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One–parametric optimization: jumps in the set of generalized critical points[1)]

by

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We consider a one–parametric finite–dimensional optimization problem $P(t)$ whose describing functions are three times continuously differentiable and belong to the generic subset $\mathcal{F}$ introduced by Jongen/Jonker/Twilt. We construct feasible descent directions — so–called jumps — at bifurcation and turning points of the set of generalized critical points Σ of $P(t)$ (in former papers such jumps have already been constructed for the special case that the considered point is an endpoint of a branch consisting of local minimizers). By using these jumps together with pathfollowing methods several connected components of Σ can be followed in order to compute solution points of $P(\bar{t})$ (e.g. stationary points or local minimizers) for finitely many parameter values $\bar{t}$ in a given interval.

Key words: One–parametric optimization, pathfollowing methods, generalized critical points, jumps, feasible descent direction, generic class $\mathcal{F}$ by Jongen/Jonker/Twilt

1. Introduction

Let R^n be the n–dimensional Euclidean space and $C^k(R^n, R), k \geq 1$ the space of k–times continuously differentiable functions.
We consider the following nonlinear optimization problem depending on the real parameter t:

$$P(t) \qquad \text{Minimize } f(\cdot, t) \quad \text{subject to } M(t),$$

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where the feasible set $M(t)$ is defined by

$$M(t) = \{x \in R^n | h_i(x,t) = 0, i \in I, g_j(x,t) \leq 0, j \in J\}$$

as well as $I = \{1, \ldots, m\}, m < n$, $J = \{1, \ldots, s\}$ and $(f, h_i, g_j, i \in I, j \in J) \in C^3(R^n \times R, R)^{1+m+s}$.

Furthermore, let $z = (x,t)$ and $Df(\bar{z}) = (D_x f(\bar{z}), D_t f(\bar{z}))$ denote the gradient (row vector) and $D_x f(\bar{z})$, $D_t f(\bar{z})$ the corresponding partial derivatives of f at $\bar{z} = (\bar{x}, \bar{t})$. $D^2 f(\bar{z})$ and $D^3 f(\bar{z})$ are analogously defined.
For $\bar{x} \in M(\bar{t})$ let $J_0(\bar{z}) = \{j \in J | g_j(\bar{z}) = 0\}$ be the index set of active inequality constraints at $\bar{z}$.
According to the definitions in Jongen, Jonker, Twilt 1986A, 1986B, a point $\bar{z}$ is called a *generalized critical point (g.c. point)* if $\bar{x} \in M(\bar{t})$ and the set $\{D_x f(\bar{z}), D_x h_i(\bar{z}), i \in I, D_x g_j(\bar{z}), j \in J_0(\bar{z})\}$ is linearly dependent. By Σ we denote the set of all generalized critical points. If x is a local minimizer of $P(t)$, then $(x,t) \in \Sigma$. In Jongen, Jonker, Twilt 1986A, 1986B the C^3_s–open and dense subset $\mathcal{F} \subset C^3(R^n \times R, R)^{1+m+s}$ is defined, where C^3_s means the strong (or Whitney-) C^3–topology (cf. e.g. Hirsch 1976, Jongen, Jonker, Twilt 1986C). Under the assumption

$$(f, h_i, g_j, i \in I, j \in J) \in \mathcal{F} \tag{1.1}$$

the local structure of Σ is completely described in Jongen, Jonker, Twilt 1986A, 1986B by defining five different types of g.c. points, i.e. Σ can be divided in five disjoint sets and every g.c. point is of exactly one of the five Types 1,2,3,4 and 5.
For more details on the set $\mathcal{F}$ we refer to Jongen, Jonker, Twilt 1986A, 1986B.

<u>Throughout this paper we assume (1.1).</u>

A point $\bar{z} \in \Sigma$ is called a *stationary point* if there exist real numbers λ_i, $i \in I$, $\mu_j \geq 0$, $j \in J_0(\bar{z})$ such that

$$D_x f(\bar{z}) + \sum_{i \in I} \lambda_i D_x h_i(\bar{z}) + \sum_{j \in J_0(\bar{z})} \mu_j D_x g_j(\bar{z}) = 0.$$

By Σ^s we denote the set of all stationary points.
In Guddat, Guerra, Jongen 1990 some concepts and algorithms were presented for the computation of an appropriately fine discretization of the interval $[0,1]$ (or another compact parameter interval):

$$0 = t_0 < \ldots < t_{l-1} < t_l < \ldots < t_N = 1$$

and corresponding local minimizers $x(t_l)$, $l = 1, \ldots, N$. There, so–called path-following — or continuation — methods have been mainly used.
When following a branch of local minimizers it may happen that this branch has an endpoint w.r.t. the set $\Sigma^l = \{z \in \Sigma | x \text{ is a local minimizer of } P(t)\}$,

i.e. a singularity appears. For this reason in Guddat, Guerra, Jongen 1990, Guddat, Jongen, Nowack 1988 it was investigated for which singularities one can construct a jump to another connected component in Σ^l. At points of Type 2 that are turning points in Σ^s, at points of Type 3 and in one of two possible situations at points of Type 4 jumps have been proposed. Algorithms based on these investigations are motivated e.g. by

