Control and Cybernetics

vol. **23** (1994) No. 1/2

An approach to well–posedness in vector optimization: consequences to stability

by

Ewa Bednarczuk

Systems Research Institute
Polish Academy of Sciences
Warsaw
Poland

We propose several definitions of well–posedness for vector optimization problems in topological vector spaces. These definitions are based on the properties of ε–minimal solutions to vector optimization problems and can be viewed as generalizations of the classical approach to well–posedness existing in scalar optimization. In the resulting classes of well–posed problems stability of minimal solutions is investigated.

1. Introduction

The notion of well–posedness and its various generalizations appear to be very fruitfull in scalar optimization especially in investigating different stability and sensitivity problems. Well–posedness plays also an important role in establishing convergence of algorithms for solving scalar optimization problems.

In vector optimization there is no a commonly accepted definition of well–posed problem. Some attempts in this direction has been already done, see eg Lucchetti 1987, Bednarczuk 1987, Bednarczuk 1989.

In this paper we define well–posedness in vector optimization via ε–minimal solutions. This can be viewed as a generalization of the ideas from scalar optimization. The definitions we introduce are analysed mainly from the point of view of their usefulness in establishing stability under perturbations of solutions to vector optimization problems.

Let Y be a topological vector space ordered by a partial ordering relation $\preceq$ generated by a closed convex pointed cone $\mathcal{K}$, $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$ with nonempty interior, $int\mathcal{K} \neq \emptyset$, and $x \preceq y \overset{\text{def}}{\Longleftrightarrow} y - x \in \mathcal{K}$.

Let $f : X \to Y$ be a function defined on a topological space X, and let $A_0 \subset X$ be a subset of X.

The minimization problem

$$(P) \qquad \mathcal{K} - \text{minimize } f \textit{ on } A_0$$

is defined as the problem of finding the set $S(f, A_0|\mathcal{K})$ of all **minimal solutions**, ie. all $\underline{x} \in A_0$ such that there is no $x \in A_0$ satisfying $f(\underline{x}) - f(x) \in \mathcal{K} \setminus \{0\}$. The image of the set $S(f, A_0)$ under the mapping f is called the **minimal set** of (P) and is denoted by $Min(f, A_0|\mathcal{K})$.

If, in the above definitions, instead of cone $\mathcal{K}$ we use cone $\mathcal{K}\prime = \{0\} \cup int\mathcal{K}$, we obtain **weak minimal solutions** $WS(f, A_0|\mathcal{K})$ and **weak minimal points** $WMin(f, A_0|\mathcal{K})$. Whenever possible we shall use the simplified notations S, Min, WS, $WMin$.

In the sequel we shall often use ε**–minimal solutions** of (P), as defined eg. in Kutateladze 1976 and Loridan 1984. We recall that a point $\underline{r} \in A_0$ is an ε**–minimal solution** of (P) if there is no $r \in A_0$ such that $f(\underline{r}) - \varepsilon - f(r) \in \mathcal{K} \setminus \{0\}$. The set of all ε–minimal solutions will be denoted by $S_\varepsilon(f, A_0|\mathcal{K})$ and the set of all the ε–minimal points (ie. the image of the $S_\varepsilon(f, A_0|\mathcal{K})$ under f) will be denoted by $Min_\varepsilon(f, A_0|\mathcal{K})$, or shortly S_ε, Min_ε.

2. Domination property and its variants

It has been already observed by many authors that in stability and sensitivity analysis for vector optimization problems some properties are important which are specific for these kind of problems. Among those properties is the domination property.

DEFINITION 2.1 *The* **domination property** (DP) *holds for* $A \subset Y$ *if*

$$A \subset Min(A|\mathcal{K}) + \mathcal{K}.$$

The domination property is widely used to investigate vector optimization problems in finite dimensional spaces (see eg. Henig 1986). In infinite–dimensional problems it was investigated by Luc 1989, Luc 1990, and in the context of stability in vector optimization by Tanino, Sawaragi, Nakayama 1985.

We have introduced another property which can be regarded as a domination-type property. This property has been investigated in a more detailed way in Bednarczuk 1992B.

DEFINITION 2.2 (Bednarczuk 1992B) *We say that A has the* **containment property** *(CP) if the following condition is satisfied: for each* 0*–neighbourhood* W *in* Y *there exists a* 0*–neighbourhood* V *in* Y *such that for each* $y \in A$ *we have*

$$y \in Min(A|\mathcal{K}) + W \quad or \quad y + V \subset Min(A|\mathcal{K}) + \mathcal{K}.$$

Equivalently, (CP) holds for A if for each 0–neighbourhood W there exists a 0–neighbourhood O such that for each $y \in A$

$$y \in W$$

or

$$y = \eta + k, \quad \text{where } \eta \in Min(f, A_0|\mathcal{K}), \quad k + O \in \mathcal{K}.$$

Observe that (CP) does not imply the equality $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$.

EXAMPLE 2.3 *Let $\mathcal{R}_2^+$ denote the nonnegative orthant in the plane. Let*

$$A = \{(x,y)|\ 1 \leq x \leq 2,\ y = -x+2\} \cup (0,1) \cup \{(x,y)|x \geq 3/2, y = 1/2\}.$$

The set A has the containment property but $WMin(A|\mathcal{R}_2^+) = \{(x,y) \in |\ y = -x+2,\ 1 \leq x \leq 2\} \cup (0,1)$, and $Min(A|\mathcal{R}_2^+) = WMin(A|\mathcal{R}_2^+) \setminus (1,1)$.

