Some remarks on convex duality in normed spaces with and without compactness

by

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We discuss old and new results in convex duality theory related to inf–compactness and coercivity properties of the dual of a convex minimization problem. When coercivity fails we investigate the weaker property of asymptotic well behavior which is characterized in terms of a continuity property of its conjugate function. The characterizations of asymptotic well behavior given in Auslender and Crouzeix (1989) are extended to the case of reflexive Banach spaces.

Keywords: Convex duality, inf–compactness, coercivity, Palais–Smale condition, asymptotic well-behaved convex functions.

1. Introduction

The goal of this paper is to discuss old and new results in convex duality theory, connected with different inf–compactness properties of the dual of a convex minimization problem.

The theory of convex analysis, an outgrowth of Werner Fenchel's work, was mainly achieved in the 60's with the work of Moreau and Rockafellar. We refer to Moreau (1966) for the general facts concerning duality pairings between locally convex topological vector spaces (l.c.t.v.s.), conjugate functions, and subgradients.

The cornerstone of convex analysis is *duality theory* which provides a systematic means of associating a dual problem with any convex minimization problem.

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To be precise, let us recall the abstract setting of convex duality. Given a couple of l.c.t.v.s. (X, τ) and (Y, σ), and a closed proper convex function $\varphi : X \times Y \longrightarrow]-\infty, +\infty]$, the *primal problem* is

$$(P) \qquad \inf\{\varphi(x, 0) : x \in X\},$$

and the *primal marginal function*

$$v(y) := \inf\{\varphi(x, y) : x \in X\}.$$

Interpreting $y \in Y$ as a parameter, problem (P) can be considered as being "convexly" imbedded into a family of optimization problems.

To describe the dual problem let the spaces X and Y be paired with the l.c.t.v.s. (X^*, τ^*) and (Y^*, σ^*) respectively. The conjugate of φ, denoted $\varphi^* : X^* \times Y^* \longrightarrow]-\infty, +\infty]$, induces the *dual problem*

$$(D) \qquad \inf\{\varphi^*(0, y^*) : y^* \in Y^*\},$$

and a corresponding *dual marginal function*

$$w(x^*) := \inf\{\varphi^*(x^*, y^*) : y^* \in Y^*\}.$$

Since $\varphi^{**} = \varphi$, dualizing (D) with respect to the perturbation function φ^* leads back to the primal problem.

The primal marginal function v is convex and a direct computation shows that $v^*(y^*) = \varphi^*(0, y^*)$, so that problem (D) consists in minimizing v^*.

The central issue in duality theory is the absence of *duality gap* (*i.e.* the equality $w(0) = -v(0)$) and the existence of solutions for (P) and (D). These properties rely on stability assumptions usually connected with different types of constraint qualification conditions.

In the "classical" duality theorems, the stability assumption corresponds to the continuity at 0 of the marginal function v. By Moreau's theorem, Moreau (1964), this implies that the function v^* to be minimized in the dual problem is inf–compact. In §2 we provide a (partially) new and unified proof of two such "classical" theorems in the case of Banach spaces: Robinson (1976), Corollary 1, and Rockafellar (1974), Theorem 18(c), and we discuss the relation with inf–compactness of the dual problem.

In §3 we turn the attention to a particular situation where the dual problem may fail to have this inf–compactness property: the Attouch–Brézis duality theorem. We supplement the original result in Attouch and Brézis (1986) by showing that even if the dual problem is not inf–compact, it is the inverse image of an inf–compact convex minimization problem by a continuous linear mapping. We also extend Attouch–Brézis' theorem by considering different hypothesis on the underlying spaces.

We observe in §4 that in the classical as well as Attouch–Brézis' duality theorems, the function v^* satisfies a weak compactness property: every stationary sequence is minimizing. This property, called *asymptotic well behavior* in

Auslender and Crouzeix (1989), may be interpreted as a weakening of the Palais–Smale condition which requires that all stationary sequences be bounded.

In §4 we obtain the main result of this paper characterizing the asymptotic well behavior of v^* in terms of a continuity property of its conjugate v^{**}. In a certain sense this is the analog of Moreau's theorem which relates inf–compactness of v^* with continuity of v^{**} at 0.

We conclude the paper in §5, by extending the characterizations of the asymptotic well behavior given in Auslender and Crouzeix (1989), to the case of reflexive Banach spaces.

In this paper we focus on the properties of the marginal function v and its conjugate v^*. Obviously, the perfect symmetry between primal and dual allows us to obtain similar results for w and w^*. Moreover, by considering a "trivial" perturbation function $\varphi(x,y) \equiv h(y)$, the results can be used to derive properties for a general convex function h.

We shall assume that (Y,σ) is a normed space, there exists a norm $\|\cdot\|$ in Y whose induced topology is σ. The corresponding dual norm in Y^* will be denoted $\|\cdot\|_*$ but, unless otherwise stated, we will not assume reflexivity so that the norm topology in Y^* may not be compatible with the duality.