(i) solving optimization problems depending naturally on a parameter (e.g. time, temperature), cf. e.g. Guddat, Guerra, Jongen 1990, Guddat, Römisch, Schultz 1992;

(ii) the so-called globalization of locally convergent algorithms, cf. e.g. Gollmer at al. 1993, Guddat 1987 and global optimization, cf. e.g. Guddat, Guerra, Jongen 1990;

(iii) solving problems in multiobjective optimization, cf. e.g. Guddat, Guerra, Nowack 1991, Guddat at al. 1985, in stochastic optimization, cf. e.g. Guddat at al. 1985, and in multilevel optimization, cf. e.g. Tammer 1987.

The goal of this paper is to generalize these jumps for the same singularities in the sets Σ^s or even Σ. Of course, this is an extension in order to find more connected components in Σ^s or Σ.

A motivation for doing this comes from the fact that sometimes (cf. (ii) and (iii)) one is interested in following as many connected components of Σ (restricted to a parameter interval) as possible, cf. e.g. Guddat, Guerra, Jongen 1990, too.

The paper is organized as follows. In Section 2 we give some preliminary remarks and lemmas as well as describe briefly a g.c. point of Type 1.

In Section 3 we present the jumps and give some concluding remarks.

2. Auxiliary results

For our investigations we need the following well-known constraint qualification. We say that the *Linear Independence Constraint Qualification (LICQ)* is satisfied at $\bar{x} \in M(\bar{t})$ if the set $\{D_x h_i(\bar{z}),\ i \in I, D_x g_j(\bar{z}), j \in J_0(\bar{z})\}$ is linearly independent.

REMARK 1 *Let $\bar{x} \in M(\bar{t})$, $\tilde{J} \subset J_0(\bar{z})$ and the set $\{D_x h_i(\bar{z}), i \in I, D_x g_j(\bar{z}), j \in \tilde{J}\}$ be linearly independent. Then, the set $\{z \in R^{n+1} |\ h_i(z) = 0, i \in I, g_j(z) = 0, j \in \tilde{J}\}$ is locally in an open neighbourhood of $\bar{z}$ an $(n+1) - m - |\tilde{J}|$-dimensional C^3-manifold (where $|\cdot|$ denotes the cardinality). For local investigations in a neighbourhood of $\bar{z}$ we can restrict ourselves w.l.o.g. (i.e. without loss of generality) to this manifold; i.e. to the space $R^{n+1-m-|\tilde{J}|}$ after an appropriate C^3-coordinate transformation and after deleting the constraints $h_i(z) = 0$, $i \in I$, $g_j(z) = 0$, $j \in \tilde{J}$.*

In the following lemma let A be an (n,n)–matrix and B an (n,n')–matrix. Furthermore, set

$$B^{\perp} = \{x \in R^n | B^T x = 0\}$$

and denote by $\text{In}(A) = (n(A), p(A), o(A))$ the so–called inertia tripel of A, where $n(A), p(A)$ resp. $o(A)$ are the number of negative, positive resp. zero eigenvalues (with counted multiplicities) of A. Now, let V be a matrix whose columns form a basis of $B^{\perp}$. According to Sylvester's law the tripel $\text{In}(V^T AV)$ is independent on the choice of V and, thus, we can define

$$\text{In}(A|B^{\perp}) = \text{In}(V^T AV).$$

LEMMA. *Let the matrices A, B be defined as above and*

$$E = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}.$$

a) (cf. Ouelette 1981) If A is nonsingular then

$$In(E) = In(A) + In(-B^T AB).$$

b) (Jongen et al. 1987, Theorem 2.1). If $rank(B) = n''$, then

$$In(E) = In(A|B^{\perp}) + (n'', n'', n' - n'').$$

REMARK 2 *Let us recall the notation of Kojima's strong stability (cf. Kojima 1980) for an optimization problem having only equality constraints. We consider the problem*

$$(\tilde{P}) \qquad \textit{Minimize } \tilde{f}(y) \textit{ subject to } \{y \in R^{\tilde{n}} |\, \tilde{h}_i(y) = 0, i = 1, \ldots, \tilde{m}\},$$

where $(\tilde{f}, \tilde{h}_i, i = 1, \ldots, \tilde{m}) \in C^2(R^{\tilde{n}}, R)^{1+\tilde{m}}$ as well as $\alpha \in R^{\tilde{m}}$, $\alpha = (\alpha_1, \ldots \alpha_{\tilde{m}})$ and $\tilde{L}(y, \alpha) .= \tilde{f}(y) + \sum_{i=1}^{\tilde{m}} \alpha_i \tilde{h}_i(y)$.