DEFINITION 2.4 *We say that A has the* **strong domination property** (SDP) *if there exists a closed convex cone $\mathcal{P}$, $\mathcal{P}\setminus\{0\} \subset int\mathcal{K}$ such that $A \subset Min(A|\mathcal{K}) + \mathcal{P}$ and cone $\mathcal{P}$ has the property that for each 0–neighbourhood $\mathcal{O}$ in Y there exists a 0–neighbourhood $\mathcal{W}$ in Y such that for each $p \in \mathcal{P}$ we have*

$$p \in \mathcal{O} \ \text{ or } \ p + \mathcal{W} \subset \mathcal{K}. \qquad (*)$$

In other words,

$$(\mathcal{P} \setminus \mathcal{O}) + \mathcal{W} \subset \mathcal{K}. \qquad (*)$$

Observe that in condition $(*)$ the choice of $\mathcal{W}$ is independent of p which means that cone $\mathcal{P}$ is placed sufficiently deeply in $int\mathcal{K}$.

It is easy to see that (SDP) implies (CP) .

Obviously, (SDP) implies (DP) and for any cone $\mathcal{P}$, $\mathcal{P} \setminus \{0\} \subset int\mathcal{K}$ we have the inclusion $Min(A|\mathcal{K}) \subset Min(A|\mathcal{P})$. When A has the strong domination property the opposite inclusion holds. Namely, we have

PROPOSITION 2.5 *If a subset A of Y has the strong domination property, then for any convex closed cone $\mathcal{C}$ such that $\mathcal{P} \subset \mathcal{C} \subset \mathcal{K}$ we have $Min(A|\mathcal{C}) = Min(A|\mathcal{K})$. In particular, $Min(A|\mathcal{K}) = WMin(A|\mathcal{K})$, ie. the set of minimal points is equal to the set of weakly minimal points.*

PROOF. Suppose on the contrary that $Min(A|\mathcal{C}) \setminus Min(A|\mathcal{K}) \neq \emptyset$, ie. there exists $\underline{x} \in Min(A|\mathcal{C})\setminus Min(A|\mathcal{K})$. We also have $\underline{x} \in Min(A|\mathcal{P})$. Since the strong domination property holds, there exists $\underline{y} \in Min(A|\mathcal{K})$ such that $\underline{x} \in \underline{y} + \mathcal{P}$. Hence, it must be $\underline{x} = \underline{y}$ which contradicts the fact that $\underline{x} \notin Min(A|\mathcal{K})$. ■

This proposition can be interpreted as a certain kind of stability of solutions under perturbations of the domination structure.

REMARK 2.6 *Let $Y = R$. Then any subset A of Y such that $Min(A|\mathcal{K}) \neq \emptyset$ has the domination property, the containment property, and the strong domination property with $\mathcal{P} = \mathcal{K}$.*

REMARK 2.7 *Let Y be a Banach space, $Y = (Y, \|\cdot\|)$. Then condition $(*)$ means that cone $\mathcal{P}$ allows plastering $\mathcal{K}$. Let us recall that cone $\mathcal{P}$ allows plastering $\mathcal{P}_1$ if a cone $\mathcal{P}_1$ can be found such that every nonzero element $x_0 \in \mathcal{P}$ is an interior point of $\mathcal{P}_1$ and x_0 is contained in $\mathcal{P}_1$ together with a spherical neighbourhood of radius $b\|x_0\|$, where b does not depend on the element x_0 (see eg. Krasnosel'skii 1964). In the context of vector optimization this kind of conical neighbourhoods of cones where considered by Wierzbicki 1977.*

REMARK 2.8 *Let us observe that if the strong domination property is satisfied, then $Min(A|\mathcal{K}) = GE(A|\mathcal{P})$, where $GE(A|\mathcal{P})$ denote the set of properly efficient points as defined by Henig 1986 and Luc 1989.*

3. Definitions and preliminary results

We adopt the standard definitions of lower ($l.s.c.$) and upper ($u.s.c.$) semicontinuities as defined eg. by Kuratowski 1966. Following Nikodem 1986 we use $\mathcal{K}$–semicontinuities. We say that a multifunction $F : X \rightrightarrows Y$ is **$\mathcal{K}$–upper Hausdorff continuous** ($\mathcal{K} - u.H.c.$) at x_0 if for every 0–neighbourhood V in Y there exists a neighbourhood W of x_0 in X such that

$$F(x) \subset F(x_0) + V + \mathcal{K}$$

for all $x \in W$. $\{0\}$–upper Hausdorff continuity is called **upper Hausdorff continuity**. In the context of vector optimization $\mathcal{K}$–semicontinuities has been used by Sterna–Karwat 1989.

We also use the following variants of lower semicontinuity. A multifunction $F : X \rightrightarrows Y$ is **inf–lower continuous** ($i.l.c.$) at (x_0, y_0) if for each 0–neighbourhood V in Y there exists a neighbourhood W of x_0 such that for each $x \in W$ one has $F(x) \cap (y_0 + V + \mathcal{K}) \neq \emptyset$. A multifunction F is **sup–lower continuous** ($s.l.c.$) at (x_0, y_0) if one has $F(x) \cap (y_0 + V - \mathcal{K}) \neq \emptyset$ for all $x \in W$. The above definitions were introduced by Penot, Sterna–Karwat 1986 (see also Penot, Sterna–Karwat 1989). Moreover, a multifunction $F : X \rightrightarrows Y$ is **uniformly sup–lower continuous** at x_0 if for each 0–neighbourhood V in Y there exists a neighbourhood W of x_0 such that for each $y_0 \in F(x_0)$ we have $F(x) \cap (y_0 + V - \mathcal{K}) \neq \emptyset$. This property can also be considered on proper subsets of $F(x_0)$. In a similar way we can also define uniform inf–lower continuity.

