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On the partition of real symmetric matrices according to the multiplicities of their eigenvalues

by

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In this paper we will be concerned with a partition of the set of all real symmetric $n \times n$–matrices into strata corresponding to the multiplicities of the eigenvalues. It will be shown that this stratification is Whitney Regular. Moreover, we derive an explicit formula for the codimension of the strata in terms of the multiplicities involved. The transversality theory of R. Thom leads to generic perturbation results for the eigenvalues of one–parameter families of real symmetric matrices. The connections with sensitivity results in parametric optimization are investigated.

1. Introduction

The present paper aims to analyse the partition of the set of all real symmetric $n \times n$–matrices, according to the multiplicities[1] of their eigenvalues. We will show that this partition forms a so-called Whitney Regular Stratification.

The formal definition of Whitney Regular Stratification is rather complicated, but roughly speaking, it means a subdivision into locally finite, mutually disjoint, smooth (C^∞) manifolds (strata) which stick together in such a regular way that the local topological type of the partition remains constant along each stratum.

For us, the relevance of such a stratification result relies on the possibility of applying certain "general-position" arguments (Thom's transversality theorem). In fact, as a corollary we will obtain the following stability/approximation result on the set of all families $A(t)$ of real symmetric $n \times n$–matrices which depend smoothly on a real parameter t.

[1] Since we deal with symmetric matrices, no distinction will be made between the algebraic and geometric multiplicity.

STABILITY AND APPROXIMATION PROPERTY: *Given any such one-parameter family of matrices, say $\overline{A}(t), t \in \mathbb{R}$, with the property that there are only simple eigenvalues for all t. Then, this property will be maintained under sufficiently small, but for the rest arbitrary perturbations (to be specified below) on the coefficients of $\overline{A}(\cdot)$. On the other hand, let the family $A(t), t \in \mathbb{R}$, be arbitrary. Then suitable, but arbitrarily small perturbations (to be specified below) will result into a family $\hat{A}(t)$ with only simple eigenvalues for all $t \in \mathbb{R}$.*

The above statement can be interpreted as a sensitivity result for the following one–parameter family of optimization problems $\mathcal{P}(t), t \in \mathbb{R}$,

$$\mathcal{P}(t) \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A(t) x \text{ , subject to } \frac{1}{2} x^T x - 1 = 0,$$

where x^T stands for the transpose of x. In fact, we will see that generically (to be explained below) for all t the problems $\mathcal{P}(t)$ do only exhibit so–called non–degenerate critical points.

Finally, we prove that — in contradistinction with the latter situation — for problems of the type

$$\mathcal{R}(t) \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A(t) x + b^T(t) x \quad \text{subject to} \quad \frac{1}{2} x^T x - 1 = 0,$$

with $A(\cdot)$ as above and $b(\cdot)$ a smooth map from $\mathbb{R}$ to $\mathbb{R}^n$, the occurrence of degenerate critical points cannot be excluded, even not in the generic case.

This paper is organized is follows:
In Section 2 we fix our notations and formulate our Stratification Theorem. The proof of this theorem is given in Section 3, whereas we present the corollaries in Section 4. Finally in Section 5 we discuss some aspects of the problems $\mathcal{R}(t)$.

2. Stratification Theorem

Let $\mathcal{A}_n$ stand for the set of all real symmetric $n \times n$–matrices. Obviously, this set may be identified with the Euclidean space $\mathbb{R}^K$, where (by symmetry) $K = \frac{1}{2} n(n+1)$. The multiplicities of the eigenvalues of a matrix A in $\mathcal{A}_n$ are denoted by $m_j(A), j = 1, \cdots, l$, where l is the number of distinct eigenvalues of A. So we have $m_1(A) + \cdots + m_l(A) = n$. Now we introduce the symbol $\sigma(A)$:

$$\sigma(A) = \{m_1(A), \cdots, m_l(A)\}.$$

Let σ be any partition of n into strictly positive integers, say $m_j, j = 1, \cdots, k$. So,