We denote by $\langle\cdot,\cdot\rangle$ both the duality pairing between Y and Y^* and also between X and X^*. $B(y,r)$ will denote the ball of radius r centered at y, and $d(y,S)$ will stand for the distance from $y \in Y$ to the set $S \subset Y$. Similar conventions apply in X. Finally, for a convex function h we shall denote $S_\alpha(h)$ its (lower) level set

$$S_\alpha(h) := \{z : h(z) \leq \alpha\}.$$

2. The convex duality theorem

As pointed out in the introduction, the primal marginal function v is convex and $v^*(y^*) = \varphi^*(0,y^*)$. Hence problem (D) consists in minimizing v^*, that is to say

$$w(0) = \inf\{v^*(y^*) : y^* \in Y^*\} = -v^{**}(0), \tag{1}$$

and y^* solves (D) if and only if $0 \in \partial v^*(y^*)$ so that

$$S(D) = \partial v^{**}(0). \tag{2}$$

The fundamental link between (P) and (D) is given in the next theorem. The proof can be found in virtually any book on convex analysis so we just give a brief sketch.

THEOREM 2.1 (Basic Duality Theorem) *The primal marginal function v is subdifferentiable at 0 if and only if the optimal values of (P) and (D) satisfy $w(0) = -v(0)$ and the infimum in the dual is attained. In such a case the* *solution set of* (D) *coincides with the subdifferential of the primal marginal function,*

$$S(D) = \partial v(0).$$

PROOF. The result follows by combining (1) and (2) with the implications

$$\partial v(0) \neq \phi \quad \Rightarrow \quad v^{**}(0) = v(0),$$
$$v^{**}(0) = v(0) \quad \Rightarrow \quad \partial v(0) = \partial v^{**}(0),$$

which hold for any convex function v. ∎

A *duality theorem* is any result ensuring the subdifferentiability of v at 0. An important case is when v is continuous at 0 since then $\partial v(0)$ is nonempty and $\sigma(Y^*, Y)$–compact. Since v is convex, the continuity at 0 is equivalent to v being bounded from above in a neighborhood of 0. Thus, the simplest (yet extremely useful) criteria for such continuity is given by

$$(H_1) \qquad \text{there exists } \ x_0 \in X \ \text{ such that } \ \varphi(x_0, \cdot) \ \text{ is continuous at } \ y = 0.$$

A more general condition for continuity of v can be derived in the case where Y is a Banach space.

THEOREM 2.2 *Suppose* (Y, σ) *is a Banach space and assume that either (a) the space* (X, τ) *is Banach, or (b) the space* (X^*, τ^*) *is normed. If*

$$(H_2) \qquad \mathbb{R}_+ \mathrm{dom}(v) = Y,$$

then v *is continuous at 0.*

PROOF. Let us fix $\alpha > v(0)$ and choose $x_0 \in X$ with $\varphi(x_0, 0) < \alpha$. Let B be the closed unit ball of X (for the norm of dual space if we are assuming (X^*, τ^*) normed). The set

$$C := \{(x, y) : \varphi(x, y) \leq \alpha, x \in x_0 + B\}$$

is a nonempty closed convex subset of $X \times Y$, and v is bounded from above by α on the projection C_Y of C onto Y. The conclusion will follow if we show that C_Y is a neighborhood of 0.

To this end we observe that under assumption (H_2) the set C_Y is absorbing: for any $y \in Y$ there exists $t > 0$ such that $ty \in \mathrm{dom}(v)$, thus we can choose $x \in X$ with $\varphi(x, ty) < +\infty$ and then for $\epsilon > 0$ small enough we will have

$$(1 - \epsilon)x_0 + \epsilon x \in x_0 + B$$
$$\varphi((1 - \epsilon)(x_0, 0) + \epsilon(x, ty)) \leq (1 - \epsilon)\varphi(x_0, 0) + \epsilon\varphi(x, ty) \leq \alpha$$

showing that $\epsilon ty \in C_Y$ for all $\epsilon > 0$ small enough.

Since Y is Banach it follows that $\overline{C_Y}$ is a neighborhood of 0, so that all which remains to prove is that

$$\mathrm{int}(\overline{C_Y}) = \mathrm{int}(C_Y). \tag{3}$$

When (X, τ) is Banach, property (3) follows from Robinson (1976), Lemma 1. When (X^*, τ^*) is normed, Alaoglu's theorem implies that B is $\sigma(X, X^*)$-compact from which it follows that C_Y is closed and (3) holds trivially. ■

The above theorem is essentially known. Under assumption *(a)* it was proved by Robinson (1976), Cor. 1, while under assumption *(b)* it is a slight extension of Rockafellar (1974), Thm. 18 (c) (it is assumed there that (X^*, τ^*) is a Banach space).