Now, let

- *$\tilde{y}$ be a g.c. point of $(\tilde{P})$,*
- *(LICQ) be satisfied at $\tilde{y}$,*
- *$\tilde{\alpha}_i, i = 1, \ldots, \tilde{m}$ be the uniquely determined solutions of $D_y \tilde{L}(\tilde{y}, \alpha) = 0$ and*
- *$\tilde{B} = \{y \in R^{\tilde{n}} | D_y \tilde{h}_i(\tilde{y}) y = 0\}$.*

According to Kojima 1980, Theorem 3.5, $\tilde{y}$ is strongly stable if and only if $o(D_y^2 \tilde{L}(\tilde{y}, \tilde{\alpha}) | \tilde{B}) = 0$.

Now, we return to our one–parametric problem $P(t)$ and the above mentioned five types of g.c. points introduced in Jongen, Jonker, Twilt 1986B. Define

$$L(z, \lambda, \mu) = f(z) + \sum_{i \in I} \lambda_i h_i(z) + \sum_{j \in J_0(z)} \mu_j g_j(z)$$

with $\lambda \in R^m$, $\mu \in R^{|J_0(z)|}$ (λ_i, μ_j are the components of λ, μ, resp.) as well as

$$B(z) = \{\xi \in R^n | D_x h_i(z)\xi = 0, i \in I, D_x g_j(z)\xi = 0, j \in J_0(z)\}$$

for $x \in M(t)$.
A g.c. point $\bar{z}$ is of *Type 1* if the following conditions are fulfilled:

- (LICQ) is satisfied at $\bar{x} \in M(\bar{t})$.
- The strong complementarity holds for the uniquely determined solution $(\bar{\lambda}, \bar{\mu})$ of $D_x L(\bar{z}, \lambda, \mu) = 0$, i.e. $\bar{\mu}_j \neq 0, j \in J_0(\bar{z})$.
- $o(D_x^2 L(\bar{z}, \bar{\lambda}, \bar{\mu}) | B(\bar{z})) = 0$.

In Jongen, Jonker, Twilt 1986B the local structure of Σ in a neighbourhood of a g.c. point $\bar{z}$ of Type 1 is described; the Implicit Function Theorem implies the existence of an open neighbourhood $U(\bar{t})$ of $\bar{t}$ and uniquely determined C^1–functions

$$(\hat{x}, \hat{\lambda}, \hat{\mu}) : t \in U(\bar{t}) \mapsto (\hat{x}(t), \hat{\lambda}(t), \hat{\mu}(t)) \in R^{n+m+|J_0(\bar{z})|}$$

such that $(\hat{x}, \hat{\lambda}, \hat{\mu})(\bar{t}) = (\bar{x}, \bar{\lambda}, \bar{\mu})$ and Σ is locally in a neighbourhood of $\bar{z}$ just the set $\{(\hat{x}(t), t) | \, t \in U(\bar{t})\}$. Thus, a g.c. point of Type 1 is neither a bifurcation nor a turning point (w.r.t. t) of Σ^s or Σ and, therefore, we are not interested in a jump at such a point when using a forward strategy (cf. JUMP I in Guddat, Guerra, Jongen 1990).

3. Jumps

In this section let $\bar{z} \in \Sigma$. If (LICQ) is satisfied at $\bar{x} \in M(\bar{t})$ then $(\bar{\lambda}, \bar{\mu}) \in R^{m+|J_0(\bar{z})|}$ denotes the unique solution of $D_x L(\bar{z}, \lambda, \mu) = 0$.

The jump at a g.c. point of Type 2

According to Jongen, Jonker, Twilt 1986B, a g.c. point $\bar{z}$ of Type 2 has the following property:

$$\text{(LICQ) is satisfied at } \bar{x} \in M(\bar{t}) \text{ and } |\{j \in J_0(\bar{z}) | \bar{\mu}_j = 0\}| = 1. \tag{2.1}$$

Since the construction of the jumps will be done locally in a neighbourhood of $\bar{z}$, and having Remark 1 in mind we can presume under (2.1) w.l.o.g. that $I = \emptyset$ and $J_0(\bar{z}) = \{1\}$.
For a further characterization of a g.c. point of Type 2 set

$$\begin{aligned} L^0(z, \mu) &= \begin{pmatrix} D_x^T f(z) \\ \mu - g_1(z) \end{pmatrix} \quad \text{and} \\ L^1(z, \mu) &= \begin{pmatrix} D_x^T f(z) + \mu D_x^T g_1(z) \\ -g_1(z) \end{pmatrix}. \end{aligned}$$

