coincide, let us note that since $v_\delta \to v_0$, for $\delta = (a,b)$ sufficiently small

$$d(t)v_\delta(t) + e(t) + b(t) = d(t)v_0(t) + e(t) + b(t) + d(t)(v_\delta(t) - v_0(t)) > 0$$

for $t \in \Omega^\epsilon$. Hence constraints (2.19) can be changed as follows:

$$d(t)v(t) + e(t) + b(t) \begin{cases} = 0 & \text{for} \quad t \in \Xi^\eta, \\ \geq 0 & \text{for} \quad t \in [0,1] \setminus \Xi^\eta. \end{cases} \tag{2.20}$$

Certainly this mechanism has worked because we assured a "margin of freedom" choosing $\epsilon > 0$ in the definiton of M. For $\epsilon = 0$ it would not work anymore.

Assume for a moment that (2.17) holds in the sense of $L^\infty(0,1)$–topology, i.e., Λ_0 is a L^∞–neighborhood of μ_0 and $K^+ = \{\mu \in L^\infty(0,1) | \ \mu(t) \geq 0\}$. If $\nu_\delta \to \mu_0$ in $L^\infty(0,1)$ (which does not follow from (2.9)!) then, for δ sufficiently small we would have $\nu_\delta > 0$ for $t \in \Xi^\eta$. This means that on the set Ξ^η the equality type constraints could be substituted by inequality constraints and (2.20) could be rewritten as

$$d(t)v(t) + e(t) + b(t) \geq 0 \quad \text{for } t \in [0,1], \tag{2.21}$$

which shows that for δ sufficientlly small problems (L_δ) and $(\widetilde{L}_\delta)$ coincide. ■

3. Linear–quadratic problem with norm discrepancy

Example 2.2 illustrates the fundamental role of the topology in which Problem (L_δ) is considered. This role can be seen in several points:

(i) Stability of constraint qualifications (2.14) is satisfied in $L^\infty(0,1)$, but it would fail in $L^2(0,1)$.

(ii) Coercivity condition (2.16) never holds in $L^\infty(0,1)$, but it could be satisfied in $L^2(0,1)$.

(iii) Condition (2.17) (stability of coercivity) is never satisfied in the space $(L^\infty(0,1))^*$, but it could be satisfied in $L^\infty(0,1)$.

The phenomenon that constraint qualifications are satisfied in a stronger topology and coercivity condition in a weaker one is called the *norm discrepancy* and it is typical for optimal control problems (cf. Ioffe,1979; Maurer,1981). The presence of the norm discrepancy creates a serious difficulty in stability analysis of optimal control problems, especially for nonlinear systems. We will discuss it in details in this and next sections.

To cope with this difficulty we will work with two norm simultaneously, using the so called *two–norm approach*, Malanowski (1993), Malanowski (1994), in which regularity of the solutions plays a crucial role.

In this section we will apply this approach to linear–quadratic problems. To do so we have to reformulate Problem (L_δ).

Let us denote by $\widehat{K}$ the closure of K in $\widehat{Y}$. Furthermore we assume that

$$Q \in L(Z,Z) \cap L(\widehat{Z},\widehat{Z}), \quad D \in L(Z,Y) \cap L(\widehat{Z},\widehat{Y}), \quad q \in Z, \quad e \in Y. \tag{3.1}$$

For $\delta \in \widehat{X} = \widehat{Z} \times \widehat{Y}$ we consider along (L_δ) the following problem:

$$(\widehat{L}_\delta) \qquad \begin{array}{l} \min_{w \in \widehat{Z}} \left\{ \frac{1}{2}(Qw, w)_{\widehat{Z}} + (q + a, w)_{\widehat{Z}} \right\} \\ \text{subject to} \\ Dw + e + b \in \widehat{K}, \end{array}$$

which is identical with (L_δ) except that it is considered in $\widehat{Z}, \widehat{Y}$ rather than in Z, Y spaces. Certainly each solution to $(\widehat{L}_\delta)$ is also a solution to (L_δ), provided that $w_\delta \in Z$.

Assume that there exists a unique solution $w_0 \in Z$ of $(\widehat{L}_0)$, i.e., also of (L_0), and the following conditions analogous to (I 1) hold:

(II 1) There exists a subspace $M \subset M_0$ closed in Z, and a linear continuous mapping

$$\Pi \in L(Y, Y) \cap L(\widehat{Y}, \widehat{Y}), \qquad \Pi : Y \to M,$$

such that

$$DZ + \Pi Y = Y, \tag{3.2}$$

$$D\widehat{Z} + \Pi\widehat{Y} = \widehat{Y}. \tag{3.3}$$

Moreover there exists a Y–neighborhood $\mathcal{Y}_0$ of y_0 such that

$$M \subset M_y \quad \text{for all } y \in \mathcal{Y}_0 \cap K. \tag{3.4}$$

By (3.2) there exists a unique Lagrange multiplier $\mu_0 \in Y^*$ associated with w_0. We assume that μ_0 is more regular namely

(II 2) $\mu_0 \in Y$.

Instead of (I 1) we introduce the following condition, which reflects the norm discrepancy:

(II 3) There exists a subspace $N \subset N_0 \subset Y^*$ and a constant $\alpha > 0$ such that

$$(Qw, w)_{\widehat{Z}} \geq \alpha \|w\|^2_{\widehat{Z}} \qquad \text{for all } w \in \{w \in \widehat{Z} \mid Dw \in N^\perp\} \tag{3.5}$$

and moreover there exists a Y–neighborhood Λ_0 of μ_0 such that

$$\Lambda_0 \cap (K^+ + N) \subset K^+. \tag{3.6}$$

By the same reasoning as in Section 2 we can find that the solution w_δ of $(\widehat{L}_\delta)$ and the associated Lagrange multipliers μ_δ are Lipschitz continuous *in the sense of $\widehat{Z}$–norm, and $\widehat{Y}$–norm*, repectively:

$$\|w_{\delta_1} - w_{\delta_2}\|_{\widehat{Z}}, \|\mu_{\delta_1} - \mu_{\delta_2}\|_{\widehat{Y}} \leq c\|\delta_1 - \delta_2\|_{\widehat{X}}, \tag{3.7}$$

provided that

$$w_{\delta_i} \in \mathcal{Y}_0 \qquad \text{and } \mu_{\delta_i} \in \Lambda_0 \qquad i = 1, 2. \tag{3.8}$$

Certainly (3.7) does not assure that (3.8) is satisfied for all variations δ from any arbitrary small ball $\widehat{\mathcal{O}}^\rho$ in $\widehat{X}$. In order to get (3.8) we should assure that convergence in the weaker norm $\widehat{X}$ implies convergence in the stronger norm X. This would hold if (w_δ, μ_δ) belong to a certain set Γ compact in X. We can not expect it for all variations δ from any ball in $\widehat{X}$, because it would mean that the mapping

$$\mathcal{P} : \widehat{X} \mapsto \widehat{X}, \qquad \mathcal{P}(\delta) = (w_\delta, \mu_\delta) \tag{3.9}$$

is compact, which is not true.

