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Multiparametric optimization: on stable singularities occuring in combinatorial partition codes

by

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We consider multiparametric differentiable optimization problems in finite dimensions. The object of our study will be the set of those Karush–Kuhn–Tucker points at which the Mangasarian–Fromovitz constraint qualification is satisfied. This set has been shown to be generically a topological manifold. It can also be decomposed into differentiable manifolds. However, the latter decomposition cannot be obtained via a combinatorial partition code of finitely generated cones (corresponding to the gradients of active constraints). In fact, such partition codes will obtain unavoidable stable singularities. We explicitly construct such a singularity, where the idea of the construction can be generalized. The state space of our example is four dimensional, with 33 inequality constraints and a 74 dimensional parameter space.
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1. Introduction

We consider the multi–parametric optimization problem $\mathcal{P}(y)$ with a parameter $y \in \mathbb{R}^p$:

$\mathcal{P}(y) \quad : \quad min\{\ f(x,y) \mid x \in M(y)\ \},$ where

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$$M(y) \;=\; \{\; x \in \mathbb{R}^n \mid h_i(x,y) = 0,\; i \in \mathcal{H} \;;\; g_j(x,y) \geq 0,\; j \in \mathcal{G} \;\}.$$

The sets $\mathcal{H} = \{1,..,\mathrm{h}\}$ and $\mathcal{G} = \{1,..,\mathrm{g}\}$ are finite index sets. All appearing functions are supposed to be of differentiability class C^s, $s \geq 1$. In this section, we assume (without specification) that the problem data $(f,h,g) = (f,h_1,..,h_{\mathrm{h}},g_1,..,g_{\mathrm{g}})$ are in general position.

A point $x \in \mathbb{R}^n$ is called a Karush–Kuhn–Tucker point of $\mathcal{P}(y)$ (shortly KKT–point) if the following conditions hold with some $\lambda \in \mathbb{R}^{\mathrm{h}}$, $\mu \in \mathbb{R}^{\mathrm{g}}$:

$$\text{KKT1} \qquad x \in M(y)\,; \quad \mu_j \geq 0\,, \quad j \in \mathcal{G}_0(x,y)\,; \quad \mu_j = 0\,, \quad j \notin \mathcal{G}_0(x,y)$$

$$\text{KKT2} \qquad D_x f = \sum_i \lambda_i \, D_x h_i + \sum_j \mu_j \, D_x g_j \;\Big|_{(x,y)}.$$

The set $\mathcal{G}_0(x,y) := \{\; j \in \mathcal{G} \mid g_j(x,y) = 0 \;\}$ denotes the set of active inequality constraints; moreover, $D_x f$ stands for the row vector of first partial derivatives with respect to x. A pair (λ,μ) satisfying $KKT1,2$ is called a Lagrange multiplier associated with (x,y). Let $\Delta(x,y)$ denote the set of all Lagrange multipliers associated with (x,y). Then, $\Delta(x,y)$ is a convex polyhedron.

Provided that some constraint qualification is fulfilled, conditions KKT1,2 are first order necessary conditions for x to be a local minimizer for $\mathcal{P}(y)$. In this paper the constraint qualification of Mangasarian–Fromovitz (shortly MFCQ) will be used. In that case, the set $\Delta(x,y)$ is compact (cf. Gauvin 1977) and gives rise to a natural definition of a combinatorial partition code. The compact set $\Delta(x,y)$ is called the Lagrange polytope.

The Mangasarian–Fromovitz constraint qualification is said to hold at (x,y) if MF1 and MF2 are satisfied:
MF1 The set $\{D_x h_i(x,y) \mid i \in \mathcal{H}\}$ is linearly independent
MF2 There exists a $v \in \mathbb{R}^n$ with $D_x h_i(x,y)\; v = 0$, $i \in \mathcal{H}$; $D_x g_j(x,y)\; v > 0$, $j \in \mathcal{G}_0(x,y)$.
The validity of MFCQ implies that the corresponding part of the Karush–Kuhn–Tucker set, say $\mathbf{\Sigma}$, is a topological manifold (cf. Günzel 1993), where

$$\mathbf{\Sigma} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p \mid x \text{ is KKT–point of } \mathcal{P}(y) \text{ and MFCQ is fulfilled}\}.$$

In general, $\mathbf{\Sigma}$ will not be a smooth set. But, it can be decomposed into differentiable manifolds exhibiting a regular fitting (Whitney regularity), (cf. Günzel at al. 1993).

For simplicity of the exposition, in the rest of the paper we delete the equality constraints, i.e. $\mathcal{H} = \emptyset$. This can be justified by MF1: a local reduction using the implicit function theorem yields a corresponding problem in $\mathbb{R}^{n-\mathrm{h}}$ without equality constraints.

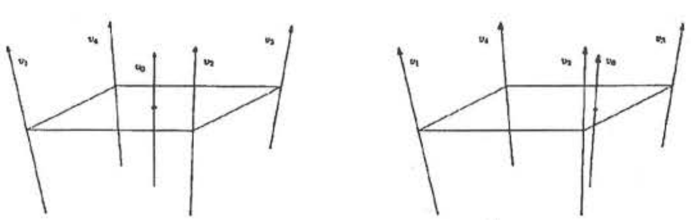
In this paper we show by means of an explicit counterexample that the decomposition of the set $\mathbf{\Sigma}$ into differentiable manifolds cannot be obtained using the following (natural) combinatorial partition code. In fact, consider the condition KKT2 with $\mathcal{H} = \emptyset$. This means that $D_x f$ lies in the cone generated by the vectors $D_x g_j$, $j \in \mathcal{G}_0$. There might be many ways for writing $D_x f$ explicitly as such a nonnegative linear combination. The extremal ones will define our partition code. This will be explained in the sequel. Let (x, y) belong to the set Σ. We call a subset $I \subset \mathcal{G}_0$ a facet–generator if there exists a Lagrange multiplier μ with $\mu_j > 0$, $j \in I$, and $D_x f = \sum_{j \in I} \mu_j D_x g_j \mid_{(x,y)}$. A facet–generator is called extremal if it does not properly contain another facet–generator. Note, that $I \subset \mathcal{G}_0(x, y)$ is an extremal facet–generator if and only if the set $\{D_x g_j \,|\, j \in I\}$ is both linearly independent and the generated nonnegative cone contains $D_x f$ in its relative interior. Moreover, a subset $I \subset \mathcal{G}_0$ is a facet–generator if and only if $\Delta(x, y) \cap \mathbb{R}^I_+$ is a relatively open facet of the Lagrange polytope $\Delta(x, y)$, where

$$\mathbb{R}^I_+ := \{\mu \in \mathbb{R}^{\mathcal{G}} \mid \mu_i > 0, i \in I;\ \mu_i = 0, i \notin I\}.$$

An extremal facet–generator then corresponds to an extremal point of the polytope $\Delta(x, y)$.