We know from Jongen, Jonker, Twilt 1986B that in case $\bar{z}$ is of Type 2, it is

$$o(D_x^2 f(\bar{z})) = o(D_x^2 f(\bar{z}) | B(\bar{z})) = 0, \tag{2.2}$$

where $B(\bar{z}) = \{\xi \in R^n | D_x g_1(\bar{z})\xi = 0\}$.
By (2.2) and the Implicit Function Theorem there exist an open neighbourhood $V(\bar{t})$ of $\bar{t}$ and uniquely determined C^1–functions

$$(x^o, \mu^o, x^1, \mu^1) : t \in V(\bar{t}) \in (x^o(t), \mu^o(t), x^1(t), \mu^1(t)) \in R^n \times R \times R^n \times R$$

with gradients $(\dot{x}^o, \dot{\mu}^o, \dot{x}^1, \dot{\mu}^1)$ such that $x^o(\bar{t}) = x^1(\bar{t}) = \bar{x}$, $\mu^o(\bar{t}) = \mu^1(\bar{t}) = 0$ and $L^o(x^o(t), \mu^o(t), t) = L^1(x^1(t), \mu^1(t), t) = 0$ for all $t \in V(\bar{t})$. According to Jongen, Jonker, Twilt 1986B, $\bar{z} \in \Sigma$ is of *Type 2* if the conditions (2.1), (2.2) and

$$D_t g_1(x^o(\bar{t}), \bar{t}) \neq 0 \tag{2.3}$$

are fulfilled. Now, let $\bar{z}$ be of Type 2. By (2.3), it follows that $\dot{\mu}^o(\bar{t}) \neq 0$ and $\dot{\mu}^1(\bar{t}) \neq 0$. If $z \in \Sigma^s$ and

$$\text{sign } \dot{\mu}^o(\bar{t}) \neq \text{sign } \dot{\mu}^1(\bar{t}) \tag{2.4}$$

then, obviously, $\bar{z}$ is a turning point of the set Σ^s (Figure 1 illustrates this situation for $\dot{\mu}^1(\bar{t}) < 0$ and $\dot{\mu}^o(\bar{t}) > 0$).

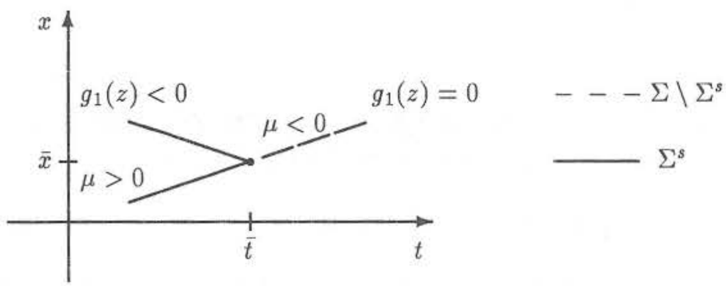

Fig. 1

In this case we are interested in a jump if we restrict the application of pathfollowing algorithms to the set Σ^s. So, assume (2.4). Then, from the index relations in Jongen, Jonker, Twilt 1986B, p. 341, resp. the above lemma we obtain

$$\text{sign}\det\begin{pmatrix} D_x^2 f(\bar{z}) & D_x^T g_1(\bar{z}) \\ -D_x g_1(\bar{z}) & 0 \end{pmatrix} \neq \text{sign}\det D_x^2 f(\bar{z}) \quad \text{resp.}$$

$$D_x g_1(\bar{z}) D_x^2 f(\bar{z})^{-1} D_x g_1(\bar{z})^T < 0. \tag{2.5}$$

Furthermore, the construction yields

$$\begin{array}{lcccl} D_x^2 f(\bar{z})\dot{x}^o(\bar{t})^T & & & + & D_{xt}^T f(\bar{z}) = 0 \\ -D_x g_1(\bar{z})\dot{x}^o(\bar{t})^T & + & \dot{\mu}^o(\bar{t}) & - & D_t g_1(\bar{z}) = 0 \end{array} \tag{2.6}$$

$$\begin{array}{lllll} D_x^2 f(\bar{z})\dot{x}^1(\bar{t})^T & + & D_x^T g_1(\bar{z})\dot{\mu}^1(\bar{t}) & + & D_{xt}^T f(\bar{z}) & = 0 \\ -D_x g_1(\bar{z})\dot{x}^1(\bar{t})^T & & & - & D_t g_1(\bar{z}) & = 0. \end{array} \tag{2.7}$$

Now, define $\dot{x} = \dot{x}^o(\bar{t}) - \dot{x}^1(\bar{t})$. Obviously, we have for $t \in V(\bar{t})$:

$$\left.\begin{array}{l} \lim\limits_{\substack{t\to\bar{t}\\ t<\bar{t}}} \frac{x^o(t)-x^1(t)}{\|x^o(t)-x^1(t)\|} = -\frac{\dot{x}^T}{\|\dot{x}\|} \quad \text{and} \\ \lim\limits_{\substack{t\to\bar{t}\\ t>\bar{t}}} \frac{x^o(t)-x^1(t)}{\|x^o(t)-x^1(t)\|} = \frac{\dot{x}^T}{\|\dot{x}\|} \end{array}\right\} \tag{2.8}$$

Formulas (2.6) and (2.7) imply

$$\frac{1}{\dot{\mu}^1(\bar{t})} D_x^2 f(\bar{z})\dot{x}^T = D_x^T g_1(\bar{z}) \tag{2.9}$$

as well as, using (2.5),

$$\dot{x} D_x^2 f(\bar{z})\dot{x}^T < 0 \quad \text{and} \quad \dot{\mu}^1(\bar{t}) D_x g(\bar{z})\dot{x}^T < 0. \tag{2.10}$$

From (2.8) and (2.10), it follows that

$$\lim_{t\to\bar{t}} \frac{x^o(t)-x^1(t)}{||x^o(t)-x^1(t)||} \tag{2.11}$$

is a feasible descent direction w.r.t. $P(\bar{t})$, i.e. it can be used for a jump at a g.c. point of Type 2 which satisfies (2.4).