$$\sigma = \{m_1, \cdots, m_k\},$$

where $m_1 + \cdots + m_k = n$. The set of all such partitions is denoted by $\mathcal{S}$. For any $\sigma \in \mathcal{S}$, the subset $\mathcal{A}_\sigma$ of $\mathcal{A}_n$ is defined as follows

$$\mathcal{A}_\sigma = \{A \in \mathcal{A}_n \mid \sigma(A) = \sigma\}.$$

Apparently, the collection $\{\mathcal{A}_\sigma\}_{\sigma \in \mathcal{S}}$ constitutes a finite partition for $\mathcal{A}_n$ into (mutually distinct) subsets. Now, we come to our main theorem.

STRATIFICATION THEOREM *The partition $\{\mathcal{A}_\sigma\}_{\sigma \in \mathcal{S}}$ forms a Whitney Regular Stratification for $\mathcal{A}_n$, i.e. the following two conditions hold.*

Condition (1) *Every $\{\mathcal{A}_\sigma\}, \sigma \in \mathcal{S}$ is a C^∞-manifold.*

Condition (2) *Any $\mathcal{A}_{\sigma_2}$ is Whitney Regular over any other $\mathcal{A}_{\sigma_1}$ at any $\overline{A} \in \mathcal{A}_{\sigma_1}$.*

Moreover, for $\sigma = \{m_1, \cdots, m_k\}$ we have

$$\text{codim } \mathcal{A}_\sigma = \sum_{j=1}^{k} [\frac{1}{2} m_j(m_j + 1) - 1].$$

where the codimension is always taken with respect to the set $\mathbb{R}^K$.

We briefly discuss the above Conditions (1) and (2). As already mentioned before, these conditions turn out to be rather complicated (especially the second one). Nevertheless, just to be complete we shall give the formal definitions, thereby following Gibson at al. 1976 and Jongen at al. 1983, 1986, in which also references on the intuitive meaning of these concepts can be found. We emphasize that both Conditions (1) and (2) are of local nature. In particular, it turns out that if we apply a strata preserving diffeomorphism, then the structures of the stratification around a point and its image point are the same. In fact, this is the only property to be used in the sequel.

Condition (1): This means that around any $\overline{A}$ in $\mathcal{A}_\sigma$ this set is locally C^∞ -diffeomorphic with an open ball in $\mathbb{R}^{d(\sigma)}$, $d(\sigma) = \dim \mathcal{A}_\sigma$; codim $\mathcal{A}_\sigma = K - \dim \mathcal{A}_\sigma$. When Condition (1) is fulfilled for $\mathcal{A}_\sigma$ we call $\mathcal{A}_\sigma$ a stratum.

Condition (2): We say that a stratum $\mathcal{A}_{\sigma_2}$ is Whitney Regular over another stratum $\mathcal{A}_{\sigma_1}$ at $\overline{A} \in \mathcal{A}_{\sigma_1}$ if for every sequence $((A_i, B_i))$ in the product space $\mathcal{A}_{\sigma_1} \times \mathcal{A}_{\sigma_2}$ with the following three properties (i), (ii), (iii)

(i) $A_i \to \overline{A}$, $B_i \to \overline{A}$,

(ii) the tangent spaces $(T_{B_i}\mathcal{A}_{\sigma_2})$ tend (in the Grassmannian sense) to a linear subspace $\mathcal{T} \subset \mathbb{R}^K$,

(iii) the lines L_i spanned by $(A_i - B_i)$ tend (in the Grassmannian sense) to a line $L \subset \mathbb{R}^K$,

the following holds: $L \subseteq \mathcal{T}$.

We conclude this section with two remarks on related results.