Let $\mathcal{J}(x, y)$ denote the set of all extremal facet–generators. The partition code $\mathcal{X} := (\mathcal{G}_0, \mathcal{J})$ is defined as a mapping from Σ to the product $2^{\mathcal{G}} \times 2^{2^{\mathcal{G}}}$, where $2^{\mathcal{G}}$ denotes the power set of $\mathcal{G}$.

Example 1.1 *Assume $n = 3$ and $\mathcal{G}_0 = \{1, 2, 3, 4\}$. Denote $v_0 := D_x f$ and $v_i := D_x g_i$.*



$\mathcal{J} = \{\{1,3\},\{2,4\}$ $\qquad$ $\mathcal{J}' = \{\{1,2,3\},\{2,3,4\}$

Figure 1.

Now,we can formulate our counterexample by means of the following theorem. The C^2–topology refers to the strong (or Whitney-) topology (cf. Hirsch 1976, Jongen, Jonker, Twilt 1986). Note that the set $\mathcal{S} := \{(u, v) \in \mathbb{R}^2 \mid uv = 0\}$ is not a (topological) manifold at the origin.

Theorem 1.1 *Let be $n = 4$, $p = 74$, $\mathrm{h} = 0$, $\mathrm{g} = 33$. Then, there exist a C^2–open set $\mathcal{F} \subset C^2(\mathbb{R}^{n+p})$ and a subset $\mathcal{I} \subset 2^{\mathcal{G}}$ such that for all problem data $(f, g) \in \mathcal{F}$ there exists a pair $(\bar{x}, \bar{y}) \in \Sigma$ with the following property:*
$\mathcal{X}^{-1}(\mathcal{G}, \mathcal{I})$ is at $(\bar{x}, \bar{y})$ diffeomorphic with $\mathcal{S} \times \{0\}$ at $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}^{76}$. ■

The diffeomorphism in the above theorem is to be understood as follows:

DEFINITION 1.1 *Let $X \subset M$ and $Y \subset N$ be subsets of smooth manifolds. Then, X is said to be diffeomorphic at $x \in X$ with Y at $y \in Y$ if there exists a local diffeomorphism $\varphi : M \to N$ sending x to y and X onto Y.*

The proof of the theorem is based on geometrical constructions for addition and multiplication due to von Staudt (cf. Hartshorne 1967). As input and output the latter computations use points on a line instead of real numbers. Certain colinearities guarantee that the resulting points in fact have an interpretation as the output of operations as addition or multiplication. Compositions of such operations make it possible to compute any polynomial function with integer coefficients. The set of colinearities determines some kind of a code. The computability of any polynomial with integer coefficients in addition with regularization arguments imply that algebraic singularities can be implanted in the space of "configurations" of points realizing a specific code. This is the main assertion of Mnëv's universality theorem for point configurations (cf. Mnëv 1988). We introduce other regularization arguments as in Mnëv 1988 such as the Connection Lemma and the Uncoupling Lemma below. In comparison with Mnëv 1988 our construction yields relatively low-dimensional examples involving singularities. The universality theorem can be proved along the same lines as presented here; this will be shown in a forthcoming paper Günzel 1994.

In Section 2 we will establish a natural relation between the latter configurations and our Lagrange polytopes. This requires a modeling in an appropriate jet-space. In Section 3 we introduce our regularization arguments and prove the theorem.

2. The model in jet-space

From now on we assume $n = 4$. In this section we regard our situation modeled in terms of jet extension and jet-space. Generally speaking, the statement of our theorem carries over to the jet-space by replacing the values $g(x, y)$, $D_x f(x, y)$, etc., by abstract variables in a target space. We will see how the jet-extension of (f, g) pulls back the singular fibre $\mathcal{S}$ (as introduced in Section 1) from the jet-space to Σ. At that point we assume the problem data to be in general position. Then it remains to establish the idea of our theorem in terms of jets. In the jet-space we study positive linear combinations of vectors $w_i \in I\!R^4$ — standing for the gradients of the constraint functions — yielding w_0, standing for the gradient of the objective function f. If none of these vectors vanishes, and the condition MF below is fulfilled, then there exists an affine hyperplane exhibiting precisely one intersection point with each set of positive multiples, say $I\!R_+ w_i$, where $I\!R_+ := \{\lambda \in I\!R | \lambda > 0\}$. This (locally) reduces the problem to the study of convex combinations of points in $I\!R^3$ yielding the intersection point with $I\!R_+ w_0$, say the origin $0 \in I\!R^3$. By means of normalization on rays emerging from the origin, the appearing vectors (denoted by v_i) can be represented by means of the intersection of the intersection of $I\!R_+ v_i$ with the union of two parallel planes, see Figure 2. These planes will play a crucial rule in the construction performed in Section 3.

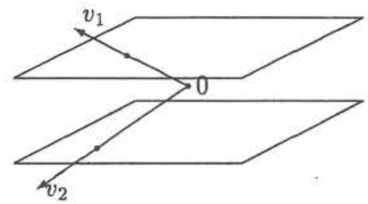


Figure 2.

For problem data (f, g) we define the following reduced 1–jet extension:

$$j^1_{(f,g)} := (g, D_x f, D_x g_1, .., D_x g_{\mathrm{g}}).$$

We omit the index (f, g) if no ambiguity can occur. The target space $Jet_{\mathbf{\Sigma}} := I\!R^{\mathrm{g}} \times I\!R^n \times (I\!R^n)^{\mathrm{g}}$ of $j^1_{(f,g)}$ is called the jet–space. Transfering the conditions KKT1,2 and MFCQ to the jet–space, yields the next conditions K1,2 and MF, respectively, for a point $(b, w, \mu) \in Jet_{\mathbf{\Sigma}} \times I\!R^{\mathrm{g}}$, where $w = (w_0, .., w_{\mathrm{g}})$:

$$(K1) \quad : \quad b \geq 0,\ \mu \geq 0 \ and \ b\mu = 0$$

$$(K2) \quad : \quad w_0 = \sum_{i \neq 0} \mu_i w_i$$

(MF) : There exists $v \in I\!R^n$ such that $w_i v > 0$ for $i = 1, .., \mathrm{g}$ with $b_i = 0$.