EXAMPLE. *Consider the problem*

$$P(t) \quad \textit{Minimize} \ \ -x^2 \ \ \textit{subject to} \ \ x - t \le 0 \quad .$$

where $x \in R^1$. *Obviously,* $(0,0)$ *is a g.c. point of Type 2 and* $\Sigma^s = \{(0,t) | t \ge 0\} \cup \{(t,t) | t \ge 0\}$ *(cf. Fig 2a).*
Using the above notations, we obtain

$$\begin{array}{l} x^o(t) = 0, x^1(t) = t \qquad \textit{for } t \textit{ sufficiently small and} \\ \lim\limits_{t\to 0} \dfrac{x^o(t)-x^1(t)}{||x^o(t)-x^1(t)||} = -1. \end{array}$$

After introducing the additional constraint $-2 - x \le 0$ *we get compact feasible sets and by applying then a jump — i.e. a feasible descent direction algorithm — we reach the branch of local minimizers* $\{(-2,t) |\, t \ge -2\}$ *(cf. Fig. 2b).*

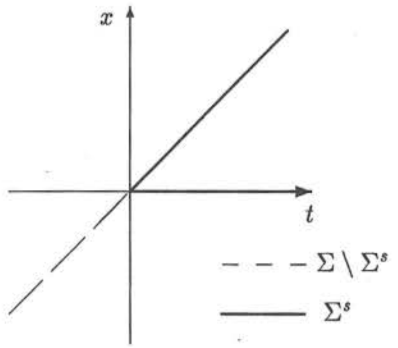

Fig. 2(a)

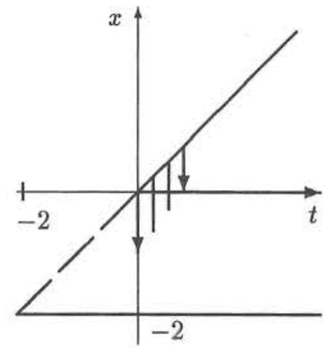

Fig. 2(b)

REMARK 3 *In order to get an approximation of the descent direction* $\lim_{t \to \bar{t}} \frac{x^o(t)-x^1(t)}{\|x^o(t)-x^1(t)\|}$ *one can compute* Σ^s *locally in a neighbourhood of* $\bar{z}$ *- i.e.* $x^o(t)$ *and* $x^1(t)$, t *near* $\bar{t}$ *— by using a pathfollowing algorithm.*
Furthermore, (2.9) and the nonsingularity of $D_x^2 f(\bar{z})$ *yield a direct formula for the direction of* $\dot{x}$. *However, in this latter case we have to compute an inverse matrix which can be — especially, if* $I \neq \emptyset$ *and* $J \neq \emptyset$ *— numerically difficult.*

The following proposals of jumps at g.c. points of Type 3 and Type 4 are generalizations of the jumps given in Guddat, Jongen, Nowack 1988 which have been constructed at endpoints of branches of local minimizers.

The jump at a g.c. point of Type 3.

Following Jongen, Jonker, Twilt 1986B, a g.c. point $\bar{z}$ of Type 3 has the following property:

$$\text{(LICQ) is satisfied at } \bar{x} \in M(\bar{t}) \text{ and } |\{j \in J_0(\bar{z}) | \bar{\mu}_j = 0\}| = 0. \tag{3.1}$$

By Remark 1, we presume under (3.1) w.l.o.g. that $I = J_0(\bar{z}) = \emptyset$. A g.c. point $\bar{z}$ is of *Type 3* if (3.1) is fulfilled and if $\bar{z}$ is a strongly stable g.c. point of the problem

$$\text{Minimize } \varphi(z) \text{ subject to } \{z \in R^{n+1} | D_x f(z) = 0\} \tag{3.2}$$

at which (LICQ) is satisfied, where $\varphi(z) = t$. Now, let $\bar{z}$ be of Type 3. Then, $\bar{z}$ is either a local minimizer (cf. Fig. 3(a)) or a local maximizer of (3.2). By (LICQ) as well as the strong stability of $\bar{z}$ w.r.t. (3.2) and the above mentioned lemma, the following conditions are fulfilled:

- The feasible set of (3.2) is locally in a neighbourhood of $\bar{z}$ a one-dimensional C^1-manifold and

- for the uniquely — up to sign and multiple — determined tangent vector $(v, 0) \in R^n \times R$ of this manifold at the point $\bar{z}$ we have $D_x^2 f(\bar{z})v = 0$, $v^T(D_x^3 f(\bar{z})v)\, v \neq 0$.