REMARK. To illustrate the difference and independence of conditions *(a)* and *(b)* in Theorem 2.2, let us consider the case $X = L^\infty(\Omega)$ with Ω an open bounded subset of $\mathbb{R}^n$. Since L^∞ is the dual of $(L^1, \|\cdot\|_1)$, the previous result under assumption *(b)* can be applied using the standard duality pairing between these two spaces. On the other hand, since L^∞ is a Banach space one may also use condition *(a)* but this time the duality to be considered is between L^∞ and its (norm) dual $(L^\infty(\Omega), \|\cdot\|_\infty)^*$ which is not so simple (see Dunford, Schwartz, 1958, p. 296 and Yosida, 1965, p. 118).

Conversely, if the primal decision space is $X = L^1(\Omega)$, case *(a)* of the above theorem applies directly to the L^1-L^∞ duality while condition *(b)* fails since L^1 is not the dual of any normed space.

The proof of Theorem 2.2 suggests that condition *(b)* could be relaxed to: *(b') there exists a convex, absorbing and $\sigma(X, X^*)$–compact subset $B \subset X$.* This condition is in fact equivalent to *(b)* as shown by

PROPOSITION 2.3 *Under assumption (b') the support function of B defines a norm on X^* whose induced topology is compatible with the duality.*

PROOF. Since B is absorbing, the subnorm $\|x^*\|_* := \sup\{\langle x^*, x\rangle : x \in B\}$ is in fact a norm. Let τ^* be the topology on X^* induced by this norm and denote by X^{**} the corresponding dual.

Clearly any $x \in X$ induces a τ^*-continuous linear functional ℓ_x on X^* by

$$\ell_x(x^*) = \langle x^*, x\rangle.$$

We must prove that, conversely, any $\ell \in X^{**}$ is of the form ℓ_x for some $x \in X$. With no loss of generality let us assume that $\|\ell\| \le 1$ and let us prove $\ell \in B^{**} := \{\ell_x : x \in B\}$.

The imbedding $x \to \ell_x$ from X into X^{**} is obviously continuous if we consider in these spaces the topologies $\sigma(X, X^*)$ and $\sigma(X^{**}, X^*)$ respectively, so that B^{**} is $\sigma(X^{**}, X^*)$–compact. If $\ell \notin B^{**}$, using the separation theorem we may find $x^* \in X^*$ such that

$$\sup_{x \in B} \ell_x(x^*) < \ell(x^*).$$

Since by definition the left hand side is $\|x^*\|_*$, the strict inequality contradicts the assumption $\|\ell\| \le 1$. ■

By Moreau's theorem, Moreau (1964), the continuity of v at 0 implies that the function v^* to be minimized in (D) is $\sigma(Y^*, Y)$–inf-compact, that is to say, it has weak* compact level sets. More precisely we have

PROPOSITION 2.4 *If* (Y, σ) *is normed, the following are equivalent:*

(a) v^* *is* $\sigma(Y^*, Y)$*-inf-compact.*
(b) v^* *is coercive, that is to say,* $\lim_{\|y^*\|_* \to \infty} v^*(y^*) = +\infty.$
(c) $\liminf_{\|y^*\|_* \to \infty} v^*(y^*)/\|y^*\|_* > 0.$
(d) v^{**} *is bounded above in a ball around* $0 \in Y$.

PROOF. The equivalence $(a) \Leftrightarrow (b)$ is a simple consequence of Alaoglu's theorem. Let us demonstrate $(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (b)$.

$(b) \Rightarrow (c)$ It suffices to consider the case v^* proper. Translating the origin, if necessary, we may further assume $v^*(0) < +\infty$. Let $r > 0$ be such that $v^*(y^*) \geq v^*(0) + 1$ whenever $\|y^*\|_* \geq r$. For each $y^* \notin B^*(0, r)$, letting $y_r^* = ry^*/\|y^*\|_*$ we get

$$v^*(0) + 1 \leq v^*(y_r^*) \leq (1 - \frac{r}{\|y^*\|_*})v^*(0) + \frac{r}{\|y^*\|_*}v^*(y^*)$$

from which we deduce

$$\liminf_{\|y^*\|_* \to \infty} \frac{v^*(y^*)}{\|y^*\|_*} \geq \frac{1}{r} > 0.$$

$(c) \Rightarrow (d)$ Take $\epsilon > 0$ and $r > 0$ such that $v^*(y^*) \geq \epsilon\|y^*\|_*$ whenever $\|y^*\|_* \geq r$. If $\|y\| \leq \epsilon/2$ we get

$$v^{**}(y) \leq \sup_{y^*}[\frac{\epsilon}{2}\|y^*\|_* - v^*(y^*)] \leq \max\{\sup_{\|y^*\|_* \leq r}[\frac{\epsilon r}{2} - v^*(y^*)], -\frac{\epsilon r}{2}\}.$$

But (c) also implies that v^* can be minorized by an affine function of the form $\langle y^*, y_0\rangle + \alpha$ with $y_0 \in Y$, and then

$$v^{**}(y) \leq \max\{\frac{\epsilon r}{2} + r\|y_0\| - \alpha, -\frac{\epsilon r}{2}\}$$

for all $y \in B(0, \epsilon/2)$.