The above conditions define the characteristic set $\mathbf{K}$:

$$\mathbf{K} = \{ \ (b, w) \in Jet_{\mathbf{\Sigma}} \mid For \ some \ \mu \in I\!R^{\mathrm{g}} : K1, K2 \ and \ MF \ \}$$

Now we have the following fundamental relation:

$$(x, y) \in \mathbf{\Sigma} \iff j^1(x, y) \in \mathbf{K}.$$

Given a point $(b, w) \in \mathbf{K}$, let $\mathcal{G}_0(b, w)$ denote the "active index set", i.e. $\mathcal{G}_0 := \{i \in \mathcal{G} \mid b_i = 0\}$. A set $I \subset \mathcal{G}_0(b, w)$ is called a facet–generator if there exists a multiplier μ with $\mu_i > 0$, $i \in I$, such that $w = \sum_{i \in I} \mu_i w_i$. A facet–generator is called extremal if it does not properly contain another facet–generator. Let $\mathcal{J}(b, w)$ denote the set of all extremal facet–generators. Put $\mathcal{X} := (\mathcal{G}_0, \mathcal{J}) : \mathbf{K} \to 2^{\mathcal{G}} \times 2^{2^{\mathcal{G}}}$. Then, on $\mathbf{\Sigma}$ we have the relations $\mathcal{X} \circ j^1 = \mathcal{X}$, and $\Delta \circ j^1 = \Delta$. From now on put $\mathrm{g} = 33$.

DEFINITION 2.1 *Let $F : X \to Y$ be a mapping between manifolds and let $M \subset Y$ denote a submanifold, all data of class C^1. We say that F meets M transversally (in Y) at point $x \in F^{-1}(M)$ if we have $DF(x)[T_x X] + T_{F(x)} M = T_{F(x)} Y$.*

ASSUMPTION* *There exists* $w \in (\mathbb{R}^4)^{1+g}$ *such that* $\mathcal{X}^{-1}(\mathcal{G},\mathcal{I})$ *is at* $(0,w)$ *diffeomorphic with* $\mathcal{S} \times \mathbb{R}^{91} \times \{0\}$ *at* $(0,0,0) \in \mathbb{R}^2 \times \mathbb{R}^{91} \times \mathbb{R}^{76}$, *where* $\mathcal{I} := \mathcal{J}(0,w)$.

Under Assumption* the assertion of the theorem can be proved as follows. Let φ denote the above local diffeomorphism $Jet_{\mathbf{\Sigma}} \to R^2 \times \mathbb{R}^{91} \times \mathbb{R}^{76}$. Put $M := \varphi^{-1}(\{0\} \times \mathbb{R}^{91} \times \{0\})$. Note that $(0,w) \in M$. The set M constitutes a smooth submanifold of $Jet_{\mathbf{\Sigma}}$ of dimension 91. Put $\mathcal{E} := \{((f,g),(x,y)) | j^1_{(f,g)}(x,y)$ meets M tranversally at $(x,y)\}$. An argument using Taylor's formula yields $\mathcal{E} \neq \emptyset$ for any dimension of the parameter space p with $dim\ \mathbb{R}^{n+p} + dim\ M \geq dim\ Jet_{\mathbf{\Sigma}}$, i.e. $4+p+91 \geq 169$, hence $p \geq 74$. Now, we fix $p = 74$ and suppose that $((\bar{f},\bar{g}),(\bar{x},\bar{y})) \in \mathcal{E}$. Then, there exist open neighborhoods $\mathcal{F}$ and $\mathcal{U}$ of $(\bar{f},\bar{g})$ and $(\bar{x},\bar{y})$, respectively, such that for any $(f,g) \in \mathcal{F}$ there exists $(x,y) \in \mathcal{U}$ with $j^1_{(f,g)}(x,y) \in \mathcal{E}$. Let $((f,g),(x,y))$ be chosen in such a way. Then, $(\varphi \circ j^1)$ meets $(\{0\} \times \mathbb{R}^{91} \times \{0\})$ tranversally in $\mathbb{R}^2 \times \mathbb{R}^{91} \times \mathbb{R}^{76}$ at the point (x,y). This implies that at this point $(\Pi_{\mathbb{R}^2}, \Pi_{\mathbb{R}^{76}}) \circ \varphi \circ j^1 : \mathbb{R}^{n+p} \to R^2 \times \mathbb{R}^{76}$ is a local diffeomorphism. Here, $\Pi_{\mathbb{R}^2}$ denotes the projection onto the first factor, etc. Obviously, the latter diffeomorphism maps $\mathbf{\Sigma}$ onto $\mathcal{S} \times \{0\}$ and $(\bar{x},\bar{y})$ to $(0,0)$. This would complete the proof of our theorem.

Now, we put $\widehat{\mathbb{R}^4} := \mathbb{R}^3 \times \mathbb{R}_+$, and consider the following diffeomorphism:

$$\begin{aligned} \psi &: (\mathbb{R}^3)^g \times \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^g_+ \to \qquad \{0\} \times \widehat{\mathbb{R}^4} \times (\widehat{\mathbb{R}^4})^g \quad \subset \quad Jet_{\mathbf{\Sigma}} \\ \psi &: \quad ((v_1,..,v_g), v_0, \lambda_0, \lambda) \quad \mapsto \left(0 \ , \ \lambda_0 \begin{pmatrix} v_0 \\ 1 \end{pmatrix} \ , \ \left(\lambda_i \begin{pmatrix} v_0 + v_i \\ 1 \end{pmatrix} \right)_{i \in \mathcal{G}} \right). \end{aligned}$$

If the origin of $\mathbb{R}^3$ is a convex combination of the vectors v_i, say $0 = \sum_{i \neq 0} \alpha_i v_i$, then $\psi(v)$ belongs to $\mathbf{K}$ with Lagrange multipliers $\mu_i = \lambda_i / \lambda_0$. In this setting, we also define (extremal) facet–generators, using convex combinations. To this end we assume $v = (v_1,..,v_g) \in (\mathbb{R}^3)^g$. A set $I \subset \mathcal{G}$ is called a facet–generator if 0 is in the convex hull of $\{v_i \mid i \in I\}$. A facet–generator is called extremal if it does not properly contain another facet–generator. The following lemma establishes the relation between codes $\mathcal{J}$ in $\mathbb{R}^4$ and $\mathbb{R}^3$.