Therefore, to assure that (w_δ, μ_δ) belong to a compact set we must restrict ourselves to a certain set Δ of more regular variations. We assume that such a set exists, namely:

(II 4) There exists a closed convex set $\Delta \subset \widehat{X}$ and a convex set $\widetilde{\Gamma}$ compact in X such that for any $\delta \in \Delta$, $(w_\delta, \mu_\delta) \in \widetilde{\Gamma}$.

From the above considerations we obtain:

THEOREM 3.1 *If assumptions (II 1)-(II 4) hold then there exist constants $\rho > 0$ and $c > 0$ such that for all $\delta \in \Delta \cap \widehat{\mathcal{O}}^\rho$ there exists a unique solution w_δ of (L_δ) and a unique associated Lagrange multiplier μ_δ such that*

$$\|w_{\delta_1} - w_{\delta_2}\|_{\widehat{Z}}, \|\mu_{\delta_1} - \mu_{\delta_2}\|_{\widehat{Y}} \leq c\|\delta_1 - \delta_2\|_{\widehat{X}} . \tag{3.10}$$

■

4. Stability analysis for nonlinear problems

This section is devoted to stability analysis of nonlinear cone-constrained optimization problems. In this analysis the stability results for linear-quadratic problems presented in Section 2 will be applied.

The same notation as in the previous section will be used. In addition, let us introduce a Banach space H, called the space of parameters, and let $G \subset H$ be an open set of admissible parameters. On $Z \times G$ there are defined two functions

$$F : Z \times G \mapsto \mathbf{R}^1, \qquad \varphi : Z \times G \mapsto Y.$$

We consider the family of the following optimization problems depending on the parameter $h \in G$:

$$(P_h) \qquad \begin{array}{l} \min_{z \in Z} F(z, h) \\ \text{subject to} \\ \varphi(z, h) \in K \subset Y. \end{array}$$

Let $\bar{h}$ be a fixed reference value of the parameter.

We assume:

(III 1) There exists a (local) solution $z_{\bar{h}}$ of $(P_{\bar{h}})$,

(III 2) $F(\cdot, \cdot)$ and $\varphi(\cdot, \cdot)$ are two times Fréchet differentiable,

(III 3) The point $z_{\bar{h}}$ is regular in the sense of Robinson, i.e.,

$$D_z\varphi(z_{\bar{h}}, \bar{h})Z - K + [\varphi(z_{\bar{h}}, \bar{h})] = Y.$$

Let us introduce the following Lagrangian associated with (P_h):

$$\mathcal{L}(\cdot,\cdot,\cdot) \,:\, Z \times Y^* \times G \mapsto \mathbf{R}^1, \qquad \mathcal{L}(z,\lambda,h) = F(z,h) - (\lambda, \varphi(z,h))_Y.$$

By (III 3) there exists a Lagrange multiplier $\lambda_{\bar{h}}$ associated with $z_{\bar{h}}$ such that the following Kuhn–Tucker conditions hold:

$$\begin{aligned} &D_z\mathcal{L}(z_{\bar{h}},\lambda_{\bar{h}},\bar{h}) = D_zF(z_{\bar{h}},\bar{h}) - D_z\varphi^*(z_{\bar{h}},\bar{h})\lambda_{\bar{h}} = 0,\\ &\left(\lambda_{\bar{h}}, \varphi(z_{\bar{h}},\bar{h})\right)_Y = 0, \qquad \lambda_{\bar{h}} \in K^+ \subset Y^*. \end{aligned}$$

We are going to investigate the existence, local uniquness and stability with respect to h of Kuhn–Tucker points $x_h = (z_h, \lambda_h) \in X$ of (P_h), i.e., of points which satisfy conditions:

$$\begin{aligned} &D_z\mathcal{L}(z_h,\lambda_h,h) = D_zF(z_h,h) - D_z\varphi^*(z_h,h)\lambda_h = 0,\\ &(\lambda_h, \varphi(z_h,h))_Y = 0, \qquad \lambda_h \in K^+ \subset Y^*. \end{aligned} \tag{4.1}$$

To this end we will use Robinson's implicit function theorem for generalized equations, Robinson (1980). To apply this theorem we need (cf. Malanowski, 1992) the following linear–quadratic approximation to $(P_{\bar{h}})$, perturbed by the parameter $\delta = (a,b) \in X^*$:

$$(LP_\delta) \qquad \begin{aligned} &\min\nolimits_{w\in Z} \left\{\tfrac{1}{2}(Qw,w)_Z + (q+a,w)_Z\right\}\\ &\text{subject to}\\ &Dw + e + b \in K, \end{aligned}$$

where

$$\begin{aligned} &Q = D^2_{zz}\mathcal{L}(z_{\bar{h}},\lambda_{\bar{h}},\bar{h}), \qquad &&D = D_z\varphi(z_{\bar{h}},\bar{h}),\\ &q = -Qz_{\bar{h}} + D_zF(z_{\bar{h}},\bar{h}), \qquad &&e = \varphi(z_{\bar{h}},\bar{h}) - Dz_{\bar{h}}. \end{aligned} \tag{4.2}$$

Following Robinson (1980), we call $(P_{\bar{h}})$ *strongly regular* at the point $(z_{\bar{h}}, \lambda_{\bar{h}})$ if:

(A) There exists $\rho > 0$ such that for all $\delta \in \mathcal{O}^\rho$ there is a unique pair $(w_\delta, \mu_\delta) \in Z \times Y^*$, where w_δ is the solution to (LP_δ) and μ_δ — the associated Lagrange multiplier and (w_δ, μ_δ) is a Lipschitz continuous function of δ.

The following fundamental theorem is due to Robinson (1980):

THEOREM 4.1 *Suppose that (A) is satisfied, then there exist a neighborhood $G_{\bar{h}} \subset G$ of $\bar{h}$ and a neighborhood $\mathcal{X}_{\bar{h}} \subset X$ of $x_{\bar{h}}$ such that for any $h \in G_{\bar{h}}$, (4.1) has a solution x_h unique in $\mathcal{X}_{\bar{h}}$, which is a Lipschitz continuous function of h.* ■

Theorem 4.1 allows us to reduce stability analysis of nonlinear problem (P_h) to such an analysis of the linearized problem (LP_δ). Cerainly (LP_δ) coincides with

Problem (L_δ) considered in Section 2. By Theorem 2.1 we find that condition (A) is satisfied if assumptions (I 1) and (I 2) hold.