Thus, v or $-v$ (for a fixed chosen v) is a (feasible) "cubic" descent direction w.r.t. $P(\bar{t})$ and can be used for a jump (cf. Fig. 3(b)). Obviously, v can be described in an analogous way to (2.11).

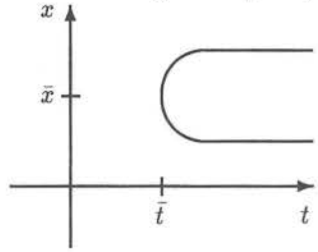

Fig. 3(a)

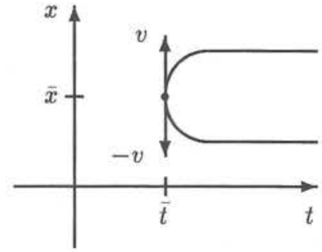

Fig. 3(b)

The jump at a g.c. point of Type 4

We know from Jongen, Jonker, Twilt 1986B, that a point $\bar{z} \in \Sigma$ of Type 4 fulfills the following conditions:

$$\left.\begin{array}{l}
\bullet\ \ 1 + |I| + |J_0(\bar{z})| \leq n \\
\bullet\ \ \dim B(\bar{z}) = n - |I| - |J_0(\bar{z})| + 1 \text{(where dim denotes the dimension), i.e. there exist uniquely - up to a common} \\
\quad \text{multiple - determined numbers } \tilde{\lambda}_i, i \in I, \tilde{\mu}_j, j \in J_0(\bar{z}) \\
\quad \text{satisfying } \sum_{i \in I} |\tilde{\lambda}_i| + \sum_{j \in J_0(\bar{z})} |\tilde{\mu}_j| > 0 \text{ and} \\
\quad \sum_{i \in I} \tilde{\lambda}_i D_x h_i(\bar{z}) + \sum_{j \in J_0(\bar{z})} \tilde{\mu}_j D_x g_j(\bar{z}) = 0. \\
\bullet\ \ \text{If } J_0(\bar{z}) \neq \emptyset, \text{ we have } \tilde{\mu}_j \neq 0, j \in J_0(\bar{z}) \text{ where } \tilde{\mu}_j, j \in J_0(\bar{z}) \\
\quad \text{are chosen as above.}
\end{array}\right\} \quad (4.1)$$

Considering Remark 1 again, we presume w.l.o.g. under (4.1) that $I = \emptyset$, $J_0(\bar{z}) = \{1\}$ and $D_x g_1(\bar{z}) = 0$. A g.c. point $\bar{z}$ is of *Type 4* if (4.1) is fulfilled and $(\beta, z) = (0, \bar{z})$ — with $\beta \in R$ — is a strongly stable g.c. point of the problem

$$\text{Minimize } \tilde{\varphi}(\beta, z) \text{ subject to } (\beta, z) \in \tilde{M} \tag{4.2}$$

at which (LICQ) is satisfied, where $\tilde{\varphi}(\beta, z) = t$ and $\tilde{M} = \{(\beta, z) \in R^{n+2} | \beta D_x f(z) + D_x g_1(z) = 0,\ g_1(z) = 0\}$. Now, let $\bar{z}$ be of Type 4. The strong stability and (LICQ) imply that $(0, \bar{z})$ is either a local minimizer or a local maximizer of (4.2) and, furthermore, that there exists a neighbourhood $W(0)$ of $\beta = 0$ as well as a uniquely determined C^1-function

$$\tilde{z} : \beta \in W(0) \mapsto \tilde{z}(\beta)(= (\tilde{x}(\beta), \tilde{t}(\beta)))$$

having the following properties:

- $\tilde{z}(0) = \bar{z}$
- $(D_\beta f(\tilde{z}(\beta)) =) D_x f(z)|_{z=\tilde{z}(\beta)} \dot{\tilde{x}}(\beta) \neq 0$ for all $\beta \in W(0)$
- Σ is — locally in a neighbourhood of $\bar{z}$ – just the set $\{\tilde{z}(\beta) | \beta \in W(0)\}$ and, by the latter property, for increasing β the function f is either increasing or decreasing along Σ.

From the index relations Jongen, Jonker, Twilt 1986B, p. 347, formula (47), we obtain that — after a possible shrinking of $W(0)$ —

- for all $\beta \in W(0) \backslash \{0\}$ at $z = \tilde{z}(\beta)$:
  $$o(D_x^2 f(z) + \frac{1}{\beta} D_x^2 g_1(z) | B(z)) = 0;$$
- for arbitrarily chosen $\beta_1 \in W(0), \beta_1 < 0$ and $\beta_2 \in W(0)$, $\beta_2 > 0$:
  $$n(D_x^2 f(z^1) + \frac{1}{\beta_1} D_x^2 g_1(z^1) | B(z^1)) = p(D_x^2 f(z^2) + \frac{1}{\beta_2} D_x^2 g_1(z^2) | B(z^2)),$$
  where $z^1 = \tilde{z}(\beta_1)$, $z^2 = \tilde{z}(\beta_2)$.