LEMMA 2.1 *On* $(\mathbb{R}^3)^{g+1} \times \mathbb{R}^{1+g}_+$ *the following relation holds:* $\mathcal{J} \circ \Pi_{(\mathbb{R}^3)^g} = \mathcal{J} \circ \psi$.

Given $\mathcal{I} \subset 2^{\mathcal{G}}$, let $\mathcal{R}(\mathcal{I}) := \{v \in (\mathbb{R}^3)^g \mid \mathcal{J}(v) = \mathcal{I}\}$ denote the "realization space" of $\mathcal{I}$. Define the following affine subspaces of $\mathbb{R}^3$: $\mathcal{A}_+ := \mathbb{R}^2 \times \{+1\}$, $\mathcal{A}_- := \mathbb{R}^2 \times \{-1\}$ and put $\mathcal{A} := \mathcal{A}_+ \cup \mathcal{A}_-$. Note that the mapping $\mathcal{J} : \mathcal{A}^g \to 2^{2^{\mathcal{G}}}$ is well defined. Let $\mathcal{R}^{\mathcal{A}}(\mathcal{I}) := \{v \in \mathcal{A}^g \mid \mathcal{J}(v) = \mathcal{I}\}$ stand for the restricted realization space of $\mathcal{I}$. The mapping $\varphi : (v,\lambda) \mapsto (\lambda_i v_i)_i$ establishes a local diffeomorphism between $\mathcal{R}^{\mathcal{A}}(\mathcal{I}) \times \mathbb{R}^g$ at $(v,(1,..,1))$ and $\mathcal{R}(\mathcal{I})$ at v.
The validity of the next Assumption** implies the validity of Assumption* using the latter diffeomorphism φ and Lemma 2.1.

ASSUMPTION** *There exists $v \in \mathcal{A}^{g}$ such that $\mathcal{R}^{\mathcal{A}}(\mathcal{I})$ is at v diffeomorphic with $\mathcal{S} \times \mathbb{R}^{21} \times \{0\}$ at $(0,0,0) \in \mathbb{R}^{2} \times \mathbb{R}^{21} \times \mathbb{R}^{76}$, where $\mathcal{I} := \mathcal{J}(v)$.*

In fact, the diffeomorphism φ uses the product with $\mathbb{R}^{33}$ whereas ψ uses $\mathbb{R}^{4}_{+} \times \mathbb{R}^{33}_{+}$. In total, we get $21 + 33 + 4 + 33 = 91$. ■

3. Proof of the theorem

We start with an outline of the proof. The main idea consists in constructing the singularity $xy = 0$, or, equivalently $x^2 = y^2$. We use constructions due to von Staudt, where the real values $0, 1$ are exposed. Therefore, we will construct the singularity $(x - z/2)^2 = (y - z/2)^2$ still having z at our disposal. To this aim we perform a geometrical computation of the function $F(z, x) := xz - x^2$ for $z \in (2, 4)$ and $x \in (1, 2)$. Put $\mathcal{S}^F_z := \{(x, y) \mid F(z, x) = F(z, y)\}$. This set actually defines the desired singularity.

First assume z to be fixed. We will "compute" the function values $F(z, x)$ and $F(z, y)$, where x and y are the coordinates of certain points P_x and P_y on different lines L_x and L_y endowed with appropriate projective scales. The "computation" generates results $R_x \in L_x$ and $R_y \in L_y$. Comparing these results, we have $F(z, x) = F(z, y)$ if and only if R_x and R_y are colinear with a so-called "connection point" C. The connection point C defines a relation between the scales on L_x and L_y. Our aim is to get a specific code $\mathcal{I}$ for all points of $\mathcal{S}^F_z$. The code $\mathcal{I}$ depends in particular on colinearities of P_x, P_y and the connection point C. The latter points are colinear if and only if $x = y$. Of course, $\mathcal{S}^F_z$ contains also a pointed line of points (x, y) with $x \neq y$. In order to guarantee that the latter points have the same code, we need two copies of $\mathbb{R}^2$, namely $\mathcal{A}_+$ and $\mathcal{A}_-$ as introduced in Section 2. As the result we will obtain the implantation of $\mathcal{S}^F_z$ into a realization space $\mathcal{R}^{\mathcal{A}}(\mathcal{I}) \subset \mathcal{A}^{g}$. The next lemma shows that we need not to fix the value of z. In fact, instead of $\mathcal{S}^F_z$ we implant the cylindrical set $\mathcal{S}^F$ as defined in the following lemma.

LEMMA 3.1 *Let be $a \in (1, 2)$. Put $\mathcal{S}^F := \{(x, y, z) \in \mathbb{R}^3 \mid F(z, x) = F(z, y)\}$. Then, $\mathcal{S}^F$ is at $(a, a, 2a)$ diffeomorphic with $\mathcal{S} \times \mathbb{R}$ at $(0, 0, 0)$.*

PROOF.
$$\begin{aligned} F(z,x) = F(z,y) &\iff z(x-y) = (x+y)(x-y) \\ &\iff x - y = 0 \text{ or } x + y - z = 0 \\ &\iff u = 0 \text{ or } v = 0, \end{aligned}$$
where $u = x - y$, $v = x + y - z$, $w = x - a$ are new coordinates in $\mathbb{R}^3$. ■

Now, let be $\bar{v} \in \mathcal{A}^{g}$ and $\mathcal{I} := \mathcal{J}(\bar{v})$. Recall that the state space is four dimensional. Hence, $|I| \leq 4$ for each $I \in \mathcal{I}$. If $I \in \mathcal{I}$ has cardinality 4, then it is stable w.r.t. small perturbations, i.e. $I \in \mathcal{J}(v)$ for any v sufficiently close to $\bar{v}$. Put $\mathcal{I}_* := \{I \in \mathcal{I} | \, |I| \leq 3\}$ and $\mathcal{R}^{\mathcal{A}}_*(\mathcal{I}_*) := \{v \in \mathcal{A}^{g} | \mathcal{J}(v)_* = \mathcal{I}_*\}$. Then, the set $\mathcal{R}^{\mathcal{A}}(\mathcal{I})$ is open in $\mathcal{R}^{\mathcal{A}}_*(\mathcal{I}_*)$. Consequently, for a local analysis we can delete extremal sets of cardinality four, since $\bar{v} \in \mathcal{R}^{\mathcal{A}}(\mathcal{I}) \cap \mathcal{R}^{\mathcal{A}}_*(\mathcal{I}_*)$.