Let us note that for the sake of consistency of terminology one may reffer to $\mathcal{K}$–upper Hausdorff continuity as to inf–upper Hausdorff continuity and to $(-\mathcal{K})$–upper Hausdorff continuity as sup–upper Hausdorff continuity. A function $f : X \to Y$ is **$\mathcal{K}$–lower continuous** at x_0 if for each 0–neighbourhood W in Y there exists a 0–neighbourhood O in X such that $f(x) \in f(x_0) + W + \mathcal{K}$ for all $x \in x_0 + O$.

A function $f : X \to Y$ is $\mathcal{K}$**–upper continuous** at x_0 if for each 0–neighbourhood W in Y there exists a 0–neighbourhood O in Y such that $f(x) \in f(x_0) + W - \mathcal{K}$ for all $x \in x_0 + O$.

THEOREM 3.1 *Let X, Y and U be any topological vector spaces. Let* $f : X \to Y$ *be a* $\mathcal{K}$*–upper continuous (respectively,* $\mathcal{K}$*–lower continuous) function on X and let* $\mathcal{R} : U \rightrightarrows X$ *be a lower semicontinuous multifunction at* $u_0 \in U$*. Then the multifunction* $\mathcal{FR} : U \rightrightarrows Y$ *defined as* $\mathcal{FR}(u) = f(R(u))$ *for* $u \in U$*, is sup–lower continuous (respectively, inf–lower continuous) at* u_0*.*

PROOF. Let $y_0 \in \mathcal{FR}(u_0)$. Let us take any 0–neighbourhood Q in Y. There exists an $x_0 \in \mathcal{R}(u_0)$ such that $f(x_0) = y_0$ and, by the $\mathcal{K}$–upper continuity of f, (respectively, $\mathcal{K}$–lower continuity of f) there exists a neighbourhood W of x_0 such that $f(W) \subset y_0 + Q - \mathcal{K}$ (respectively, $f(W) \subset y_0 + Q + \mathcal{K}$). Since $\mathcal{R}$ is lower semicontinuous at u_0, there exists a neighbourhood U of u_0 such that

$$W \cap \mathcal{R}(u) \neq \emptyset \text{ for } u \in U.$$

Now, by taking any $x \in \mathcal{R}(u)$, $x \in W$, $u \in U$, we obtain that $f(x) \in \mathcal{FR}(u)$, $f(x) \in y_0 + Q - \mathcal{K}$, (respectively, $f(x) \in y_0 + Q + \mathcal{K}$) and hence $(y_0 + Q - \mathcal{K}) \cap \mathcal{FR}(u) \neq \emptyset$ (respectively, $(y_0 + Q + \mathcal{K}) \cap \mathcal{FR}(u) \neq \emptyset$) for $u \in U$. ■

For $X = R^n$, $Y = R^p$ the above result was proved by Tanino, Sawaragi and Nakayama 1985.

THEOREM 3.2 (Bednarczuk 1992A) *Let X, U be topological spaces and Y a topological vector space. Let* $f : X \to Y$ *be (uniformly) continuous on X, and let* $\mathcal{R} : U \rightrightarrows X$ *be (uniformly) lower continuous multifunction at* u_0*. Then the multifunction* $\mathcal{FR} : U \rightrightarrows Y$ *is (uniformly) lower continuous at* u_0*.*

4. Well–posedness and its basic properties

We start this section with three definitions of well–posedness for vector optimization problems. These concepts are based on the properties of ε–minimal solutions and can be viewed as generalizations of the classical approach to well–posedness existing in scalar optimization, see eg. Bednarczuk, Penot 1992B, Bednarczuk, Penot 1992A.

The main difference comparing to the scalar case and, at the same time, the main difficulty to overcome lies in the fact that in vector optimization one can hardly expect that the set $Min(f, A_0|\mathcal{K})$ be a singleton.

For $\varepsilon \in \mathcal{K}$ and $\eta \in Min(f, A_0|\mathcal{K})$ we consider the multifunction $\Pi^\eta : \mathcal{K} \rightrightarrows X$ defined as

$$\Pi^\eta(\varepsilon) = \{x \in A_0 |\ f(x) \preceq \eta + \varepsilon\}.$$

For $\varepsilon = 0$ we have $\bigcup_{\eta \in Min} \Pi^\eta(0) = S$, where $\Pi^\eta(0)$ is the subset of the solution set containing the elements $x \in S(f, A_0|\mathcal{K})$ for which $f(x) = \eta$. The sets $\Pi^\eta(\varepsilon)$ has been already used by Aśić, Dugośija 1986 to investigate some stability properties of sequences of vector optimization problems.

For $\varepsilon \in \mathcal{K}$ we define the multifunction $\Pi : \mathcal{K} \rightrightarrows X$ by the formula

$$\Pi(\varepsilon) = \bigcup_{\eta \in Min(f, A_0|\mathcal{K})} \Pi^\eta(\varepsilon) = A_0 \cap f^{-1}\{Min(f, A_0|\mathcal{K}) + \varepsilon - \mathcal{K}\}.$$

We call this multifunction the ε–**minimal–solution multifunction.**

Basic properties

1. For any ε_1, $\varepsilon_2 \in \mathcal{K}$, $\varepsilon_1 \preceq \varepsilon_2$, we have $\Pi^\eta(\varepsilon_1) \subseteq \Pi^\eta(\varepsilon_2)$ (convexity of $\mathcal{K}$ is important here).

2. If f is continuous and A_0 is closed, then $\Pi^\eta(\varepsilon)$ is closed for any $\eta \in Min(A_0|\mathcal{K})$, and any $\varepsilon \in \mathcal{K}$, (closedness of $\mathcal{K}$ is important here).

Now the definition of well–posed vector optimization problems can be introduced in the following way.