In Guddat, Jongen, Nowack 1988 the case of $\{\tilde{z}(\beta) | \beta > 0, \beta \in W(0)\} \subset \Sigma^l$ is investigated (i.e. $\tilde{x}(\beta)$ is a local minimizer of $P(\tilde{t}(\beta))$ and $n(D_x^2 f(\tilde{z}(\beta)) + \frac{1}{\beta} D_x^2 g_1(\tilde{z}(\beta)) | B(\tilde{z}(\beta))) = 0$ for $\beta > 0$, $\beta \in W(0)$). They distinguished two possibilities:

If $D_x f(\bar{z}) \dot{\tilde{x}}(0) < 0$, i.e. f is decreasing for increasing β, the corresponding connected component of $M(t)$ shrinks for $t \to \bar{t}$ to a point. Therefore, there exists no jump to another connected component of Σ.

If $D_x f(\bar{z}) \dot{\tilde{x}}(0) > 0$, then for every $\beta < 0$, $\beta \in W(0)$ the point $\tilde{x}(\beta)$ is no local minimizer of $P(\tilde{t}(\beta))$ and, hence, there exists a feasible descent direction w.r.t $P(\tilde{t}(\beta))$ at $\tilde{x}(\beta)$ (note, that $M(\tilde{t}(\beta))$ is locally in a neighbourhood of $\tilde{x}(\beta)$ a topological manifold, since (LICQ) is satisfied at $\tilde{x}(\beta)$) and a corresponding descent algorithm *will not reach the other branch* $\{\tilde{z}(\beta) | \beta > 0, \beta \in W(0)\}$.

The latter argument can also be used in the following case: Assume now, that $\tilde{x}(\beta)$ is no local minimizer of $P(\tilde{t}(\beta))$ for all $t \in W(0) \backslash \{0\}$.

If $D_x f(\bar{z}) \dot{\tilde{x}}(0) < 0$ (resp. > 0) then for every $\tilde{\beta} > 0$, $\tilde{\beta} \in W(0)$ (resp. $\tilde{\beta} < 0$, $\tilde{\beta} \in W(0)$) there exists a feasible "quadratic" descent direction ξ w.r.t. $P(\tilde{t}(\tilde{\beta}))$ at $\tilde{x}(\tilde{\beta})$ — satisfying $\xi^T [D_x^2 f(\tilde{z}(\tilde{\beta})) + \frac{1}{\tilde{\beta}} D_x^2 g_1(\tilde{z}(\tilde{\beta}))] \xi < 0$, $D_x g_1(\tilde{z}(\tilde{\beta})) \xi \leq 0$ — and a corresponding descent algorithm will not reach the other branch $\{\tilde{z}(\beta) | \beta < 0, \beta \in W(0)\}$, cf. Fig. 4 (resp. $\{\tilde{z}(\beta) | \beta > 0, \beta \in W(0)\}$). *So, we always have a jump in an arbitrarily small chosen neighbourhood of a g.c. point $\bar{z}$ of Type 4, if $\bar{z}$ is not an endpoint of a branch consisting of local minimizers.*

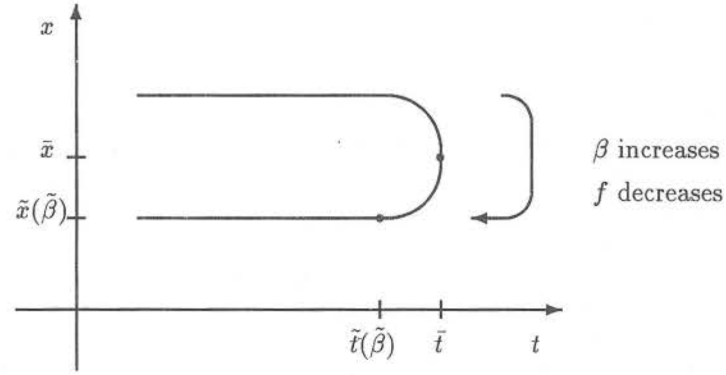

Fig. 4

Concluding Remarks

REMARK 4 *As already mentioned in Guddat, Guerra, Jongen 1990, there is no proposal for a jump at a g.c. point of Type 5 being a turning point of* Σ.