In virtue of the preparations in Section 2 it suffices to prove the following proposition.

PROPOSITION 3.1 *There exists* $v \in \mathcal{A}^{33}$ *such that* $\mathcal{R}_*^{\mathcal{A}}(\mathcal{I}_*)$ *is at* v *diffeomorphic with* $\mathcal{S}^F \times I\!R^{20} \times \{0\}$ *at* $((a,a,2a),0,0) \in (I\!R^2 \times I\!R) \times I\!R^{20} \times I\!R^{76}$, *where* $\mathcal{I}_* := \mathcal{J}(v)_*$ *and* $a \in (1,2)$.

Before proving Proposition 3.1, we need some preliminaries.

Given $\bar{v} \in \mathcal{A}^g$, we have $\mathcal{J}(v)_* \subset \mathcal{J}(\bar{v})_*$ for v sufficiently close to $\bar{v}$. The following lemma allows to perturb $\bar{v}$ in such a way that $\mathcal{J}(\bar{v})_* \setminus \mathcal{J}(v)_*$ can be appropriately controlled. To this end we extend a mapping $\varphi : \mathcal{A}_+ \to \mathcal{A}_+$ to a mapping $\mathcal{A} \to \mathcal{A}$ by $x \mapsto -\varphi(-x)$ for $x \in \mathcal{A}_-$. We again denote the extended mapping by φ. A subset $\mathcal{V} \subset \mathcal{A}$ is called affine if $-\mathcal{V} = \mathcal{V}$ and $\mathcal{V} \cap \mathcal{A}_+$ is an affine subspace of $\mathcal{A}_+$. Given $\varphi : \mathcal{A} \to \mathcal{A}$, we define a mapping $\varphi : \mathcal{A}^g \to \mathcal{A}^g$ by $\varphi(v_1, .., v_g) := (\varphi(v_1), .., \varphi(v_g))$. The proof of the next lemma is easy, and will be omited (cf. Figure 3).

LEMMA 3.2 (Uncoupling Lemma) *Let be* $\mathcal{V} \subset \mathcal{A}$ *an affine subset and let* $v^1 \in \mathcal{V}^{k_1}$, $v^2 \in (\mathcal{A} \setminus \mathcal{V})^{k_2}$, $v^3 \in (\mathcal{A} \setminus \mathcal{V})^{k_3}$. *Then, there exists an affine mapping* $\varphi : \mathcal{A}_+ \to \mathcal{A}_+$ *arbitrarily close to the identity such that* $\varphi|_{\mathcal{V}} = id|_{\mathcal{V}}$ *and:*

$$\mathcal{J}(v^1, v^2, \varphi(v^3))_* \;=\; \mathcal{J}(v^1, v^2)_* \cup \mathcal{J}(v^1, v^3)_*$$

In Figure 3, we use the isomorphism $\mathcal{A}_+ \equiv I\!R^2$. Points in $\mathcal{A}_+$ are represented by means of the symbol • whereas ∘ stands for points v with $(-v) \in \mathcal{A}_-$.

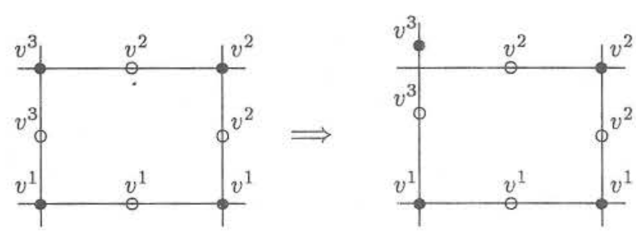


Figure 3.

Projective scales

Let $\overline{I\!R}$ denote the one–point compactification of $I\!R$, i.e $\overline{I\!R} = I\!R \cup \{\infty\}$.

DEFINITION 3.1 *Let be* $a,b,c,d \in I\!R$ *such that* $ad \neq bc$. *Then, the following mapping* $\omega_{abcd} : \overline{I\!R} \to \overline{I\!R}$ *is called projective:*

$$\omega_{abcd}(x) := \begin{cases} \infty & c = 0 \text{ and } x = \infty, \text{or } cx + d = 0 \\ a/c & x = \infty \text{ and } c \neq 0 \\ \frac{ax+b}{cx+d} & else \end{cases}$$

The projective mappings on $\overline{I\!R}$ form a group, with composition as group operation. Note that any affine isomorphism (extended by $\infty \mapsto \infty$) is a projective mapping. If L is a line, i.e. an affine space of dimension one, then let $\overline{L}$ denote the one–point compactification $L \cup \{\infty\}$. The affine isomorphy of lines with $I\!R$ yields that projective mappings are well defined between the one–point compactifications of lines as well. The compactified lines together with the projective mappings form a category. A projective mapping $\omega : \overline{L} \to \overline{I\!R}$ is called a projective scale on L. Given three distinct points P_0, P_1 and P_∞ on a line L, there exists precisely one projective scale ω on L with $\omega(P_0) = 0, \omega(P_1) = 1$ and $\omega(P_\infty) = \infty$.

In the following we see how projective mappings between lines in a plane appear in a natural way. Let L_1, L_2 denote lines in $I\!R^2$ having precisely one common point. Let C be a point distinct from both lines. Then, there is exactly one point $Q_1 \in L_1$ such that the affine hull $aff(Q_1, E)$ does not meet L_2. Let $Q_2 \in L_2$ be defined analogously. Now, the following mapping $\varphi : \overline{L_1} \to \overline{L_2}$ is well defined:

$$\varphi(P_1) := \begin{cases} \infty & P_1 = Q_1 \\ Q_2 & P_1 = \infty \\ P_2 \in L_2 & P_2 \in aff(P_1, E) \end{cases}$$

The point C is called the connection point.

LEMMA 3.3 (Connection Lemma) *Let be L_1, L_2, E and φ as above. Then, φ is a projective mapping.* ■

Geometrical computations

A set of k points an affine space is called an l–colinearity if its affine hull has dimension $l < k - 1$. For abbreviation, a colinearity in $I\!R^2$ always means a 1–colinearity.