$(d) \Rightarrow (b)$ If v^{**} is bounded above by M on $B(0, \epsilon)$ then

$$v^*(y^*) \geq \sup_{y \in B(0,\epsilon)} \langle y^*, y\rangle - M = \epsilon\|y^*\|_* - M$$

from which (b) follows at once. ∎

3. Attouch–Brézis' duality theorem

In the setting of Theorem 2.2, Proposition 2.4 shows that (H_2) is restricted to the case in which the dual problem (D) is inf–compact. While this is a very favorable situation, there exist some cases where such inf–compactness is hopeless. Thus, a natural question is to which extent the Basic Duality Theorem stays valid when dropping assumption (H_2).

In 1986, Attouch and Brézis proved a duality formula for the conjugate of the sum of two convex functions under the following weakening of (H_2),

$$(H_3) \qquad E := \mathbb{R}_+ \mathrm{dom}(v) \ \text{ is a closed subspace of } Y.$$

This approach was also pursued in the subsequent papers : Auslender et al. (1993), Borwein et al. (1988), Borwein and Lewis (1992), Seetharama Gowda and Teboulle (1990), Rodrigues and Simons (1988), Volle (1992), Zalinescu (1987), where duality theorems for different duality schemes were obtained. Reference Seetharama Gowda and Teboulle (1990) contains a comparative study of different constraint qualification conditions.

When $E \neq Y$, the dual problem (D) may fail to be inf–compact. However, by considering E as the perturbation space one can see that such inf–compactness does hold "up to a projection operation". This is stated precisely in the following result which is essentially Attouch–Brézis' theorem in the abstract framework of duality via perturbation functions (we add, however, some remarks on the structure and compacity properties of the dual problem, as well as the case in which (X^*, τ^*) is a normed space). It may be worth noticing here that most duality schemes (Lagrangian duality, perturbational duality, Fenchel's duality, etc.) are equivalent in this sense that the results may be easily transferred from one setting to another. The proof we present is essentially by reduction to the framework of Theorem 2.2.

THEOREM 3.1 *Suppose that (Y, σ) is Banach and that either (X, τ) is Banach or (X^*, τ^*) is normed. If (H_3) holds, the primal marginal function v is subdifferentiable at 0, that is, we have the conclusions of the Basic Duality Theorem. Moreover, the dual problem (D) can be expressed as*

$$(D) \qquad \inf\{v_E^*(\Pi_E y^*) : y^* \in Y^*\},$$

where v_E^* is the Fenchel conjugate of the restriction of v to E, which is $\sigma(E^*, E)$–inf–compact, and $\Pi_E : Y^* \to E^*$ is the continuous linear mapping associating with a functional $y^* \in Y^*$ its restriction to E. The solution set of (D) can be expressed as the inverse image by Π_E of the weakly compact set $\partial v_E(0)$,

$$S(D) = \Pi_E^{-1}[\partial v_E(0)].$$

PROOF. Let φ_E be the restriction of φ to $X \times E$ and

$$v_E(y) := \inf\{\varphi_E(x, y) : x \in X\}$$

the associated marginal function. It is easily checked that v_E is the restriction of v to E, and more precisely

$$v(y) = \begin{cases} v_E(y) & \text{if } y \in E \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

Now, E is a Banach space and we have $\mathbb{R}_+\text{dom}(v_E) = E$ so that Theorem 2.2 applies: v_E is continuous at 0 and we have the duality relation

$$v_E(0) = -\min\{v_E^*(e^*) : e^* \in E^*\} \tag{5}$$

the minimum on the right having a nonempty $\sigma(E^*, E)$–compact solution set $\partial v_E(0)$. Moreover, the function v_E^* is $\sigma(E^*, E)$–inf–compact.

From (4), and after a direct computation, we get

$$v^*(y^*) = v_E^*(\Pi_E y^*)$$

and since (by the Hahn–Banach theorem) the mapping Π_E is surjective, property (5) can be written as

$$v(0) = -\min\{v_E^*(\Pi_E y^*) : y^* \in Y^*\} = -\min\{v^*(y^*) : y^* \in Y^*\}$$

the minimum being attained with solution set

$$\partial v(0) = \{y^* \in Y^* : \Pi_E y^* \in \partial v_E(0)\}.$$

proving the result. ■

The previous theorem shows that problem (D) is the inverse image by Π_E of the inf–compact optimization problem (5). In particular, the optimal solutions of (D) are all those functionals which are obtained by extension of the functionals e^* which solve the minimum problem (5).

REMARK. In the particular case where Y is finite dimensional, the closedness requirement in assumption (H_3) is superfluous and (H_3) is equivalent to the well known "relative interiority" condition

$$(H_4) \qquad 0 \in \text{ri}(\text{dom}(v)).$$

Moreover, in this finite dimensional setting the space (X, τ) may be any l.c.t.v.s. and need not be a Banach or the dual of a normed space.