It is easy to see that $w_0 = z_{\bar{h}}, \mu_0 = \lambda_{\bar{h}}, y_0 = Dw_0 + e = \varphi(z_{\bar{h}}, \bar{h})$. Hence, by (2.13) the subspaces $M_0 \subset Y$ and $N_0 \subset Y^*$ are given by

$$
\begin{aligned}
M_0 &= (K + [\varphi(z_{\bar{h}}, \bar{h})]) \cap (-K + [\varphi(z_{\bar{h}}, \bar{h})]), \\
N_0 &= (K^+ + [\lambda_{\bar{h}}]) \cap (-K^+ + [\lambda_{\bar{h}}]).
\end{aligned}
\tag{4.3}
$$

We assume:

(III 4) Condition (I 1) holds with M_0 given in (4.3),

(III 5) Condition (I 2) holds with N_0 given in (4.3).

It follows from Theorems 2.1 and 4.1 that if (III 1)–(III 5) hold, then for $h \in G_{\bar{h}}$ the solutions to (4.1) exist and are Lipschitz continuous functions of h.

On the other hand, it follows from Lemma 8 in Dontchev, Hager (1993) (cf. also Lemma 1 in Dunn,Tian,1992) that (III 5) together with (III 4) constitutes a (strong) second order sufficient optimality condition. In view of Lipschitz continuity of (z_h, λ_h), using the same argument as in the proof of Lemma 5.3 in Malanowski,1993, we find that conditions (III 4) and (III 5) are satisfied at (z_h, λ_h, h) for h sufficiently close to $\bar{h}$. This means that z_h is a (local) solution to (P_h) and λ_h — the associated Lagrange multiplier. Thus we obtain:

THEOREM 4.2 *If Z and Y are Hilbert spaces and assumptions (III 1)–(III 5) hold, then there exist a neighborhood $G_{\bar{h}} \subset G$ of $\bar{h}$ and a neighborhood $\mathcal{Z}_{\bar{h}} \subset Z$ of $z_{\bar{h}}$ such that for each $h \in G_{\bar{h}}$ there is a unique in $\mathcal{Z}_{\bar{h}}$ solution z_h to (P_h) and a unique associated Lagrange multiplier λ_h.*

Moreover, z_h and λ_h are Lipschitz continuous functions of h. ■

5. Nonlinear problems with norm discrepancy

The notion of norm discrepancy was introduced in Section 3. Now we are going to discuss stability of solutions to optimization problems for nonlinear systems in presence of the norm discrepancy. To illustrate better the nature of the norm discrepancy we start with the following classical example.

EXAMPLE 5.1 In (2.1) let us choose $Z = L^\infty(0,1)$ and $\widehat{Z} = L^2(0,1)$. Consider the integral functional

$$F(z) = \int_0^1 (z^2(t) - 1)^2 dt.$$

It is easy to see that F is well defined and differentiable on Z, but not on $\widehat{Z}$. However for $z \in Z$

$$(D^2_{zz} F(z)y, y) = 4 \int_0^1 (3z^2(t) - 1)y^2(t)dt$$

is a quadratic form well defined and continuous on $\widehat{Z}$.

Any function z, such that $|z(\cdot)| \equiv 1$ is a global minimizer of F on Z. Let us take as the minimizer $\widetilde{z}(\cdot) \equiv 1$. We have

$$(D^2_{zz}F(\widetilde{z})y, y) = 4\int_0^1 (3\widetilde{z}^2(t) - 1)y^2(t)dt = 8\|y\|_2^2, \tag{5.1}$$

however *does not exist* $c > 0$ such that

$$(D^2_{zz}F(\widetilde{z})y, y) \geq c\|y\|^2_\infty \quad \text{for all } y \in Z = L^\infty(0, T).$$

Just this phenomenon is called the *norm discrepancy*. Moreover, note that if for any $\epsilon \in (0, 1)$ we put

$$z_\epsilon = \begin{cases} \frac{1}{\sqrt{3}} & \text{for } t \in [0, \epsilon) \\ 1 & \text{for } t \in [\epsilon, 1] \end{cases}, \qquad y_\epsilon = \begin{cases} 1 & \text{for } t \in [0, \epsilon) \\ 0 & \text{for } t \in [\epsilon, 1] \end{cases}$$

then we have $\|\widetilde{z} - z_\epsilon\|_2 \to 0$ for $\epsilon \to 0$ and

$$(D^2_{zz}F(z_\epsilon)y_\epsilon, y_\epsilon) = 0,$$

which shows that coercitivity condition (5.1) is not stable under small perturbations in $L^2(0, 1)$, but it is stable under small perturbations in $L^\infty(0, 1)$. ■

We are going to analyse stability of solutions to (P_h) in the situation of the norm discrepancy using the two–norm approach introduced in Section 3. In a similar way as in Section 4 we assume:

(IV 1) There exists a (local) solution $z_{\bar{h}} \in Z$ of $(P_{\bar{h}})$,

(IV 2) $F(\cdot, \cdot)$ and $\varphi(\cdot, \cdot)$ are two times Fréchet differentiable on $Z \times G$.

Moreover we introduce the following conditions:

(IV 3) For each $h \in G$, $z \in Z$ and $\lambda \in Y$ the following compatibility conditions hold:

$$\begin{aligned} &D_z F(z, h) \in \widehat{Z}, \\ &D^2_{zz}F(z, h) \in L(\widehat{Z}, \widehat{Z}), \\ &D_z\varphi(z, h) \in L(\widehat{Z}, \widehat{Y}), \\ &D_z\varphi^*(z, h)\lambda \in \widehat{Z}, \\ &D^2_{zz}\varphi^*(z, h)\lambda \in L(\widehat{Z}, \widehat{Z}). \end{aligned} \tag{5.2}$$

Moreover

$$\begin{aligned} &\lim \|D^2_{zz}F(z_1, h_1) - D^2_{zz}F(z_2, h_2)\|_{L(\widehat{Z},\widehat{Z})} = 0, \\ &\lim \|D_z\varphi(z_1, h_1) - D_z\varphi(z_2, h_2)\|_{L(\widehat{Z},\widehat{Y})} = 0, \\ &\lim \|D^2_{zz}\varphi^*(z_1, h_1)\lambda_1 - D^2_{zz}\varphi^*(z_2, h_2)\lambda_2\|_{L(\widehat{Z},\widehat{Z})} = 0, \end{aligned} \tag{5.3}$$

for $[\![z_1 - z_2]\!]_Z \to 0$, $[\![\lambda_1 - \lambda_2]\!]_Y \to 0$, $[\![h_1 - h_2]\!]_H \to 0$.