DEFINITION 3.2 (Addition) *Let be $v = (A, D, F, G, K, M, P, Q, R) \in (I\!R^2)^9$ such that $A \neq F$, $P \notin aff(A, F)$ and $M \notin \{F, P\}$. Then v is called an Addition if the following colinearities are present: $\{A, D, F, Q, R\}$, $\{F, M, P\}$, $\{A, K, M\}$, $\{K, P, Q\}$, $\{F, G, K\}$, $\{D, G, M\}$, $\{G, P, R\}$.*

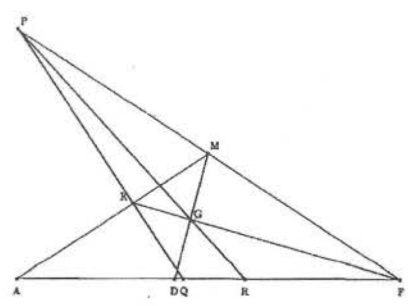


Figure 4.

LEMMA 3.4 (cf. Hartshorne 1967) *Let $v \in (\mathbb{R}^2)^9$ be an Addition and let ω denote a projective scale on $aff(A, F)$ such that $\omega(A) = 0$ and $\omega(F) = \infty$. Then we have: $\omega(R) = \omega(Q) + \omega(D)$.* ■

DEFINITION 3.3 (Multiplication)
Let be $v = (A, B, C, E, F, H, L, M, N, P, R) \in (\mathbb{R}^2)^{11}$ such that $A \neq F$, $B \notin \{A, F\}$, $P \notin aff(A, F)$ and $M \notin \{F, P\}$. Then v is called an Multiplication if the following colinearities are present: $\{A, B, C, E, F, R\}$, $\{F, M, N, P\}$, $\{A, H, L, M\}$, $\{C, H, P\}$, $\{B, H, N\}$,$\{E, L, N\}$, $\{L, P, R\}$.

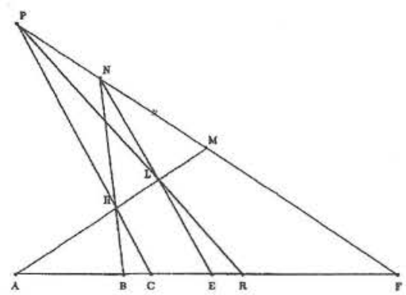


Figure 5.

LEMMA 3.5 (cf. Hartshorne 1967) *Let $v \in (\mathbb{R}^2)^{11}$ be a Multiplication and let ω denote the projective scale on $aff(A, F)$ with $\omega(A) = 0$, $\omega(B) = 1$ and $\omega(F) = \infty$. Then we have: $\omega(R) = \omega(C)\,\omega(E)$.* ■

In Figure 6, we consider the Multiplication $v = (A, B, C, C, F, H, K, M, N, P, Q)$, i.e. $\omega(Q) = \omega(C)\,\omega(C)$.

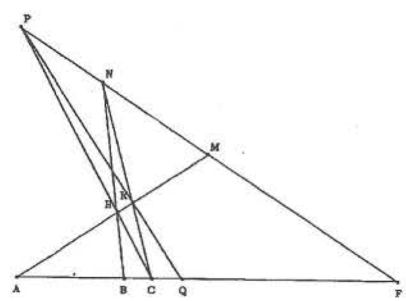


Figure 6.

Combining the latter constructions, we obtain a computation for the function F.

DEFINITION 3.4 (F–Computation)
Let $v = (A, B, C, D, E, F, G, H, K, L, M, N, P) \in (\mathbb{R}^2)^{13}$ *be such that* $A \neq F$, $B \notin \{A, F\}$, $P \notin aff(A, F)$ *and* $M \notin \{F, P\}$. *Then* v *is called an* F*–Computation if the following colinearities are present:* $\{A, B, C, D, E, F\}$, $\{F, M, N, P\}$, $\{A, H, K, L, M\}$, $\{C, H, P\}$, $\{B, H, N\}$, $\{C, K, N\}$, $\{E, L, N\}$, $\{F, G, K\}$, $\{G, L, P\}$ $\{D, G, M\}$. *An* F*–Computation is called regular, if* $\omega(C) \in (1, 2)$ *and* $\omega(E) \in (2, 4)$, *where* ω *denotes the associated projective scale (on* $aff(A, F)$*), being defined by the relations* $\omega(A) = 0$, $\omega(B) = 1$ *and* $\omega(F) = \infty$.

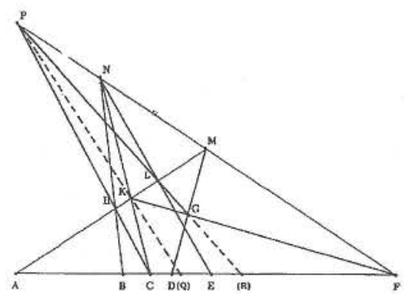


Figure 7.

COROLLARY 3.1 *Let* $v \in (\mathbb{R}^2)^{13}$ *be an* F*–Computation, and* ω *the associated projective scale. Then we have* $\omega(D) = F(\ \omega(E)\ ,\ \omega(C)\ )$. ■

Next, we discuss the l–colinearities that may occur in an F–computation. The 2–colinearities are stable, hence they are not interesting for our purposes. Suppose that there are no 0–colinearities. The set $\mathcal{W}$ of "wished" colinearities is defined to be the smallest set with the following properties. The colinearities appearing in Definition 3.4 are wished. Moreover, if $\mathcal{L}_1$ is wished and $\mathcal{L}_2 \subset \mathcal{L}_1$ is a colinearity, then $\mathcal{L}_1$ is wished as well. If $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{W}$ and $|\mathcal{L}_1 \cap \mathcal{L}_2| \geq 2$, then $\mathcal{L}_1 \cup \mathcal{L}_2 \in \mathcal{W}$. All other colinearities are called "unwished".

LEMMA 3.6 *Let* v *be a regular* F*–Computation and* ω *the associated projective scale. Then, 0–colinearities are not present and* $\{D, K, P\}$ *is the only unwished colinearity which might appear.*

PROOF. The absence of 0–colinearities is easily checked. Now, let $\mathcal{L}$ be an unwished colinearity. By definition any unwished colinearity contains an unwished colinearity of cardinality three. Hence, without loss of generality, $|\mathcal{L}| = 3$. We will prove that $\mathcal{L} = \{D, K, P\}$. For convenience, we consider four classes of points: $\mathcal{C}_1 := \{A, B, C, D, E, F\}$, $\mathcal{C}_2 := \{G\}$, $\mathcal{C}_3 := \{H, K, L, M\}$ and $\mathcal{C}_4 := \{N, P\}$ (see Figure 7). Since $\mathcal{L}$ is unwished, it has to meet three of them.

Case 1. $G \in \mathcal{L}$.