4. Asymptotic well–behaved convex functions

We have seen that hypotheses (H_1) through (H_4) imply that 0 belongs to the (relative) interior of the domain of v and correspond to some type of inf–compactness for the dual problem (D). In general, however, the set $\mathbb{R}_+\text{dom}(v)$ will only be a convex cone, so we are led to investigate the properties of (D) when 0 is a boundary point of the domain of v, and specifically when 0 is a "corner" of the domain of v.

More precisely, we will study the asymptotic well behavior of v^*.

DEFINITION 4.1 *Let (Y,σ) be a normed space. The sequence $\{y_k^*\}$ is said to be* stationary *for v^* whenever*

$$d(0, \partial v^*(y_k^*)) \to 0.$$

The function v^* is said to be asymptotic well-behaved *if every stationary sequence is minimizing, that is to say,*

$$\lim_k v^*(y_k^*) = w(0).$$

The notion of asymptotic well behaviour was first considered in Auslender and Crouzeix (1989) where, in a finite dimensional setting, different characterizations were obtained.

In order to motivate this property and to make the link with inf–compactness we briefly discuss the Palais–Smale condition. We recall that a (not necessarily convex) Fréchét differentiable function h defined on a normed space Z is said to satisfy the Palais–Smale condition if every sequence $\{z_k\}$ with $h(z_k)$ bounded and $||h'(z_k)||_*$ tending to zero, is relatively compact. Recently Shujie (1986) (see also Costa and Silva, 1991) has proved that when Z is a Banach space such a function is necessarily coercive.

In the case of the convex function v^* we modify the Palais–Smale condition as:

$(\tilde{PS})$ every stationary sequence for v^* is $\sigma(Y^*, Y)$–relatively compact.

Notice that we consider arbitrary stationary sequences regardless of whether the function values are bounded or not.

Clearly, any $\sigma(Y^*, Y)$–accumulation point of a stationary sequence for v^* will be a critical point, thus a solution of (D). It is also easy to check that a bounded stationary sequence is minimizing so that $(\tilde{PS})$ implies asymptotic well behavior.

PROPOSITION 4.2 *Let (Y,σ) be a normed space and suppose that the primal marginal function v is such that its conjugate v^* is not identically $+\infty$, that is to say, $w(0) < +\infty$. If v^* is coercive then it is bounded from below and satisfies $(\tilde{PS})$. The converse holds when (Y,σ) is reflexive.*

PROOF. If v^* is coercive then it is proper and $\sigma(Y^*, Y)$–inf–compact. Hence its infimum is attained and is not $-\infty$ proving that v^* is bounded from below.

Let us consider next a stationary sequence $\{y_k^*\}$ for v^* and choose $y_k \in \partial v^*(y_k^*)$ such that $||y_k|| \to 0$. If $\{y_k^*\}$ were unbounded, passing to a subsequence so that $||y_k^*|| \to \infty$ and using Proposition 2.4 we could find $\epsilon > 0$ such that $v^*(y_k^*) \geq \epsilon ||y_k^*||_*$ for all k large enough, and then

$$v^{**}(y_k) = \langle y_k^*, y_k \rangle - v^*(y_k^*) \leq (||y_k|| - \epsilon)||y_k^*||_*.$$

From this we get $v^{**}(y_k) \to -\infty$ so that $v^{**}(0) = -\infty$. But this implies $v^* \equiv +\infty$ contradicting our assumption.

The converse implication, in the case of (Y, σ) reflexive, can be proved by adapting the proof of Costa and Silva (1991), Thm. 4 (which is an application of Ekeland's Variational Principle). ■

The previous result shows that under the assumptions of Theorem 2.2, (H_2) implies that v^* satisfies $(\tilde{PS})$. When (H_2) is replaced by (H_3) we obtain,

PROPOSITION 4.3 *With the assumptions of Theorem* 3.1 *and* $w(0) < +\infty$, *the function* v_E^* *satisfies* $(\tilde{PS})$ *and the function* v^* *is asymptotic well-behaved.*

PROOF. From Theorem 3.1 the function v_E^* is coercive so that it satisfies $(\tilde{PS})$. In particular v_E^* is asymptotic well-behaved.

To prove the asymptotic well behavior of v^* let $\{y_k^*\}$ be a stationary sequence. It is easy to see that $\partial v^*(y_k^*) \subset \partial v_E^*(\Pi_E y_k^*)$ so that $\{\Pi_E y_k^*\}$ is stationary, hence minimizing, for v_E^*. Since $v^*(y_k^*) = v_E^*(\Pi_E y_k^*)$ it follows that $\{y_k^*\}$ is minimizing for v^* as was to be proved. ■