(IV 4) Let M_0 be given in (4.3). There exists a subspace $M \subset M_0$ closed in Z, and a linear continuous mapping

$$\Pi \in L(Y,Y) \cap L(\widehat{Y},\widehat{Y}), \qquad \Pi : Y \mapsto M,$$

such that

$$\begin{aligned} D_z\varphi(z_{\bar{h}},\bar{h})Z + \Pi Y &= Y, \\ D_z\varphi(z_{\bar{h}},\bar{h})\widehat{Z} + \Pi\widehat{Y} &= \widehat{Y}. \end{aligned} \tag{5.4}$$

Moreover there exists a Y-neighborhood $\mathcal{Y}_0$ of $y_{\bar{h}} = \varphi(z_{\bar{h}},\bar{h})$ such that

$$M \subset M_y \quad \text{for all } y \in \mathcal{Y}_0 \cap K. \tag{5.5}$$

By (IV 4) there exists a unique Lagrange multiplier $\lambda_{\bar{h}} \in Y^*$ associated with $z_{\bar{h}}$. We assume that $\lambda_{\bar{h}}$ is more regular, namely:

(IV 5) $\lambda_{\bar{h}} \in Y$.

(IV 6) There exist a subspace $N \subset N_0 \subset Y^*$ and a constant $\alpha > 0$ such that

$$(D^2_{zz}\mathcal{L}(z_{\bar{h}},\lambda_{\bar{h}},\bar{h})w,w)_{\widehat{Z}} \geq \alpha\|w\|^2_{\widehat{Z}} \tag{5.6}$$

$$\text{for all } w \in \{w \in \widehat{Z} \mid D_z\varphi(z_{\bar{h}},\bar{h})w \in N^{\perp}\}$$

and moreover there exists a Y–neighborhood Λ_0 of $\lambda_{\bar{h}}$ such that

$$\Lambda_0 \cap (K^+ + N) \subset K^+. \tag{5.7}$$

For $\delta = (a,b) \in \widehat{X}$ we consider the following linear-quadratic problem $(\widehat{LP}_\delta)$ associated with $(P_{\bar{h}})$ and analogous to $(\widehat{L}_\delta)$:

$$(\widehat{LP}_\delta) \qquad \begin{aligned} &\min_{w\in\widehat{Z}}\left\{\tfrac{1}{2}(Qw,w)_{\widehat{Z}} + (q+a,w)_{\widehat{Z}}\right\} \\ &\text{subject to} \\ &Dw + e + b \in \widehat{K}, \end{aligned}$$

In order to be able to use Theorem 3.1 we still have to assume condition (II 4), which takes on the form:

(IV 7) There exists a closed convex set $\Delta \subset \widehat{X}$ and a convex set $\widetilde{\Gamma}$ compact in X such that for any $\delta \in \Delta$, $(w_\delta,\mu_\delta) \in \widetilde{\Gamma}$, where w_δ is the solution to $(\widehat{LP}_\delta)$ and μ_δ — the associated Lagrange multiplier.

Under the above conditions, (3.10) holds for $(\widehat{LP}_\delta)$. However, it is not enough to allow application of Robinson's theorem 4.1. To see it let us briefly recall the idea of the proof of that theorem.

Define the following mappings:

$$\begin{aligned} &\mathcal{P} : \mathcal{O}^\rho \mapsto X, \qquad \mathcal{P}(\delta) := y_\delta := (w_\delta,\mu_\delta), \\ &g(\cdot,\cdot) : X \times G \mapsto X^*, \qquad \ell(\cdot,\cdot) : X \times G \mapsto X^* \\ &g(x,h) := \begin{pmatrix} D_z\mathcal{L}(z,\lambda,h) \\ \varphi(z,h) \end{pmatrix}, \\ &\ell(x,h) := g(x_{\bar{h}},\bar{h}) + D_x g(x_{\bar{h}},\bar{h})(x - x_{\bar{h}}) - g(x,h). \end{aligned}$$

By (A) the mapping $\mathcal{P}$ is well defined and Lipschitz continuous. On the other hand, by continuity of $\ell(\cdot,\cdot)$ there exist $\sigma > 0$ and $G_{\bar{h}}$ such that for all $h \in G_{\bar{h}}$, $\ell(\cdot, h)$ maps $\mathcal{B}^{\sigma}_{\bar{h}} := \{x \in X \mid \|x - x_{\bar{h}}\|_X \leq \sigma\}$ into $\mathcal{O}^{\rho}$. Hence for all $h \in G_{\bar{h}}$ there is the well defined mapping

$$\Phi_h(\cdot) : \mathcal{B}^{\sigma}_{\bar{h}} \mapsto X, \qquad \Phi_h(x) := \mathcal{P}(\ell(x, h)),$$

as it is illustrated in Figure 2.

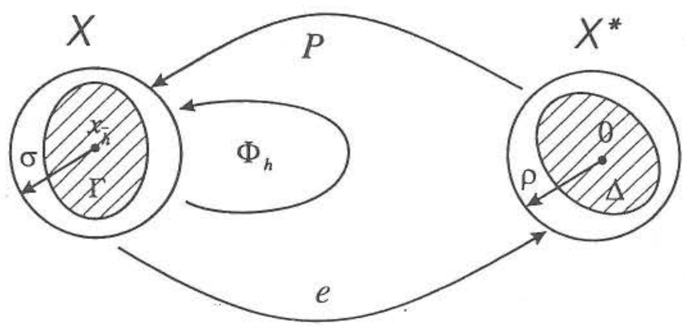

Figure 2

It can be checked that $x = \Phi_h(x)$ if and only if the Kuhn–Tucker conditions (4.1) are satisfied, i.e., solving (4.1) is equivalent to finding a fixed point of Φ_h. It is shown in Robinson (1980), that if (A) is satisfied, then for sufficiently small $\sigma > 0$ and $G_{\bar{h}}$, Φ_h is a contraction self-map on $\mathcal{B}^{\sigma}_{\bar{h}}$ for all $h \in G_{\bar{h}}$.

By the contraction principle there exists a unique fixed point x_h of Φ_h. Lipschitz continuity of x_h follows from the contraction estimates.

In the situation of the norm discrepancy, where instead of (A) we have (3.10), Φ_h is no longer a self-map on any closed ball either in X or in $\widehat{X}$.

Rather than on a ball we can try to use the fixed point argument on the set $\widetilde{\Gamma}$ defined in (IV 7). However, to be able to do so we still have to assure that $\ell(x, h) \in \Delta$ for all $x \in \widetilde{\Gamma}$ and $h \in G_{\bar{h}}$ (cf. Fig. 2).

We assume:

(IV 8) There exists $\sigma > 0$ and $G_{\bar{h}}$ such that

$$\ell(x, h) \in \Delta \qquad \text{for all } x \in \Gamma \text{ and } h \in G_{\bar{h}}, \tag{5.8}$$

where $\Gamma = \widetilde{\Gamma} \cap \widehat{\mathcal{B}}^{\sigma}_{\bar{h}}$, with $\widehat{\mathcal{B}}^{\sigma}_{\bar{h}} := \{x \in \widehat{X} \mid \|x - x_{\bar{h}}\|_{\widehat{X}} \leq \sigma\}$.