DEFINITION 4.1 *The problem* (P) *is* η–**well–posed** *if*

(i) $Min(f, A_0 \,|\, \mathcal{K}) \neq \emptyset$,

(ii) for each $\eta \in Min(f, A_0|\mathcal{K})$ *the multifunction* Π^η *is upper continuous at* $\varepsilon = 0$.

DEFINITION 4.2 *The problem* (P) *is* **well–posed** *if*

(i) $Min(f, A_0|\mathcal{K}) \neq \emptyset$,

(ii) the multifunction Π *is upper continuous at* $\varepsilon = 0$.

The set $\Pi(\varepsilon)$ contains all the ε–solutions, ie. $\Pi(\varepsilon) = S_\varepsilon$ and $\Pi(0) = S(f, A_0|\mathcal{K})$. If $Min(f, A_0|\mathcal{K}) = \{\eta\}$, ie., $Min(f, A_0|\mathcal{K})$ is a singleton, η–well–posedness coincides with well–posedness.

It is important to note that in the case of scalar optimization problems the two definitions given above coincide and both reduce to topological well–setness as defined in Bednarczuk, Penot 1992A.

The notion of well–posedness introduced below corresponds to the notion of metrically well–set problems for scalar optimization problems as defined in Bednarczuk, Penot 1992B.

DEFINITION 4.3 *The problem* (P) *is* **weakly well–posed** *if*

(i) $Min(f, A_0|\mathcal{K}) \neq \emptyset$,

(ii) the multifunction Π *is u.H.c. at* $y = 0$.

Both, well–posedness and weak well–posedness can be characterised in terms of certain sequences which we call minimizing sequences because they reduce to the usual minimizing sequences when scalar optimization problems are consider.

DEFINITION 4.4 *Let (x_n) be a sequence of feasible elements ie. $x_n \in A_0$, for n=1,... . The sequence (x_n) is said to be a* **minimizing sequence** *of the problem (P) if for each n there exist $y_n \in \mathcal{K}$ and $\eta_n \in Min(f, A_0|\mathcal{K})$ such that* $f(x_n) \preceq \eta_n + y_n$, $\lim_n y_n = 0$.

Let us note that if $Min(f, A_0|\mathcal{K})$ is a singleton, minimizing sequences satisfy the relation $f(x_n) \preceq \eta + y_n$. If $Y = R$, $\mathcal{K} = R^+$ ie. if we consider scalar optimization problems with the minimal value f_{opt}, we obtain $f(x_n) \leq f_{opt} + y_n$, and since $f_{opt} \leq f(x_n)$ we get $\lim_n f(x_n) = f_{opt}$.

PROPOSITION 4.5 *Let X and Y be topological vector spaces with Y satisfying the first countability axiom. The following conditions are equivalent:*
(i) the problems (P) is well-posed,
(ii) $Min(f, A_0|\mathcal{K}) \neq \emptyset$, any minimizing sequence (x_n), $(x_n) \subset A_0 \setminus S(f, A_0|\mathcal{K})$, contains a convergent subsequence with the limit point belonging to $S(f, A_0|\mathcal{K})$.

PROOF. $(ii) \to (i)$. Suppose on the contrary that the problem (P) is not well-posed. This means that there exists an open set Q containing $\Pi(0)$, a sequence $(y_n) \subset \mathcal{K}$ tending to 0, and some elements $x_n \in \Pi(y_n)$ such that $x_n \notin Q$. Hence, there exists a sequence $\eta_n \in Min(f, A_0|\mathcal{K})$ such that $\eta_n - f(x_n) + y_n \in \mathcal{K}$. But it must be also $cl((x_n)) \cap \Pi(0) = \emptyset$ since $x_n \notin Q$. This, however, contradicts (ii).

$(i) \to (ii)$. Follows directly from the definitions. ■

Analogously as above we can prove the following

PROPOSITION 4.6 *The following conditions are equivalent:*
(i) the problem (P) is weakly well-posed,
(ii) $Min(f, A_0|\mathcal{K}) \neq \emptyset$, any minimizing sequence (x_n), $(x_n) \subset A_0 \setminus S(f, A_0|\mathcal{K})$ has the property that for every neighbourhood W of zero

$$x_n \in S(f, A_0|\mathcal{K}) + W$$

for all n sufficiently large.

The relation between η–well posedness and well posedness is investigated in the following proposition.

PROPOSITION 4.7 *Suppose that $Min(f, A_0|\mathcal{K})$ is compact. If, for each $\eta \in Min(f, A_0|\mathcal{K})$, Π^η is upper continuous at $\varepsilon = 0$, then Π is upper continuous at $\varepsilon = 0$.*

PROOF. Let us take any open set $Q \supset \Pi(0)$. Since, for all $\eta \in Min(f, A_0|\mathcal{K})$, Π^η is upper continuous at $\varepsilon = 0$, there exist 0–neighbourhoods O^η such that

$$\Pi^\eta(O^\eta) = A_0 \cap f^{-1}\{\eta + O^\eta - \mathcal{K}\} \subset Q.$$

Hence,

$$\bigcup_{\eta\in Min(f,A_0|\mathcal{K})} \Pi^{\eta}(O^{\eta}) = A_0 \cap f^{-1}(\bigcup_{\eta\in Min} \{\eta + O^{\eta} - \mathcal{K}\} \subset Q.$$