Case 1.1. $P \in \mathcal{L}$: $\{G, L, P\}$ is wished, thus the third point coincides with R. This is not possible by $\omega(R) = \omega(E)\omega(C) > \omega(E)$.

Case 1.2. $N \in \mathcal{L}$: Now, $aff(\mathcal{L})$ meets $aff(A, M)$ in (L, M) and it meets $aff(A, F)$ in (E, F). But then it does not contain any point of $\mathcal{C}_3$ or $\mathcal{C}_1$.

Case 1.3. $\mathcal{L}$ meets $\mathcal{C}_1$ and $\mathcal{C}_3$: By simultaneous reasons: $K, L \notin \mathcal{L}$. Finally, $(A, F) \cap aff(G, H) = \emptyset$.

Case 2. $\mathcal{L}$ meets $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$: The intersection of $\mathcal{L}$ with $\mathcal{C}_4$ cannot be N, since all colinearities involving $\mathcal{C}_3$ and N are wished. Hence, $P \in \mathcal{L}$. But then, the only candidate for a point from $\mathcal{C}_3$ being included in an unwished colinearity should be K. The point from $\mathcal{C}_1 \cap \mathcal{L}$ has to coincide with Q. Since $\omega(Q) = \omega(C)^2$, we have $Q \in (C, E)$. Hence, the only candidate from $\mathcal{C}_1$ for coinciding with Q is D. ■

If we choose $S \in (L, M)$, $T \in (F, M)$, $U \in (C, H)$ and $V \in (C, K)$, then, these four points extend the set of wished colinearities in a natural way. One can choose the latter points such that no additional unwished colinearities appear. In the sequel, S, T, U, V are always chosen in such a way.

Proof of Proposition 3.1

Let v be a regular F-Computation and S, T as above in $\mathbb{R}^2 \equiv \mathcal{A}_+$. Next, we define $\hat{v} \in \mathcal{A}^{15}$:

$$\hat{v} := (A, B, C, -D, E, -F, G, -H, -K, -L, -M, N, P, S, T).$$

By means of the following relation, the points $\hat{A}, \hat{B}, \hat{C}, ..$ are well defined: $\hat{v} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}, .., \hat{T})$ (see also Figure 8). Such $\hat{v}$ shall be called an $\hat{F}$-Computation. Note that $\{\hat{D}, \hat{K}, \hat{P}\} \notin \mathcal{J}(\hat{v})$. In a regular F-Computation the order of the points on the lines belonging to wished colinearities is fixed. This, together with Lemma 3.6 implies the existence of $\hat{\mathcal{I}}_*$ such that the set of $\hat{F}$-Computations is an open subset of $\mathcal{R}^{\mathcal{A}}_*(\hat{\mathcal{I}}_*)$.

Analogously, we define $\tilde{v} \in \mathcal{A}^{17}$:

$$\tilde{v} := (A, -B, C, D, -E, F, -G, H, K, L, M, -N, -P, -S, -T, -U, -V),$$

where v is a regular F-Computation and S, T, U, V as above. Let be $\tilde{v} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, .., \tilde{V})$ (see Figure 9). This defines an $\tilde{F}$-Computation $\tilde{v}$. Here, we also have $\{\tilde{D}, \tilde{K}, \tilde{P}\} \notin \mathcal{J}(\tilde{v})$. Hence, the set of $\tilde{F}$-Computations is open in $\mathcal{R}^{\mathcal{A}}_*(\tilde{\mathcal{I}}_*)$ for a specific $\tilde{\mathcal{I}}_*$.

In the following we sketch the latter computations. In order to draw points in $\mathcal{A}$, we use the convention of Figure 3.

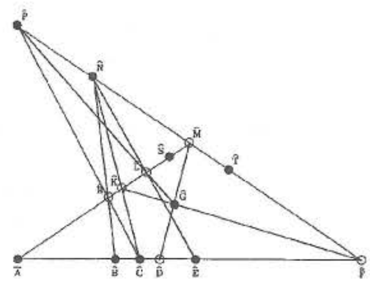


Figure 8.

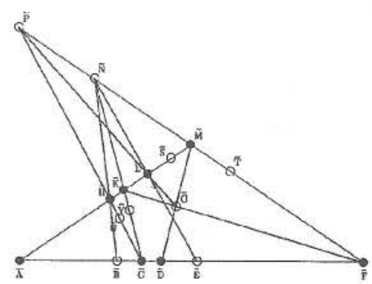


Figure 9.

We say that $\bar{v} = (\bar{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}, X_+, X_-) \in \mathcal{A}^{13}$ is a Connection if $\bar{A}, \hat{B}, \hat{C}, -\hat{D}, \hat{E}, -\hat{F}, -\tilde{B}, \tilde{C}, \tilde{D}, -\tilde{E}, \tilde{F}, X_+ \in \mathcal{A}_+$ are distinct points, if $X_- = -X_+$, $X_+ \notin aff(\bar{A}, -\hat{F})$, $\tilde{F} \in (-\hat{F}, -X_-)$ and if in $\mathcal{A}_+ \equiv I\!R^2$ the following colinearities are present: $\{\bar{A}, \hat{B}, \hat{C}, -\hat{D}, \hat{E}, -\hat{F}\}$, $\{\bar{A}, -\tilde{B}, \tilde{C}, \tilde{D}, -\tilde{E}, \tilde{F}\}$, $\{\hat{B}, -\tilde{B}, X_+\}$, $\{-\hat{D}, \tilde{D}, -X_-\}$, $\{\hat{E}, -\tilde{E}, X_+\}$, $\{-\hat{F}, \tilde{F}, -X_-\}$, see Figure 10. There is a $\bar{\mathcal{I}}_*$ such that the set of Connections is open in $\mathcal{R}_*^{\mathcal{A}}(\bar{\mathcal{I}}_*)$.

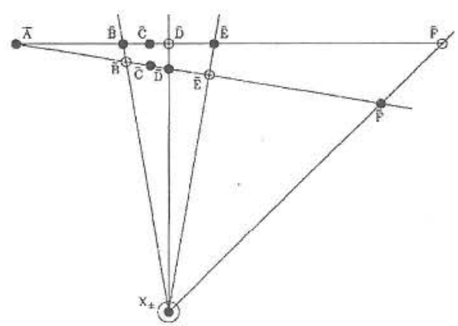


Figure 10.