By (IV 7) and (IV 8) we can repeat the fixed point argument on the set Γ and we obtain (cf, Corollary 5.2 in Malanowski,1993):

THEOREM 5.2 *If (IV 1)-(IV 8) are satisfied then there exist a neighborhood* $G_{\bar{h}} \subset G$ *of* $\bar{h}$ *and constants* $\sigma > 0$ *and* $c > 0$ *such that for any* $h \in G_{\bar{h}}$ *there exists a solution* $x_h = (z_h, \lambda_h)$ *of (4.1), unique in* $\Gamma = \widetilde{\Gamma} \cap \widehat{\mathcal{B}}^{\sigma}_{\bar{h}}$ *and*

$$\|z_{h_1} - z_{h_2}\|_{\widehat{Z}}, \|\lambda_{h_1} - \lambda_{h_2}\|_{\widehat{Y}} \leq c \, [\![h_1 - h_2]\!]_H \tag{5.9}$$

for all $h_1, h_2 \in G_{\bar{h}}$. ■

To show that z_h and λ_h are actually the solutions and the Lagrange multipliers of (P_h) we have to show that a sufficient optimality condition is satisfied at z_h.

First of all note that by (5.9) and by compactness of the set $\Gamma \subset X$ we have:

$$\lim_{h \to \bar{h}} [\![z_h - z_{\bar{h}}]\!]_Z = 0, \qquad \lim_{h \to \bar{h}} [\![\lambda_h - \lambda_{\bar{h}}]\!]_Y = 0.$$

Hence, using (IV 6), (IV 7) and the argument similar to that in the proof of Lemma 5.3 in Malanowski,1993, we find that condition (5.6) is preserved under small perturbations of h, i.e., there exists a neighborhood $G_{\bar{h}}$ of $\bar{h}$ such that for each $h \in G_{\bar{h}}$

$$(D^2_{zz}\mathcal{L}(z_h, \lambda_h, h)w, w)_{\widehat{Z}} \geq \frac{\alpha}{2}\|w\|^2_{\widehat{Z}} \tag{5.10}$$

for all $w \in \{w \in \widehat{Z} \mid D_z\varphi(z_h, h)w \in N^{\perp}\}$

On the other hand, it follows from (5.7) that there exists a constant $\kappa > 0$ such that

$$(\lambda_h, y)_Y \geq \kappa \, [\![y]\!]_{Y^*} \qquad \text{for all } y \in K \cap N. \tag{5.11}$$

Moreover, for any $y \in Y$ we have

$$\|y\|^2_{\widehat{Y}} = (y, y)_{\widehat{Y}} \leq [\![y]\!]_{Y^*} [\![y]\!]_Y, \quad \text{i.e.,}$$

$$\|y\|^2_{\widehat{Y}} / [\![y]\!]_{Y^*} \leq [\![y]\!]_Y. \tag{5.12}$$

In view of (5.11) and (5.12) it follows from Theorem 1 in Dontchev et al. (1994), that (5.10) constitutes a sufficient optimality condition and that there exist a constant $\beta > 0$ and a constant $\varsigma > 0$ such that

$$F(z, h) \geq F(z_h, h) + \beta\|z - z_h\|^2_{\widehat{Z}} \tag{5.13}$$

for all $z \in \mathcal{Z}^{\varsigma}_h$ feasible for (P_h) and for all $h \in G_{\bar{h}}$, where

$$\mathcal{Z}^{\varsigma}_h = \{z \in Z \mid [\![z - z_h]\!]_Z < \varsigma\}.$$

From Theorem 5.2 and from (5.13) we obtain:

COROLLARY 5.3 *If (IV 1)-(IV 8) are satisfied then there exist a neighborhood* $G_{\bar{h}} \subset G$ *of* $\bar{h}$ *and a neighborhood* $\mathcal{Z}_{\bar{h}} \subset Z$ *of* $z_{\bar{h}}$ *such that for each* $h \in G_{\bar{h}}$ *there is a unique in* $\mathcal{Z}_{\bar{h}}$ *solution* z_h *of* (P_h) *and a unique associated Lagrange multiplier* λ_h. *Moreover, there exists a constant* $c > 0$ *such that*

$$\|z_{h_1} - z_{h_2}\|_{\widehat{Z}}, \|\lambda_{h_1} - \lambda_{h_2}\|_{\widehat{Y}} \leq c \, [\![h_1 - h_2]\!]_H$$

for all $h_1, h_2 \in G_{\bar{h}}$. ■

6. Optimal control problems

In this section the obtained abstract stability results will be applied to state and control constrained optimal control problems for nonlinear ordinary differential equations.

As before $G \subset H$ denotes an open set of admissible parameters and for each $h \in G$ we consider the following optimal control problem:

$$
(O_h) \qquad \begin{array}{ll}
\multicolumn{2}{l}{\text{find } (u_h, x_h) \in L^\infty(0,T;\mathbf{R}^m) \times W^{1,\infty}(0,T;\mathbf{R}^n) \text{ such that}} \\
\multicolumn{2}{l}{F(u_h, x_h, h) = \min_{u,x} \left\{ F(u,x,h) := \int_0^T f^0\left(u(t), x(t), h\right) dt \right\}} \\
\multicolumn{2}{l}{\text{subject to}} \\
\dot{x}(t) = f\left(u(t), x(t), h\right) & \text{for a.a. } t \in [0,T], \\
x(0) = x_0(h), & \\
\theta\left(u(t), h\right) \le 0 & \text{for a.a. } t \in [0,T], \\
v\left(x(t), h\right) \le 0 & \text{for all } t \in [0,T],
\end{array}
$$

where

$$\theta(\cdot,\cdot) : \mathbf{R}^m \times G \mapsto \mathbf{R}^k, \quad v(\cdot,\cdot) : \mathbf{R}^n \times G \mapsto \mathbf{R}^l.$$

In order to represent (O_h) in the form (P_h) we put :

$$
\begin{array}{l}
Z = L^\infty(0,T;\mathbf{R}^m) \times W^{1,\infty}(0,T;\mathbf{R}^n), \\
Y = L^\infty(0,T;\mathbf{R}^n) \times \mathbf{R}^n \times L^\infty(0,T;\mathbf{R}^k) \times W^{1,\infty}(0,T;\mathbf{R}^l), \\
K = K_1 \times K_2 \times K_3 \times K_4, \\
K_1 = \{0\}, \qquad K_2 = \{0\}, \\
K_3 = \left\{ u \in L^\infty(0,T;\mathbf{R}^k) \mid u^i(t) \ge 0, \ i = 1,...,k \ \text{ for a.a. } t \in [0,T] \right\}, \\
K_4 = \left\{ x \in W^{1,\infty}(0,T;\mathbf{R}^l) \mid x^j(t) \ge 0, \ j = 1,...,l \ \text{ for all } t \in [0,T] \right\} \\
F(z,h) = F(u,x,h), \\
\varphi(z,h) = \left(\dot{x} - f(u,x,h), \ x(0) - x_0(h), \ -\theta(u,h), \ -v(x,h)\right).
\end{array}
$$