Since $Min(f, A_0|\mathcal{K})$ is assumed to be compact and $\bigcup_{\eta\in Min}\{\eta+O_1^{\eta}\}$ is a covering of $Min(f, A_0|\mathcal{K})$, where O_1^{η} are 0–neighbourhoods such that $O_1^{\eta} + O_1^{\eta} \subset O^{\eta}$, we can choose a finite number of points η_i, $i = 1, ..., k$, and neighbourhoods $O_1^{\eta_i} = O_1^i$ such that

$$Min(f, A_0|\mathcal{K}) \subset \bigcup_{i=1}^{k} \eta_i + O_1^i,$$

and

$$Min(f, A_0|\mathcal{K}) + \bigcap_{i=1}^{k} O_1^i \subset \bigcup_{i=1}^{k} \eta_i + O^i.$$

Consequently,

$$A_0 \cap f^{-1}\{Min(f, A_0|\mathcal{K}) + \bigcap_{i=1}^{k} O_1^i - \mathcal{K}\} \subset A_0 \cap f^{-1}(\bigcup_{\eta\in Min} \{\eta + O^{\eta} - \mathcal{K}\}) \subset Q,$$

which completes the proof. ■

5. Conditions for well–posedness in the objective space

The question of well–posedness in the objective space amounts to the question of upper continuity of ε–minimal points with respect to ε.

Let A be a subset in the objective space Y. The multifunction $\tilde{\Pi} : \mathcal{K} \rightrightarrows Y$,

$$\tilde{\Pi}(\varepsilon) = \bigcup_{\eta\in Min(A|\mathcal{K})} \{y \in A \mid y \preceq \eta + \varepsilon\}$$

is called the ε–minimal point multifunction.

Obviously, $\tilde{\Pi}(0) = Min(A|\mathcal{K})$.

PROPOSITION 5.1 (Bednarczuk 1992A) *If (DP) holds for A, then $\tilde{\Pi}$ is $\mathcal{K}$–u.H.c. at $\varepsilon = 0$.*

PROPOSITION 5.2 (Bednarczuk 1992A) *If (CP) holds for A, then $\tilde{\Pi}$ is u.H.c. at $\varepsilon = 0$.*

In finite dimensional space Y for closed subsets A such that $Min(A|\mathcal{K})$ is compact we can prove the above proposition under weaker assumptions.

PROPOSITION 5.3 *Suppose that Y is a finite dimensional space. Let A be a closed subset of Y and $Min(A|\mathcal{K})$ be compact. If (DP) holds for A and $WMin(A|\mathcal{K}) = clMin(A|\mathcal{K})$, then $\tilde{\Pi}$ is u.H.c. at $\varepsilon = 0$.*

PROOF. Suppose on the contrary, that $\tilde{\Pi}$ is not u.H.c. at $\varepsilon = 0$. This means that there exists a 0–neighbourhood $\bar{W}$ and a sequence ε_n, $\lim_{n\to\infty} \varepsilon_n = 0$, such that $\tilde{\Pi}(\varepsilon_n) \not\subset \tilde{\Pi}(0) + \bar{W}$, ie., for some sequence $\{y_n\}$, $y_n \preceq \eta_n + \varepsilon_n$, $\eta_n \in Min(A_0|\mathcal{K})$, $y_n \notin \tilde{\Pi}(0) + \bar{W}$.

By (DP), for each n, $y_n = \bar{\eta}_n + \bar{k}_n$, where $\bar{k}_n \in \mathcal{K}$, and $\bar{k}_n \notin \bar{W}$.

By the compactness of $Min(A_0|\mathcal{K})$, there exist converging sequences $\{\bar{\eta}_{n_m}\} \subset \{\bar{\eta}_n\}$, and $\{\eta_{n_m}\} \subset \{\eta_n\}$, with the limit points, $\bar{\eta}_0$ and η_0, respectively. This implies that, for all m sufficiently large, we have

$$\bar{\eta}_0 + \varepsilon_0 \preceq k_{n_m} \preceq \eta_0 + \varepsilon_0 ,$$

where $\varepsilon_0 \in \text{int}\mathcal{K}$. This proves boundedness of the sequence $\{k_{n_m}\}$. Thus, we can choose a converging subsequence from $\{k_{n_m}\}$, and without loosing generality we can assume that $\{k_{n_m}\}$ itself converges to a certain $k_0 \in \mathcal{K}$. Moreover, it must be $k_0 \in \partial\mathcal{K}$, since otherwise $\bar{\eta}_0$ would not be minimal.

Since A is closed $\bar{\eta}_0 + k_0 \in A$. Moreover, $(\bar{\eta}_0 + k_0 - \text{int}\mathcal{K}) \cap A = \emptyset$. This proves that $\bar{\eta}_0 + k_0 \in WMin(A|\mathcal{K})$ and clearly, $\bar{\eta}_0 + k_0 \notin Min(A|\mathcal{K})$, contradictory to the assumption. ■

There exist examples showing that without the compactness assumption the domination property and the equality $WMin(A_0|\mathcal{K}) = \text{cl}Min(A_0|\mathcal{K})$ are not sufficient for upper Hausdorff continuity of the ε–minimal point multifunction.

Moreover, the condition $WMin(A|\mathcal{K}) = Min(A|\mathcal{K})$ is not necessary for upper Hausdorff continuity of $\tilde{\Pi}$ at $\varepsilon = 0$ as shows the example below.

EXAMPLE 5.4 *Let $Y = R^2$, $\mathcal{K} = R^2_+$ and $A_0 = R^2_+$. $Min(A_0|\mathcal{K}) = \{0\}$ and is not equal $WMin(A_0|\mathcal{K})$. However, $\tilde{\Pi}$ is u H.c. at $\varepsilon = 0$.*

This example can be easily generalized to non–unique set of minimal points.

REMARK 5.5 *If $Min(A|\mathcal{K})$ is compact, the upper Hausdorff continuity of $\tilde{\Pi}$ coincides with its upper continuity.*

6. Stability results

Let U be a topological space.