This proposition partially extends the finite-dimensional result, Auslender et al. (1993), Thm. 3.1. As a matter of fact, it was shown in the latter that when $0 \in \mathrm{ri}(\mathrm{dom}(v))$ a stationary sequence $\{y_k^*\}$ is not only minimizing but converges towards the set of optimal solutions of (D)

$$\lim_{k \to \infty} d(y_k^*, S(D)) = 0.$$

In the present setting all we can assert is that there exist weak* accumulation points of $\{\Pi_E y_k^*\}$ and they are solutions of

$$\min\{v_E^*(e^*) : e^* \in E^*\}.$$

To complete the previous discussion let us mention the following

PROPOSITION 4.4 *Suppose that* v^* *is such that every minimizing sequence is bounded. Then* v^* *is coercive.*

PROOF. By contradiction suppose we can find a sequence $\{y_k^*\}$ with $\|y_k^*\|_* \to +\infty$ and $v^*(y_k^*)$ bounded above, let us say by $M \in \mathbb{R}$. Let us choose a minimizing sequence x_k^* and define

$$z_k^* = (1 - \alpha_k)x_k^* + \alpha_k y_k^*$$

where $\alpha_k = 1/\sqrt{\|y_k^* - x_k^*\|_*}$. By our assumption $\{x_k^*\}$ is bounded so that α_k is well defined and tends to 0. Now,

$$\|z_k^* - x_k^*\|_* = \alpha_k \|y_k^* - x_k^*\|_* = \sqrt{\|y_k^* - x_k^*\|_*} \;\to\; +\infty$$

so that $\|z_k^*\|_* \to +\infty$. However,

$$v^*(z_k^*) \le (1 - \alpha_k)v^*(x_k^*) + \alpha_k v^*(y_k^*) \le (1 - \alpha_k)v^*(x_k^*) + \alpha_k M$$

which implies that $\{z_k^*\}$ is an unbounded minimizing sequence contradicting our assumption. ■

Among the papers where the notion of asymptotic well behavior has been studied, let us mention the characterizations given in Angleraud (1992) and also the link between asymptotic well–behavior, well posedness and well conditioning presented in Lemaire (1992), the latter in the infinite dimensional case.

In what follows we give a new characterization of the asymptotic well behavior, which is closely connected with the results in Angleraud (1992). Let us briefly describe some of the latter. First of all in Angleraud (1992), Thm. 3 it is shown that a sublinear function h is asymptotic well–behaved if and only if the domain of its conjugate h^* is *locally conical at 0*. We recall that a convex subset A of a normed space is said to be locally conical at 0 if $0 \in \overline{A}$ and there exists $\epsilon > 0$ such that

$$\operatorname{cone}(A) \cap B(0,\epsilon) \subset A.$$

The condition $0 \in \overline{A}$ was not a part of the definition of local conicity considered in Angleraud (1992) but here we will only consider this case.

For a general convex function h its asymptotic well behavior implies the local conicity of $\overline{\operatorname{dom}(h^*)}$, Angleraud (1992), Thm. 6. Conversely, Angleraud (1992), Thm. 7, if $\operatorname{dom}(h^*)$ is locally conical at 0, a sufficient condition for asymptotic well behavior of h is that h^* be bounded above on a neighborhood of 0, or more precisely

$$\sup\{h^*(z^*) : z^* \in \operatorname{dom}(h^*) \cap B(0,\epsilon)\} < +\infty \tag{6}$$

for some $\epsilon > 0$. These conditions are obviously satisfied in the situations described in Theorems 2.2 and 3.1. We also point out that, as mentioned in Angleraud (1992), they are not necessary in general.

Our next result characterizes the asymptotic well behavior in terms of a continuity property of the conjugate function, in the same way as Moreau's theorem relates inf–compactness of a convex function with the continuity of its Fenchel conjugate at 0.

THEOREM 4.5 *Let (Y,σ) be a normed space and suppose that v^* is proper. If*

$$\lim_{\substack{\|y\|\to 0\\ y\in \operatorname{dom}(\partial v^{**})}} [2v^{**}(y) - v^{**}(2y)] = v^{**}(0) \tag{7}$$

then v^* is asymptotic well-behaved. The converse holds if $\operatorname{dom}(\partial v^{**})$ *is locally conical at 0.*

PROOF. Let $\{y_k^*\}$ be a stationary sequence for v^* and select $y_k \in \partial v^*(y_k^*)$ with $\|y_k\| \to 0$. We recall that the infimum of v^* on Y^* is $w(0) = -v^{**}(0)$ so that using Fenchel's inequality $v^{**}(y_k) + \langle y_k^*, 2y_k - y_k\rangle \le v^{**}(2y_k)$, we get

$$w(0) \le v^*(y_k^*) = \langle y_k^*, y_k\rangle - v^{**}(y_k) \le v^{**}(2y_k) - 2v^{**}(y_k).$$

The conclusion follows: if (7) holds then $\{y_k^*\}$ is a minimizing sequence for v^*.