As in Section 3 we introduce the Hilbert spaces

$$
\begin{array}{l}
\widehat{Z} = L^2(0,T;\mathbf{R}^m) \times W^{1,2}(0,T;\mathbf{R}^n), \quad \text{and} \\
\widehat{Y} = L^2(0,T;\mathbf{R}^n) \times \mathbf{R}^n \times L^2(0,T;\mathbf{R}^k) \times W^{1,2}(0,T;\mathbf{R}^l).
\end{array}
$$

As in (2.1) we treat $\widehat{Y}$ as the pivot space putting $\widehat{Y} = \widehat{Y}^*$, but for technical reasons it will be more convenient not to identify $\widehat{Z}$ with $\widehat{Z}^*$ but to introduce another pivot space:

$$\widehat{V} = L^2(0,T:\mathbf{R}^m) \times M^2(0,T:\mathbf{R}^n).$$

We have

$$Z \subset \widehat{Z} \subset \widehat{V} = \widehat{V}^* \subset \widehat{Z}^* \subset Z^*$$

and the considerations of Sections 3 and 5 remain virtually intact, if in (IV 3) the spaces $\widehat{Z}$, $L(\widehat{Z},\widehat{Z})$ and $L(H,\widehat{Z})$ are substituted by $\widehat{V}$, $L(\widehat{Z},\widehat{V})$ and $L(H,\widehat{V})$, respectively.

In order to apply Theorem 5.2 we have to verify all assumptions of this theorem.

To simplify notation let us put:

$$
\begin{aligned}
A(t) &:= D_x f(u_{\bar{h}}(t), x_{\bar{h}}(t), \bar{h}),\\
B(t) &:= D_u f(u_{\bar{h}}(t), x_{\bar{h}}(t), \bar{h}),\\
\Theta(t) &:= D_u \theta(u_{\bar{h}}(t), \bar{h}),\\
\Upsilon(t) &:= D_x \upsilon(x_{\bar{h}}, \bar{h}).
\end{aligned}
$$

We assume:

(V 1) There exists a (local) solution $(u_{\bar{h}}, x_{\bar{h}})$ of $(O_{\bar{h}})$, which satisfies the following regularity condition:
$(u_{\bar{h}}, x_{\bar{h}}) \in C^0(0,T;\mathbf{R}^m) \times C^1(0,T;\mathbf{R}^n)$,

(V 2) $f^0(\cdot,\cdot,\cdot)$, $f(\cdot,\cdot,\cdot), \theta(\cdot,\cdot), \upsilon(\cdot,\cdot)$ and $D_x\upsilon(\cdot,\cdot)$ are two times Fréchet differentiable in all arguments, and the respective derivatives are locally Lipschitz continuous in u, x,
$x_0(\cdot)$ is Fréchet differentiable.

(V 3) h does not depend on t.

Certainly (V 1) implies (IV 1). On the other hand, it is easy to check that (IV 2) and (IV 3) are satisfied by (V 2).

Let us pass to constraint qualifications. For $\epsilon > 0$ we introduce the sets:

$$
\begin{aligned}
\Psi^i_\epsilon &:= \left\{t \in [0,T] \mid -\theta^i(u_{\bar{h}}(t), \bar{h}) > \epsilon\right\}, \quad i = 1, ..., k,\\
\Omega^j_\epsilon &:= \left\{t \in [0,T] \mid -\upsilon^j(x_{\bar{h}}(t), \bar{h}) > \epsilon\right\}, \quad j = 1, ..., l,
\end{aligned}
$$

and define the following continuous functions (see Figure 3):

$$
\psi^i_\epsilon(t) = \begin{cases} \theta^i(u_{\bar{h}}(t), \bar{h}) + \epsilon & \text{if } \ t \in \Psi^i_\epsilon \\ 0 & \text{if } \ t \notin \Psi^i_\epsilon, \end{cases}
$$

and

$$
\omega^j_\epsilon(t) = \begin{cases} \upsilon^j(x_{\bar{h}}(t), \bar{h}) + \epsilon & \text{if } \ t \in \Omega^j_\epsilon \\ 0 & \text{if } \ t \notin \Omega^j_\epsilon. \end{cases}
$$

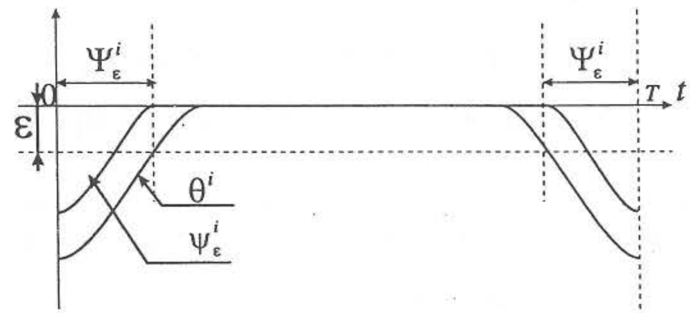

Figure 3

We introduce the following $(k \times k)$ and $(l \times l)$ diagonal matrices

$$U_\epsilon(t) = \text{diag } \psi^i_\epsilon(t) \qquad T_\epsilon(t) = \text{diag } \omega^j_\epsilon(t)$$

and define $(k+l) \times (m+k+l)$ matrices

$$V_\epsilon(t) = \begin{bmatrix} \Theta(t) & U_\epsilon(t) & 0 \\ \Upsilon(t)B(t) & 0 & T_\epsilon(t) \end{bmatrix}.$$

In addition to (V 1)–(V 3) we assume:

(V 4) $\upsilon^j(x_0(\bar h), \bar h) < 0$, for $j = 1, 2, ..., l$,

(V 5) there exists $\eta > 0$ such that

$$|V_0(t)V_0^*(t)\zeta| \geq \eta\ |\zeta| \qquad \text{for all } \zeta \in \mathbf{R}^{k+l} \text{ and all } t \in [0, T].$$

Roughly speaking condition (V 5) has the meaning that all gradients of the active control constraints $\theta^i(u_{\bar h}(t), \bar h) = 0$ and all gradients of the active state constraints $\upsilon^j(x_{\bar h}(t), \bar h) = 0$, transformed into the space $\mathbf{R}^m$ by means of the mapping $B^*(t) : \mathbf{R}^n \to \mathbf{R}^m$, are jointly linearly independent, uniformly on $[0, T]$.