Assume the existence of

$$v = (\bar{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\hat{G}\hat{H}\hat{K}\hat{L}\hat{M}\hat{N}\hat{P}\hat{S}\hat{T}\tilde{B}\tilde{C}\tilde{D}\tilde{E}\tilde{F}\tilde{G}\tilde{H}\tilde{K}\tilde{L}\tilde{M}\tilde{N}\tilde{P}\tilde{S}\tilde{T}\tilde{U}\tilde{V}X_+X_-) \in \mathcal{A}^{33}$$

such that the corresponding $\hat{v}$ is an $\hat{F}$–Computation, $\tilde{v}$ is an $\tilde{F}$–Computation and $\bar{v}$ is a Connection.

Note that at this stage we perform simultaneously two computations in $\mathcal{A}_+$. This might produce new colinearities between the two different parts of our computation scheme. Such new colinearities will be eliminated by means of the Uncoupling Lemma. First, we apply the Uncoupling Lemma to the triple $v^1 := \hat{v} \cap \bar{v}$, $v^2 := v \setminus \hat{v}$, $v^3 := \hat{v} \setminus \bar{v}$ with $\mathcal{V} := aff(\bar{A}, \hat{F})$. Note, that we have $v^1 \cup v^3 = \hat{v}$ and $v^1 \cup v^2 = \tilde{v} \cup \bar{v}$. Hence, we may assume without loss of generality that $\mathcal{J}(v)_* = \hat{\mathcal{I}}_* \cup \mathcal{J}(\tilde{v} \cup \bar{v})_*$.

Now, we neglect $\hat{v} \setminus \bar{v}$ and apply the Uncoupling Lemma to the triple $v^1 := \tilde{v} \cap \bar{v}$, $v^2 := (\tilde{v} \cup \bar{v}) \setminus \tilde{v}$, $v^3 := \tilde{v} \setminus \bar{v}$ with $\mathcal{V} := aff(\bar{A}, \tilde{F})$. Note, that $v^1 \cup v^3 = \tilde{v}$ and $v^1 \cup v^2 = \bar{v}$. Moreover, all extremal facet–generators involving points from $\hat{v} \setminus \bar{v}$ remain present, since they are contained in $\hat{v} \cup \bar{v}$ (which is not affected). Hence, without loss of generality we have $\mathcal{J}(v)_* = \hat{\mathcal{I}}_* \cup \tilde{\mathcal{I}}_* \cup \bar{\mathcal{I}}_* =: \mathcal{I}_*$.

In order to complete the proof of Proposition 3.1 we have to show the existence of some v satisfying the above assumption together with the desired diffeomorphism. Consider the following order within the points of v:

$$\bar{A} X_+ X_- \hat{F} \tilde{F} \hat{B} \tilde{B} \hat{P} \tilde{P} \hat{M} \tilde{M} \hat{S} \tilde{S} \hat{T} \tilde{T} \hat{E} \tilde{E} \hat{C} \hat{H} \hat{N} \hat{K} \hat{L} \hat{G} \hat{D} \tilde{C} \tilde{U} \tilde{H} \tilde{N} \tilde{V} \tilde{H} \tilde{L} \tilde{G} \tilde{D}.$$

Starting with $\bar{A}$ (two degrees of freedom), etc., we built up v stepwise in the given order such that $\mathcal{J}(v)_* \in \{\mathcal{I}_*\ ,\ \mathcal{I}_* \setminus \{\hat{D}, \tilde{D}, X_-\}\}$. After the choice of $\tilde{B}$ we have projective scales $\hat{\omega}$ and $\tilde{\omega}$ on $aff(\bar{A}, -\hat{F})$ and $aff(\bar{A}, \tilde{F})$, respectively, such that $\hat{\omega}(\bar{A}) = 0$, $\hat{\omega}(\hat{B}) = 1$, $\hat{\omega}(-\hat{F}) = \infty$, $\tilde{\omega}(\bar{A}) = 0$, $\tilde{\omega}(-\tilde{B}) = 1$ and $\tilde{\omega}(\tilde{F}) = \infty$. We choose $\hat{E}$ such that $\hat{\omega}(\hat{E}) \in (2,4)$. Finally, let $\hat{C}, \tilde{C}$ be such that $\hat{\omega}(\hat{C}), \tilde{\omega}(\tilde{C}) \in (1,2)$. By the specific choice of projective scales, an application of the Connection Lemma to $\bar{v}$ implies $\hat{\omega}(\hat{E}) = \tilde{\omega}(-\tilde{E})$. Moreover, we have the following fundamental relation:

$$\{\hat{D}, \tilde{D}, X_-\} \in \mathcal{J}(v)_* \iff \hat{\omega}(-\hat{D}) = \tilde{\omega}(\tilde{D}) \tag{1}$$

Let N denote the space of v constructed so far. In fact, N is locally diffeomorphic with $(\mathbb{R}^2 \times \mathbb{R}) \times \mathbb{R}^{20}$. The latter diffeomorphism φ can be chosen in such a way that the first three coordinates of the image $(\varphi_1, \varphi_2, \varphi_3)(v)$ coincide with $(\hat{\omega}(\hat{C}), \tilde{\omega}(\tilde{C}), \hat{\omega}(\hat{E}))$.

For any $v \in N$, the involved $\hat{v}, \tilde{v}$ are $\hat{F}-$ and $\tilde{F}$–Computations. This yields:

$$\hat{\omega}(-\hat{D}) = F(\hat{\omega}(\hat{E}), \hat{\omega}(\hat{C})) \tag{2}$$

$$\tilde{\omega}(\tilde{D}) = F(\tilde{\omega}(-\tilde{E}), \tilde{\omega}(\tilde{C})) \tag{3}$$

For any $v \in N$, the relations 1, 2 and 3 imply the next relations:

$$\mathcal{J}_*(v) = \mathcal{I}_* \iff \varphi(v) \in \mathcal{S}^F \times \mathbb{R}^{20}$$

Without loss of generality, for a specific $v \in N$ we have chosen $\hat{C}$ and $\tilde{C}$ such that $\tilde{\omega}(\tilde{C}) = \hat{\omega}(\hat{C}) = 1/2\ \hat{\omega}(-\hat{E})$. Again in virtue of the relations 1–3, we have $(\varphi_1, \varphi_2, \varphi_3)(v) = (a, a, 2a) \in \mathcal{S}^F$, where $a := \hat{\omega}(\hat{C})$. This completes the proof of Proposition 3.1. ∎

REMARK 3.1 *The forestanding construction can be generalized in order to implant any algebraic singularity for polynomials with integer coefficients. The number of scales coincides with the number of variables in the polynomial. This will be shown in the forthcoming paper Günzel 1994.*

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