In a neighbourhood of an arbitrary fixed parameter value $u_0 \in U$ we shall investigate the family of parametric problems of the form

$$P(u) \qquad \mathcal{K} - minimize \quad f \;\; on \;\; \mathcal{R}(u).$$

The *performance multifunction* $\mathcal{P}$, $\mathcal{P} : U \rightrightarrows Y$ is defined as

$$\mathcal{P}(u) = Min(f, \mathcal{R}(u)|\mathcal{K}).$$

The *solution multifunction* $\mathcal{S}$, $\mathcal{S} : U \rightrightarrows X$ is given by the formula

$$\mathcal{S}(u) = S(f, \mathcal{R}(u)|\mathcal{K}).$$

$\mathcal{R} : U \rightrightarrows X$ is called the *feasible set multifunction.* We have

$$\mathcal{R}(u_0) = A_0, \quad \mathcal{P}(u_0) = Min(f, A_0|\mathcal{K}), \quad \mathcal{S}(u_0) = S(f, A_0|\mathcal{K}).$$

The theorem below is a refinement of Theorem 6.3 of Bednarczuk 1992A.

THEOREM 6.1 *Suppose that*
f is continuous on X,
$\mathcal{R}$ is upper continuous at u_0,
(P) is well–posed,
the performance multifunction $\mathcal{P}$ is $(-\mathcal{K})$–upper Hausdorff continuous at u_0.
Then $\mathcal{S}$ is upper continuous at u_0.

PROOF. Let Q be an open set containing $\mathcal{S}(u_0)$. By well–posedness, there exists a 0–neighbourhood O such that

$$\bigcup_{\eta \in Min(f, A_0|\mathcal{K})} \{x \in A_0 |\ f(x) \preceq \eta + \varepsilon\} \subset Q,$$

for $\varepsilon \in \mathcal{K}$, and $\varepsilon \in O$. Let

$$\mathcal{L}(O) = Min(f, A_0|\mathcal{K}) + O - \mathcal{K}.$$

Hence,

$$A_0 \cap f^{-1}(\mathcal{L}(O \cap \mathcal{K})) \subset Q,$$

and consequently,

$$A_0 \subset Q \cup [X \setminus f^{-1}(\mathcal{L}(O \cap \mathcal{K}))].$$

Let $\varepsilon \in \text{int}\mathcal{K} \cap O$. Hence, there exists a 0–neighbourhood $\bar{O}$ such that $\varepsilon - \bar{O} \subset \mathcal{K}$, ie. $\bar{O} \subset \varepsilon - \mathcal{K} \subset (\mathcal{K} \cap O) - \mathcal{K}$. From the last inclusion we get that

$$A_0 \subset Q \cup (X \setminus f^{-1}(\mathcal{L}(\bar{O}))).$$

Let $\bar{O}_1$ be a 0–neighbourhood such that $\bar{O}_1 + \bar{O}_1 \subset \bar{O}$. Since

$$\text{cl}\mathcal{L}(\bar{O}_1) \subset \mathcal{L}(\bar{O}_1) + \bar{O}_1 \subset \mathcal{L}(\bar{O}),$$

we have

$$A_0 \subset Q \cup [X \setminus f^{-1}\{\mathcal{L}(\bar{O})\}] \subset Q \cup [X \setminus f^{-1}\{\text{cl}\mathcal{L}(\bar{O}_1)\}].$$

Since f is continuous on X, the set $f^{-1}\{\text{cl}\mathcal{L}(\bar{O}_1)\}$ is closed and hence, its complement is open.

By the upper continuity of $\mathcal{R}$ at u_0, there exists a neighbourhood U_1 of u_0 such that

$$\mathcal{R}(u) \subset Q \cup [X \setminus f^{-1}\{\mathrm{cl}\mathcal{L}(\bar{O}_1)\}].$$

Moreover, there exists a neighbourhood U_2 of u_0 such that

$$Min(f, \mathcal{R}(u)|\mathcal{K}) \subset Min(f, A_0|\mathcal{K}) + \bar{O}_1 - \mathcal{K}.$$

Let us take now any $x \in \mathcal{S}(u)$, for $u \in U = U_1 \cap U_2$. By the last inclusion

$$f(x) \in Min(f, A_0|\mathcal{K}) + \bar{O}_1 - \mathcal{K}$$

which means that $f(x) \in \mathcal{L}(\bar{O}_1)$ and consequently, $x \in f^{-1}(\mathcal{L}(\bar{O}_1)) \subset f^{-1}\{\mathrm{cl}\mathcal{L}(\bar{O}_1)\}$. Thus, since $u \in U_1$, it must be $x \in Q$, which completes the proof. ■

This result fully corresponds to what has been obtained for scalar optimization problems, see eg. Lemma 2.7 of Bednarczuk, Penot 1992A. Indeed, according to previous remarks, in scalar case well-posedness reduces to topological well-setness of Bednarczuk, Penot 1992A. Moreover, $(-\mathcal{K})$-upper Hausdorff continuity of the performance multifunction is simply the upper semicontinuity of the optimal value function.

In view of the Proposition 4.7, we can refrase the above theorem in the following way.

THEOREM 6.2 *Suppose that*
f is continuous on X,
$\mathcal{R}$ is upper continuous at u_0,
$Min(f, A_0|\mathcal{K})$ is compact,
(P) is η-well-posed,
$\mathcal{P}$ is $(-\mathcal{K})$-upper Hausdorff continuous at u_0.
Then the solution multifunction $\mathcal{S}$ is upper continuous at u_0.

Analogously, for upper Hausdorff continuity of $\mathcal{S}$ we get the following refinement of Theorem 6.4 of Bednarczuk 1992A.