To prove the converse let us suppose that $\text{dom}(\partial v^{**})$ is locally conical at 0, and consider a sequence $\{y_k\} \subset \text{dom}(\partial v^{**})$ such that $\|y_k\| \to 0$. The local conicity of $\text{dom}(\partial v^{**})$ implies that $2y_k \in \text{dom}(\partial v^{**})$ for k large enough, so we may choose $y_k^* \in \partial v^{**}(2y_k)$. By convexity we have

$$\begin{aligned} v^{**}(0) \ \geq \ 2v^{**}(y_k) - v^{**}(2y_k) &= v^{**}(2y_k) + 2[v^{**}(y_k) - v^{**}(2y_k)] \\ &\geq v^{**}(2y_k) + 2\langle y_k^*, y_k - 2y_k \rangle \\ &= v^{**}(2y_k) - \langle y_k^*, 2y_k \rangle \\ &= -v^*(y_k^*). \end{aligned}$$

Since $2y_k \in \partial v^*(y_k^*)$ it follows that $\{y_k^*\}$ is a stationary sequence for v^* and then it is minimizing. Thus the right hand side above tends to $-w(0) = v^{**}(0)$ and property (7) follows. ■

REMARK. In the previous theorem one may replace condition (7), both for the direct as well as the converse implications and with essentially the same proof, by

$$\lim_{\substack{\|y\| \to 0 \\ y \in \text{dom}(\partial v^{**})}} [\lambda v^{**}(y) - v^{**}(\lambda y)] = (\lambda - 1) v^{**}(0)$$

for any $\lambda > 1$.

It is clear that (7) holds when v^{**} is continuous at 0 relative to its domain (assumed to be locally conical at 0). However, this assumption is not necessary as shown by the counterexamples in Angleraud (1992).

Concerning the hypothesis "$\text{dom}(\partial v^{**})$ *is locally conical at 0*", let us observe that when (Y, σ) is a Banach space Bronsted–Rockafellar's theorem implies that $\overline{\text{dom}(\partial v^{**})} = \overline{\text{dom}(v^{**})}$, and then Angleraud (1992), Thm. 6 shows that this closure is locally conical as soon as v^* is asymptotic well behaved. However, it is not clear whether the asymptotic well behavior of v^* implies $\text{dom}(\partial v^{**})$ is locally conical at 0.

5. The reflexive case

We conclude this work by addressing the question of extending the results in Auslender and Crouzeix (1989) to the infinite dimensional setting. More precisely we assume from now on that (Y, σ) is a reflexive Banach space and we characterize the asymptotic well behavior of v^* in terms of the following quantities : let $m := \inf\{v^*(y^*) : y^* \in Y^*\}$ be the optimal value of the dual problem and define for each $\alpha > m$,

$$\begin{aligned} r(\alpha) &:= \inf\{\|y\| : y \in \partial v^*(y^*), v^*(y^*) = \alpha\} \\ l(\alpha) &:= \inf\{[v^*(y^*) - \alpha]/d(y^*, S_\alpha(v^*)) : v^*(y^*) > \alpha\}. \end{aligned}$$

To prove the main result of this section we need the following lemmas.

LEMMA 5.1 *Let $\alpha > m$ and $y^* \notin S_\alpha(v^*)$ with $v^*(y^*) < +\infty$. If $x^* \in S_\alpha(v^*)$ is a projection of y^* onto $S_\alpha(v^*)$ (there exists at least one!) then we have $v^*(x^*) = \alpha$ and we may find $x \in Y$ and $\beta > 0$ such that $-\beta x \in \partial v^*(x^*)$ and*

$$\langle x, x^* - y^* \rangle = \|x\| \, \|x^* - y^*\|_*. \tag{8}$$

PROOF. The function $z^* \to \frac{1}{2}\|z^* - y^*\|_*^2$ is a closed convex function which is $\sigma(Y^*, Y)$–inf–compact (by Alaoglu's theorem). Hence there exists at least a projection x^* of y^* onto the $\sigma(Y^*, Y)$–closed set $S_\alpha(v^*)$, that is to say, a solution of

$$(P_\alpha) \qquad \min\{\frac{1}{2}\|z^* - y^*\|_*^2 : v^*(z^*) \le \alpha\}.$$

Clearly $v^*(x^*) \le \alpha$. On the other hand, for $t \in (0,1)$ we have $x^* + t(y^* - x^*) \notin S_\alpha(v^*)$ and then

$$\alpha \le (1-t)v^*(x^*) + tv^*(y^*).$$

Letting $t \downarrow 0$ we deduce $v^*(x^*) \ge \alpha$ hence $v^*(x^*) = \alpha$ as claimed.

We now perturb (P_α) by considering the perturbation function

$$\varphi(z^*, t) = \begin{cases} \frac{1}{2}\|z^* - y^*\|_*^2 & \text{if } v^*(z^*) \le \alpha - t \\ +\infty & \text{otherwise.} \end{cases}$$

The perturbation parameter is $t \in \mathbb{R}$ and we obtain the dual problem

$$(D_\alpha) \qquad \min_{\mu \ge 0} \mu[\alpha - v_\mu^*(y^*)],$$

where

$$v_\mu^*(y^*) := \inf_{z^* \in Y^*} [v^*(z^*) + \frac{1}{2\mu}\|z^* - y^*\|_*^2].$$

Since $\alpha > m$, Theorem 2.2 can be applied to deduce that (D_α) has an optimal solution $\mu \ge 0$.