By (V 2) and (V 5) (cf. Lemma 7.2 in Malanowski,1993), there exists $\epsilon > 0$ such that

$$|V_\epsilon(t)V_\epsilon^*(t)\zeta| \geq \frac{\eta}{2}\ |\zeta| \qquad \text{for all } \zeta \in \mathbf{R}^{k+l} \text{ and all } t \in [0, T]. \tag{6.1}$$

The subspace $M \subset Y$ needed in (IV 4) is define as follows:

$$M = M_1 \times M_2 \times M_3 \times M_4,$$

where

$$\begin{aligned}
M_1 &= \{0\}, \qquad\qquad M_2 = \{0\},\\
M_3 &= \left\{u \in L^\infty(0,T;\mathbf{R}^k) \mid u^i(t) = 0 \text{ for a.a. } t \in [0,T] \setminus \Psi^i_{\frac{\epsilon}{2}},\ \ i = 1,...,k\right\},\\
M_4 &= \left\{x \in W^{1,\infty}(0,T;\mathbf{R}^l) \mid x^j(t) = 0 \text{ for all } t \in [0,T] \setminus \Omega^j_{\frac{\epsilon}{2}},\ \ j = 1,...,l\right\},
\end{aligned}$$

and ϵ is given in (6.1).
It is easy to see that for any $\epsilon > 0$, $M \subset M_0$, where M_0 is given by (4.3). Moreover, if we choose as $\mathcal{Y}_0 \subset Y$ the open ball of radius $\epsilon/4$ about $-\varphi(z_{\bar h}, \bar h)$, then (5.5) holds.
The mapping $\Pi : Y \mapsto M$ is defined by

$$\begin{aligned}
&\Pi_1 = 0, && \Pi_2 = 0,\\
&(\Pi_3 y)(t) = U_{\frac{\epsilon}{2}}(t)u(t), && (\Pi_4 y)(t) = T_{\frac{\epsilon}{2}}(t)x(t).
\end{aligned}$$

It follows from Lemma 4.3 in Malanowski (1994), that for the above difinition of M, condition (V 4) and (V 5) imply (5.4), i.e., (IV 4) is fully satisfied.
Let us define the following Lagrangian associated with (O_h):

$$\begin{aligned}
\mathcal{L} \ :\ & L^\infty(0,T;\mathbf{R}^m) \times W^{1,\infty}(0,T;\mathbf{R}^n) \times (L^\infty(0,T;\mathbf{R}^n))^* \times\\
& \times \mathbf{R}^n \times (L^\infty(0,T;\mathbf{R}^k))^* \times (W^{1,\infty}(0,T;\mathbf{R}^l))^* \times G \mapsto \mathbf{R}^1,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(u,x,q,\rho,\kappa,\nu,h) \ &= \ F(u,x,h) + (q, \dot x - f(u,x,h)) +\\
&+ \ \langle \rho, x(0) - x_0(h)\rangle + (\kappa, \theta(u,h)) +\\
&+ \ \langle \nu(0), v(x(0),h)\rangle + (D_x v^*(x,h)\dot\nu, f(u,x,h)).
\end{aligned}$$

By (IV 4) there exist unique Lagrange multipliers $(q_{\bar h}, \rho_{\bar h}, \kappa_{\bar h}, \nu_{\bar h})$ associated with $(u_{\bar h}, x_{\bar h})$. Using (V 1) and (V 5) it can be shown (cf. Corollary 4.6 in Hager,1979), that the multipliers are more regular. Namely:

$$q_{\bar h} \in C^1(0,T;\mathbf{R}^n), \quad \rho_{\bar h} \in \mathbf{R}^n, \quad \kappa_{\bar h} \in C^0(0,T;\mathbf{R}^k), \quad \nu_{\bar h} \in C^1(0,T;\mathbf{R}^l),$$

i.e., in particular (IV 5) holds.
Let us define the following augmented Hamiltonian:

$$\begin{aligned}
&\mathcal{H}(t) = f^0\left(u_{\bar h}(t), x_{\bar h}(t), \bar h\right) - \langle q_{\bar h}(t), f\left(u_{\bar h}(t), x_{\bar h}(t), \bar h\right)\rangle +\\
&+\langle \kappa_{\bar h}(t), \theta(u_{\bar h}(t), \bar h)\rangle + \langle D_x v^*(x_{\bar h}(t), \bar h)\dot\mu_{\bar h}(t), f(u_{\bar h}(t), x_{\bar h}(t), \bar h)\rangle.
\end{aligned}$$

In order to introduce a second order sufficient optimality condition, for $\eta > 0$ we define the sets

$$\Xi^i_\eta = \{t \in [0,T] \mid \kappa^i_{\bar h}(t) > \eta\}.$$

Let us introduce the following subspaces:

$$\begin{aligned}
&U_\eta(t) = \{u \in \mathbf{R}^m \mid \langle \theta^i(t), u\rangle = 0 \quad \text{for } t \in \Xi^i_\eta, \quad i = 1,2,...,k\},\\
&\mathcal{U}_\eta = \{u \in L^2(0,T;\mathbf{R}^m) \mid u(t) \in U_\eta(t) \quad \text{for a.a. } t \in [0,T]\}.
\end{aligned}$$

We assume:

(V 6) There exists $\gamma > 0$ such that

$$\langle u, D^2_{uu}\mathcal{H}(t)u\rangle \geq \gamma |u|^2 \tag{6.2}$$

for all $u \in U_0(t)$ and all $t \in [0,T]$. Moreover

$$\int_0^T \left([u^T(t), x^T(t)] \begin{bmatrix} D^2_{uu}\mathcal{H}(t) & D^2_{ux}\mathcal{H}(t) \\ D^2_{xu}\mathcal{H}(t) & D^2_{xx}\mathcal{H}(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} \right) dt \geq$$

$$\geq \gamma \left(\|u\|_2^2 + \|x\|_{1,2}^2 \right), \tag{6.3}$$

for all pairs $(u, x) \in \mathcal{U}_0 \times W^{1,2}(0,T;\mathbf{R}^n)$ satisfying

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ x(0) &= 0. \end{aligned}$$

Note that it follows from (V 6) that there exists $\eta > 0$ such that (6.2) and (6.3) hold with γ substituted by $\gamma/2$ for the subspaces $U_\eta(t)$ and $\mathcal{U}_\eta$, respectively (see Dontchev et al.,1994). Using this fact it can be shown that condition (IV 6) is satisfied with the subspaces $N = N_1 \times N_2 \times N_3 \times N_4$ defined as follows:

$$\begin{aligned} &N_1 = L^\infty(0,T;\mathbf{R}^n), && N_2 = \mathbf{R}^n, \\ &N_3 = \{u \in L^\infty(0,T;\mathbf{R}^m) \mid u(t) \in [U_{\frac{\eta}{2}}(t)]^\perp\}, && N_4 = \{0\}. \end{aligned}$$

Note that the cone K_4^+ polar to K_4 is given by the closure in $(W^{1,\infty}(0,T;\mathbf{R}^l))^*$–topology of the cone

$$\begin{aligned} \widehat{K}_4^+ \;=\; &\{\nu \in W^{1,2}(0,T;\mathbf{R}^l) | \nu^j(t) \geq 0,\ \dot{\nu}^j(t) \text{ — is non-increasing}, \\ &\text{and } 0 \leq \dot{\nu}^j(t) \leq \nu^j(0),\quad j = 1,2,...,l\}. \end{aligned}$$

Hence we can not expect that (5.7) is satisfied for a non-trivial subspace N. For that reason we have chosen $N_4 = \{0\}$.