THEOREM 6.3 *Suppose that*
f is uniformly continuous on X,
$\mathcal{R}$ is upper Hausdorff continuous at u_0,
(P) is weakly well-posed,
the performance multifunction $\mathcal{P}$ is $(-\mathcal{K})$-upper Hausdorff continuous at u_0.
Then the solution multifunction $\mathcal{S}$ is upper Hausdorff continuous at u_0.

PROOF. Let W be any 0–neighbourhood. By weak well–posedness of (P), there exists a 0–neighbourhood O such that

$$\Pi(\varepsilon) \subset \Pi(0) + W_1 ,$$

where W_1 is a 0–neighbourhood in Y such that $W_1 + W_1 \subset W$, and $\varepsilon \in O$, and $\varepsilon \in \mathcal{K}$.

In other words,

$$A_0 \cap f^{-1}\{Min(f, A_0|\mathcal{K}) + (O \cap \mathcal{K}\} - \mathcal{K}) \subset \mathcal{S}(u_0) + W_1 .$$

Hence,

$$A_0 \subset [\mathcal{S}(u_0) + W_1] \cup [X \setminus f^{-1}(\mathcal{L}(O \cap \mathcal{K}))] ,$$

where, as previously,

$$\mathcal{L}(O) = Min(f, A_0|\mathcal{K}) + O - \mathcal{K} .$$

By the same arguments as in Theorem 6.1 we can get rid of the intersection $O \cap \mathcal{K}$ and pass to a certain 0–neighbourhood $\bar{O}$. More precisely, we get

$$A_0 \subset [\mathcal{S}(u_0) + W_1] \cup [X \setminus f^{-1}(\mathcal{L}(\bar{O}))] , \qquad (*)$$

Let us consider now the second term of $(*)$. By the uniform continuity of f, there exists a 0–neighbourhood $\bar{W}$ such that

$$f([X \setminus f^{-1}(\mathcal{L}(\bar{O}))] + \bar{W}) \subset f(X \setminus f^{-1}(\mathcal{L}(\bar{O}))) + \bar{O}_1 ,$$

where a 0–neighbourhood O_1 is such that $\bar{O}_1 + \bar{O}_1 \subset \bar{O}$. Since

$$f(X \setminus f^{-1}(\mathcal{L}(\bar{O}))) + \bar{O}_1 \subset (Y \setminus \mathcal{L}(\bar{O})) + \bar{O}_1 \subset Y \setminus \mathcal{L}(\bar{O}_1) ,$$

we get

$$[X \setminus f^{-1}(\mathcal{L}(\bar{O}))] + \bar{W} \subset f^{-1}(Y \setminus \mathcal{L}(\bar{O}_1)) . \qquad (**)$$

Let us take the 0–neighbourhood $\bar{W} \cap W_1$. By the upper–Hausdorff continuity of $\mathcal{R}$, there exists a neighbourhood U_1 of u_0 such that

$$\mathcal{R}(u) \subset A_0 + \bar{W} \cap W_1 .$$

for $u \in U_1$.

On the other hand, by $(*)$, we have

$$A_0 + \bar{W} \cap W_1 \subset [\mathcal{S}(u_0) + W_1 + \bar{W} \cap W_1] \cup [(X \setminus f^{-1}(\mathcal{L}(\bar{O}))) + \bar{W} \cap W_1],$$

and by $(**)$,

$$A_0 + \bar{W} \cap W_1 \subset [\mathcal{S}(u_0) + W_1 + \bar{W} \cap W_1] \cup f^{-1}(Y \setminus \mathcal{L}(\bar{O}_1)) .$$

Since

$$f^{-1}(Y \setminus \mathcal{L}(\bar{O}_1)) \subset X \setminus f^{-1}(\mathcal{L}(\bar{O}_1))$$

we get

$$A_0 + \bar{W} \cap W_1 \subset [\mathcal{S}(u_0) + W_1 + \bar{W} \cap W_1] \cup [X \setminus f^{-1}(\mathcal{L}(\bar{O}_1))] .$$

Hence,

$$\mathcal{R}(u) \subset [\mathcal{S}(u_0) + W_1 + W_1 \cap \bar{W}] \cup [X \setminus f^{-1}(\mathcal{L}(\bar{O}_1))]$$

for $u \in U_1$. By $(-\mathcal{K})$–upper Hausdorff continuity of $\mathcal{P}$ at u_0, there exists a neighbourhood U_2 of u_0 such that

$$Min(f, \mathcal{R}(u)|\mathcal{K}) \subset Min(f, A_0|\mathcal{K}) + \bar{O}_1 - \mathcal{K} .$$

Now, by taking any $x \in \mathcal{S}(u)$, for $u \in U_1 \cap U_2$ we get

$$f(x) \in Min(f, A_0|\mathcal{K}) + \bar{O}_1 - \mathcal{K} = \mathcal{L}(\bar{O}_1) ,$$

which means that

$$x \notin X \setminus f^{-1}(\mathcal{L}(\bar{O}_1)) .$$

Consequently, we obtain $x \in \mathcal{S}(u_0) + W_1 + \bar{W} \cap W_1$, and finally $x \in \mathcal{S}(u_0) + W$. ■

Till now no sufficient conditions have been formulated for $(-\mathcal{K})$–upper Hausdorff continuity of the performance multifunction $\mathcal{P}$. In Bednarczuk 1992A we have formulated the following sufficient conditions for the upper Hausdorff continuity of $\mathcal{P}$ which is obviously a stronger property.

THEOREM 6.4 (Theorem 4.2 Bednarczuk 1992A) *If*
f is continuous on X,
$\mathcal{R}$ is upper continuous at u_0,
$\mathcal{FR}$ is sup–lower continuous at u_0, uniformly on $Min(f, A_0|\mathcal{K})$,
(CP) holds for $f(A_0)$,
then the performance multifunction $\mathcal{P}$ is upper Hausdorff continuous at u_0.