We observe next that the solution x^* of (P_α) also minimizes the function

$$\Gamma(z^*) := \frac{1}{2}\|z^* - y^*\|_*^2 + \mu(v^*(z^*) - \alpha).$$

Indeed, for any $z^* \in Y^*$ we have

$$\Gamma(z^*) \ge \mu(v_\mu^*(y^*) - \alpha),$$

and since μ is optimal for (D_α), x^* optimal for (P_α) and since the optimal values are related as $-v(D_\alpha) = v(P_\alpha)$, we deduce

$$\Gamma(z^*) \ge \frac{1}{2}\|x^* - y^*\|_*^2 = \Gamma(x^*)$$

where the last equality follows since $v^*(x^*) = \alpha$.

The fact that x^* minimizes Γ implies $\mu > 0$ since otherwise we would have $x^* = y^*$ which is impossible since $y^* \notin S_\alpha(v^*)$.

Moreover, the optimality of x^* gives $0 \in \partial\Gamma(x^*)$. Taking $z^* = y^*$ we see that there exists a point where v^* is finite and $\frac{1}{2}\|\cdot - y^*\|_*^2$ is continuous (w.r.t. the norm topology on Y^* which is compatible with the duality since we are assuming (Y, σ) reflexive). We may then use the calculus rule for the subdifferential of a sum, to assert the existence of $x \in \partial[\frac{1}{2}\|\cdot - y^*\|_*^2](x^*)$ with $-x \in \mu\partial v^*(x^*)$. It is not difficult to prove that the first condition implies (8) so we may conclude by taking $\beta = 1/\mu$. ∎

LEMMA 5.2 *If $m < \alpha < \alpha'$ then $r(\alpha) \leq l(\alpha) \leq r(\alpha') \leq l(\alpha')$, and we have the alternative characterization*

$$r(\alpha) = \inf\{\|y\| : y \in \partial v^*(y^*), v^*(y^*) \geq \alpha\}.$$

PROOF. $r(\alpha) \leq l(\alpha)$. We must prove that for each y^* such that $v^*(y^*) > \alpha$ we have

$$r(\alpha) \leq \frac{v^*(y^*) - \alpha}{d(y^*, S_\alpha(v^*))}.$$

This inequality is evident if $v^*(y^*) = +\infty$. Otherwise let us denote by x^* a projection of y^* onto $S_\alpha(v^*)$ and let x, β be given as in the previous lemma. Then we have

$$\frac{v^*(y^*) - \alpha}{d(y^*, S_\alpha(v^*))} = \frac{v^*(y^*) - v^*(x^*)}{\|y^* - x^*\|_*}$$

and since $-\beta x \in \partial v^*(x^*)$ and using (8) we deduce

$$\frac{v^*(y^*) - \alpha}{d(y^*, S_\alpha(v^*))} \geq \frac{\langle -\beta x, y^* - x^*\rangle}{\|y^* - x^*\|_*} = \|\beta x\| \geq r(\alpha).$$

$l(\alpha) \leq r(\alpha')$. Let y^* be such that $v^*(y^*) = \alpha'$ and let $y \in \partial v^*(y^*)$. Denoting as before x^* a projection of y^* onto $S_\alpha(v^*)$ we get

$$l(\alpha) \leq \frac{v^*(y^*) - v^*(x^*)}{\|y^* - x^*\|_*} \leq \frac{-\langle y, x^* - y^*\rangle}{\|y^* - x^*\|_*} \leq \|y\|$$

and we may conclude by taking the infimum over all $y \in \partial v^*(y^*)$ and all y^* with $v^*(y^*) = \alpha'$.

The alternative characterization of r follows from the monotonicity of r. ∎

We may now prove,

THEOREM 5.3 *If (Y, σ) is reflexive, the following are equivalent:*

(*a*) *v^* is asymptotic well-behaved.*

(*b*) *$r(\alpha) > 0$ for all $\alpha > m$.*

(*c*) *$l(\alpha) > 0$ for all $\alpha > m$.*

(*d*) *Every stationary sequence* $\{y_k^*\}$ *for* v^* *with* $v^*(y_k^*)$ *bounded from above is minimizing.*

PROOF. The implication $(a) \Rightarrow (d) \Rightarrow (b)$ and the equivalence between (b) and (c) are obvious from the definition of r and the previous lemma respectively.

To prove $(b) \Rightarrow (a)$ we proceed by contradiction: if $\{y_k^*\}$ is a stationary sequence which is not minimizing we may find $\alpha > m$ and a subsequence, still denoted $\{y_k^*\}$, such that $v^*(y_k^*) \geq \alpha$. The alternative characterization of $r(\alpha)$ in the previous lemma implies $r(\alpha) = 0$ contradicting (b). ■

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