To apply Theorem 5.3 it remains to verify assumptions (IV 7) and (IV 8) concerning regularity of the solutions. To this end, for a given value of the parameter

$$\delta = (a_1, a_2, b_1, b_2, b_3, b_4) \in \widehat{Z} \times \widehat{Y} = \widehat{X},$$

let us introduce the linear-quadratic problem $(\widehat{LO}_\delta)$ corresponding to $(O_{\bar{h}})$ and analogous to $(\widehat{LP}_\delta)$:

$(\widehat{LO}_\delta)$ find $(w_\delta, z_\delta) \in L^2(0,T;\mathbf{R}^m) \times W^{1,2}(0,T;\mathbf{R}^n)$ such that

$$I(w_\delta, z_\delta, \delta) = \min_{(w,z)} \{I(w,z,\delta) :=$$

$$:= \int_0^T \left(\frac{1}{2} [w^T(t), z^T(t)] \begin{bmatrix} D^2_{uu}\mathcal{H}(t) & D^2_{ux}\mathcal{H}(t) \\ D^2_{xu}\mathcal{H}(t) & D^2_{xx}\mathcal{H}(t) \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} + \right.$$

$$\left. + (q_1^T(t) + a_1^T(t))w(t) + (q_2^T(t) + a_2^T(t))z(t) \right) dt \Big\},$$

subject to

$$
\begin{aligned}
&\dot{z}(t) - A(t)z(t) - B(t)w(t) + e_1(t) + b_1(t) = 0,\\
&z(0) + e_2 + b_2 = 0,\\
&\Theta(t)w(t) + e_3(t) + b_3(t) \le 0,\\
&\Upsilon(t)z(t) + e_4(t) + b_4(t) \le 0.
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &= -D^2_{uu}\mathcal{H}u_{\bar{h}} - D^2_{ux}\mathcal{H}x_{\bar{h}} + D_u f^0(u_{\bar{h}}, x_{\bar{h}}, \bar{h}),\\
q_2 &= -D^2_{xu}\mathcal{H}u_{\bar{h}} - D^2_{xx}\mathcal{H}x_{\bar{h}} + D_x f^0(u_{\bar{h}}, x_{\bar{h}}, \bar{h}),\\
e_1 &= -\dot{x}_{\bar{h}} + Ax_{\bar{h}} + Bu_{\bar{h}},\\
e_2 &= -x_{\bar{h}}(0),\\
e_3 &= \theta(u_{\bar{h}}, \bar{h}) - \Theta u_{\bar{h}},\\
e_4 &= \upsilon(x_{\bar{h}}, \bar{h}) - \Upsilon x_{\bar{h}}.
\end{aligned}
$$

Note that conditions (V 1)–(V 6) imply the following regularity of the solution and Lagrange multipliers of $(O_{\bar{h}})$ (cf. Lemma 5.2 in Malanowski,1994):

$$
(u_{\bar{h}}, \dot{x}_{\bar{h}}, \dot{q}_{\bar{h}}, \kappa_{\bar{h}}, \dot{\nu}_{\bar{h}}) \qquad \text{are Lipschitz continuous on } [0,T]. \tag{6.4}
$$

By (V 2) and (6.4) the functions $q_1, q_2, e_1, e_2, e_3, \dot{e}_4$ are uniformly bounded and Lipschitz continuous on $[0,T]$. Let $l_1 > 0$ be the Lipschitz modulus joint for all these functions, i.e.,

$$
l_1 = \max\{||\dot{q}_1||_\infty, ||\dot{q}_2||_\infty, ||\dot{e}_1||_\infty, ||\dot{e}_3||_\infty, ||\ddot{e}_4||_\infty\}.
$$

Moreover let

$$
m_1 = \max\{||q_1||_\infty, ||q_2||_\infty, ||e_1||_\infty, |e_2|, ||e_3||_\infty, ||\dot{e}_4||_\infty\}.
$$

As the set of variations Δ we choose the set of uniformly bounded and Lipschitz continuous functions $\delta \in \widehat{X}$ with the bound m_1 and the Lipschitz modulus l_1. This set is convex and compact in X.
Let $(r_\delta, \eta_\delta, \pi_\delta, \chi_\delta)$ denote the Lagrange multipliers associated with (w_δ, z_δ). It is shown in Malanowski (1994) (cf. Propositions 6.6 and 6.7) that, after possible shrinking of Δ, $\mathbf{x}_\delta := (w_\delta, \dot{z}_\delta, \dot{r}_\delta, \pi_\delta, \dot{\chi}_\delta)$ are bounded and Lipschitz continuous uniformly with respect to $\delta \in \Delta$. Let m_2 and l_2 be the bound and the Lipschitz modulus, respectively for all these functions.

The set $\widetilde{\Gamma}$ needed in (IV 7) is chosen as the set of functions $\mathbf{x} \in \widehat{X}$ with the above bound and Lipschitz mudulus. Certainly $\widetilde{\Gamma}$ is convex and by the Arzeli–Ascola theorem it is compact in X, i.e., condition (IV 7) is satisfied. It is shown in Section 7 of Malanowski (1994), that also condition (IV 8) holds. Hence all assumptions of Theorem 5.2 and Corollary 5.3 are satisfied and by that corollary we obtain:

THEOREM 6.1 *If assumptions (V 1)–(V 6) are satisfied then there exist a neighborhood $G_{\bar{h}} \subset G$ of $\bar{h}$ and a neighborhood $Z_{\bar{h}} \subset Z$ of $(u_{\bar{h}}, x_{\bar{h}})$, such that for each $h \in G_{\bar{h}}$ there is a unique in $Z_{\bar{h}}$ solution (u_h, z_h) of (O_h) and a unique associated Lagrange multipliers $(q_h, \rho_h, \kappa_h, \nu_h)$.*

Moreover, there exists a constant $c > 0$ such that

$$\|u_{h_1} - u_{h_2}\|_2, \|x_{h_1} - x_{h_2}\|_{1,2} \leq c \; [\![h_1 - h_2]\!]_H,$$

$$\|q_{h_1} - q_{h_2}\|_{1,2}, |\rho_{h_1} - \rho_{h_2}|, \|\kappa_{h_1} - \kappa_{h_2}\|_2, \|\nu_{h_1} - \nu_{h_2}\|_{1,2} \leq c \; [\![h_1 - h_2]\!]_H,$$

for all $h_1, h_2 \in G_{\bar{h}}$. ■

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