REMARK 5 *The used characterizations of g.c. points of the Types 2, 3 and 4 are different — but, of course, equivalent — to the original definitions in Jongen, Jonker, Twilt 1986B. Our characterizations and their equivalence to the original ones have been published in Rückmann 1988.*

REMARK 6 *Recall, that we assumed (1.1). In Rückmann, Tammer 1992, we proposed a perturbation of the nonlinear one-parametric problem* $P(t)$ *in order to ensure, that the function vector of the perturbed problem belongs to* $\mathcal{F}$. *We proved, that for fixed* $(f, h_i, g_j, i \in I, j \in J) \in C^3(R^{n+1}, R)^{1+m+s}$ *every Lebesgue measurable subset of*

$$\left\{ \frac{(A, b_1, \ldots, b_{m+s}, c, d_1, \ldots, d_{m+s}) \in R^{\frac{1}{2}n(n+1)+n(m+s+1)+m+s} \mid}{\mid (f(x,t)+x^TAx+c^Tx, h_i(x,t)+b_i^Tx+d_i, g_j(x,t)+b_{m+j}^Tx+d_{m+j}, i \in I, j \in J) \notin \mathcal{F}} \right\}$$

has the Lebesgue measure zero, where $A \in R^{\frac{1}{2}n(n+1)}$ *(*$R^{\frac{1}{2}n(n+1)}$ *is the space of symmetric real* (n,n)*-matrices),* $b_i \in R^n$, $i = 1, \ldots, m+s$, $c \in R^n$ *and* $d_i \in R$, $i = 1, \ldots, m+s$.

References

GOLLMER R., J. GUDDAT, F. GUERRA, D. NOWACK AND J.-J. RÜCKMANN (1993) Pathfollowing methods in nonlinear optimization I : penalty embedding. In : J. Guddat et al. (eds.), *Parametric Optimization and Related Topics III*, Peter Lang Verlag, Frankfurt a. M., Bern, New York, pp. 163–214.

GUDDAT J. (1987) Parametric optimization: pivoting and predictor–corrector continuation, a survey. In : J. Guddat et al. (eds.), *Parametric Optimization and Related Topics*, Akademie–Verlag, Berlin, pp. 125–162.

GUDDAT J., F. GUERRA AND H. TH. JONGEN (1990) Parametric Optimization: Singularities, Pathfollowing and Jumps. John Wiley, Chichester and Teubner, Stuttgart.

GUDDAT J., F. GUERRA AND D. NOWACK (1991) Multiobjective optimization: singularities, pathfollowing and jumps. In: G. Fandel, H. Gehring (eds.), *Operations Research*, Springer–Verlag, Berlin, Heidelberg, New York, pp. 81–108.

GUDDAT J., F. GUERRA, K. TAMMER AND K. WENDLER (1985) Multiobjective and Stochastic Optimization Based on Parametric Optimization. Akademie–Verlag, Berlin.

GUDDAT J., H. TH. JONGEN AND D. NOWACK (1988) Parametric optimization: pathfollowing with jumps. In: A. Gomez et al. (eds.), *Approximation and Optimization, Lecture Notes in Mathematics* **1354**, Springer–Verlag, Berlin, Heidelberg, New York, pp. 43–53.

GUDDAT J., W. RÖMISCH AND R. SCHULTZ (1992) Some applications of mathematical programming techniques in optimal power dispatch. *Computing* **49**, pp. 193–200.

HIRSCH M.W. (1976) Differential Topology. Springer–Verlag, Berlin, Heidelberg, New York.

JONGEN H. TH., P. JONKER AND F. TWILT (1986) One–parameter families of optimization problems: equality constraints. *J. Optimization Theory and Appl.* **48**, pp. 141–161.

JONGEN H. TH., P. JONKER AND F. TWILT (1986) Critical sets in parametric optimization. *Mathematical Programming* **34**, pp. 333–353.

JONGEN H. TH., P. JONKER AND F. TWILT (1986) Nonlinear Optimization in R^n, II. Transversality, Flows, Parametric Aspects. Peter Lang Verlag, Frankfurt a.M., Bern, New York.

JONGEN H. TH., T. MÖBERT, J.-J. RÜCKMANN AND K. TAMMER (1987) On inertia and Schur complement in optimization. *Linear Algebra and Appl.* **95**, pp. 97–109.

KOJIMA M. (1980) Strongly stable stationary solutions in nonlinear programs. In: S.M. Robinson (ed.), *Analysis and Computation of Fixed Points*, Academic Press, New York, pp. 93–138.

OUELETTE D.V. (1981) Schur complements and statistics. *Linear Algebra and Appl.* **36**, pp. 187–295.

RÜCKMANN J.-J. (1988) Einparametrische nichtkonvexe Optimierung: Strukturuntersuchungen und eine Verallgemeinerung des Einbettungsprinzips. Dissertation, Technische Hochschule Leipzig, Sektion Mathematik und Informatik.

RÜCKMANN J.-J. AND K. TAMMER (1992) On linear-quadratic perturbations in one-parametric nonlinear optimization. *Systems Science* **18** 1, pp. 37–48.

TAMMER K. (1987) The application of parametric optimization and imbedding for the foundation and realization of a generalized primal decomposition approach. In: J. Guddat et al. (eds.), *Parametric Optimization and Related Topics*, Akademie–Verlag, Berlin, pp. 376–386.