With the help of this theorem we obtain slightly weaker forms of our main results, Theorem 6.1 and Theorem 6.3.

Taking in account Theorem 6.4 we get the following weaker variant of Theorem 6.1.

THEOREM 6.5 *Suppose that*
f is continuous on X,
$\mathcal{R}$ is upper continuous at u_0,

$\mathcal{FR}$ is sup–lower continuous at u_0, uniformly on $Min(f, A_0|\mathcal{K})$,
(CP) holds for $f(A_0)$,
(P) is well–posed.
Then the solution multifunction $\mathcal{S}$ is upper continuous at u_0.

By applying Theorem 6.4 to Theorem 6.2 we get the following stronger conditions for problems with compact sets of minimal points.

THEOREM 6.6 *Suppose that*
f is continuous on X,
$\mathcal{R}$ is upper continuous at u_0,
$\mathcal{R}$ is lower continuous at u_0,
$Min(f, A_0|\mathcal{K})$ is compact,
(CP) holds for $f(A_0)$,
(P) is η–well posed.
Then $\mathcal{S}$ is upper continuous at u_0.

For upper Hausdorff continuity the following conditions result from Theorem 6.3.

THEOREM 6.7 *Suppose that*
f is uniformly continuous on X,
$\mathcal{R}$ is upper Hausdorff continuous at u_0,
$\mathcal{R}$ is uniformly lower continuous at u_0,
(CP) holds for $f(A_0)$,
(P) is weakly well–set.
Then the solution multifunction $\mathcal{S}$ is upper Hausdorff continuous at u_0.

Since in Theorem 6.7 the function f is assumed to be uniformly continuous on X, we can apply Theorem 3.2 to get rid of the indirect assumption that $\mathcal{FR}$ is uniformly sup–lower continuous at v_0. Let us note that in general we cannot do that in Theorem 6.5 unless we make some additional assumption.

In vector optimization we can recover the results concerning the performance multifunction from that concerning the solution multifunction. In fact, in view of Proposition 5.2 we see that under the containment property (CP) vector optimization problems are weakly well–posed in the objective space and hence we recover Theorem 6.4 from Theorem 6.7.

References

AŚIĆ M.D. AND DUGOŚIJA D. (1986) Uniform convergence and Pareto optimality, *Optimization* **17**, no.6, pp. 723–729

BEDNARCZUK E. (1989) Eps-solutions; well-posedness and stability in vector optimization, Science Report no.35

BEDNARCZUK E. (1992A) Some stability results for vector optimization problems in topological vector spaces, *Proceedings of the First World Congress of Nonlinear Analysts*, Tampa, Florida, August 1992.

BEDNARCZUK E. (1987) Well posedness of vector optimization problems, in *Recent Advances and Historical Development of Vector Optimization Problems*, Springer Verlag, New York.

BEDNARCZUK E. (1992B) On certain variants of the domination property, Technical Raport zpm/1992.

BEDNARCZUK E. AND PENOT J.-P. (1992A) On the positions of the notions of well-posed minimization problems, *Bolletino U.M.I.* **(7)** 6-B, pp. 665–683.

BEDNARCZUK E. AND PENOT J.-P. (1992B) Metrically well-set optimization problems, *Applied Mathematics and Optimization* **26**, pp. 273–285.

HENIG M.I. (1986) The domination property in multicriteria optimization, *JMAA* **(114)**, pp. 7–16.

KRASNOSEL'SKII M.A. (1964) Positive Solutions of Operator Equations, The Netherlands.

KURATOWSKI K. (1966) Topology, Academic Press, New York, Polish Scientific Publishers, Warsaw.

KUTATELADZE S.S. (1976) Convex ϵ-programming, *Doklady ANSSR* **(249)**, no.6, pp. 1048–1059, [*Soviet Mathematics Doklady* **20**(1979) no.2]

LORIDAN P. (1984) ϵ-solutions in vector minimization problems, *Journal of Optimization Theory and Applications* **43**, no.2 pp. 265–276.

LUCCHETTI R. (1987) Well posedness, towards vector optimization, in *Recent Advances and Historical Development of Vector Optimization Problems*, Springer Verlag, New York.

LUC D.T. (1989) Theory of Vector Optimization Springer Verlag, Berlin Heidelberg.

LUC D.T. (1990) Recession cones and the domination property in vector optimization, *Mathematical Programming* **(49)**, pp. 113–122.

NIKODEM K. (1986) Continuity of $\mathcal{K}$-convex set-valued functions, *Bulletin of the Polish Academy of Sciences*, ser. Mathematics **(34)** no.7-8, pp. 393–400.

PENOT J.-P. AND STERNA-KARWAT A. (1986) Parametrized multicriteria optimization: continuity and closedness of optimal multifunctions, *JMAA* **(120)** no.1, pp 150–168.

PENOT J.-P. AND STERNA-KARWAT A. (1989) Parametrized multicriteria optimization: order continuity of the marginal multifunctions, *JMAA*

(**144**) n0.1, pp. 1–15.

PERESSINI A. (1967) Ordered Topological Vector Spaces, *Harper and Row*, New York.

STERNA–KARWAT A. (1989) Convexity of the optimal multifunctions and its consequences in vector optimization, *Optimization* (**20**) no.6, pp. 799–807.

TANINO T., SAWARAGI Y., NAKAYAMA H. (1985) Theory of Multiobjective Optimization, Academic Press, New York.

WIERZBICKI A. (1977) Basic properties of scalarizing functionals for multiobjective optimization, *Mathematische Operationsforschung und Statistik, s. Optimization* **8**, pp. 55–60.