Let $\mathcal{M}_n$ be the set of all complex $n \times n$- matrices. Obviously, this set may be identified with $\mathbb{C}^{n^2}$. For any matrix M in $\mathcal{M}_n$ we denote by $s(M)$ the so-called Segre symbol. Here we will not give a formal definition of this symbol, but merely mention that $s(M)$ is essentially an array containing all information on the structure of the Jordan decomposition of M. Moreover, if $M \in \mathcal{A}_n$, then

$s(M)$ reduces to $\sigma(M)$. The following result has been stated by Arnold (cf. Arnold 1981) and proved in full detail by Gibson (cf. Gibson 1976):

THEOREM (Arnold/Gibson) *The partition of $\mathcal{M}_n$ according to the Segre symbols of its elements forms a Whitney Regular Stratification of $\mathcal{M}_n$.*

Formally, if $\mathcal{M}_n$ is replaced by $\mathcal{A}_n$, this theorem gives the statement of our Stratification Theorem. We emphasize however, that this does not mean that the Arnold/Gibson result automatically implies our Stratification Theorem. For some special cases, the formula for $codim \mathcal{A}_\sigma$ has been derived by Dellnitz and Melbourne 1992 from the results of Arnold 1981, without a rigorous proof. In the paper of Dellnitz and Melbourne, the generic eigenvalue behaviour of classes of one–parameter families of selfadjoint matrices has been investigated.

As a second remark we note that the partition of $\mathcal{A}_n$ into a Whitney Regular Stratification is certainly not unique. In fact, the partition of $\mathcal{A}_n$ according to the rank of its elements also forms a Whitney Regular Stratification, see e.g. Jongen et al. 1983, 1986. This reference also contains applications of the "rank stratification" within the field of Newton flows and the field of Parametric Optimization.

3. Proof of the Stratification Theorem

We begin by showing that $\{\mathcal{A}\}_{\sigma \in \mathcal{S}}$ forms a Whitney Regular Stratification for $\mathcal{A}_n (= \mathbb{R}^K)$. We put $\sigma = \{m_1, \ldots, m_k\}$ and give the proof in three steps.

Step 1: $\mathcal{A}_\sigma$ is a semi–algebraic subset of $\mathbb{R}^K$.
Note that a subset of an Euclidean space, say $\mathbb{R}^l$, is semi–algebraic if it is generated — in the Boolean sense — by sets of the form $\{\zeta \in \mathbb{R}^l \mid p(\zeta) > 0\}$ with $p : \mathbb{R}^l \to \mathbb{R}$ a polynomial function. Given σ, we consider the subset V_σ of $\mathbb{R}^n$ given by all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ such that:

$x_1 = \cdots = x_{m_1}, \ \ x_{m_1+1} = \cdots = x_{m_1+m_2}, \ \cdots,$
$x_{m_1+\cdots+m_{k-1}+1} = \cdots = x_n,$ and the k numbers
$x_1, x_{m_1+1}, \cdots, x_{m_1+\cdots+m_{k-1}+1}$ are mutually distinct.

Evidently, the set V_σ is semi–algebraic in $\mathbb{R}^n$. Now we consider the map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ given by $\varphi(x) = (s_1(x), \cdots, s_n(x))$, where $s_i, \ i = 1, \cdots, n,$ are the elementary symmetric functions on $\mathbb{R}^n$, i.e. $s_1(x) = x_1 + x_2 + \cdots + x_n, \cdots, s_n(x) = x_1 x_2 \cdots x_n$. Since φ is a polynomial map, it follows from the theorem of Tarski–Seidenberg (cf. Gibson at al. 1976) that the image $W_\sigma := \varphi(V_\sigma)$ is also semi–algebraic in $\mathbb{R}^n$.

Next we introduce the map $\Psi : \mathcal{A}_n \to \mathbb{R}^n$ given by $\Psi(A) = (c_1(A), \cdots, c_n(A))$, where $c_j(A) =$ (sum of all $j \times j$–principal minors of A), $j = 1, \cdots, n$. Note that the components of $\varphi(x)$ resp. $\Psi(A)$ are (up to a sign) the coefficients of the characteristic polynomials of any matrix in $\mathcal{A}_n$ with eigenvalues $x_1, \cdots, x_n$, resp.

the matrix A. This yields $\mathcal{A}_\sigma = \Psi^{-1}(W_\sigma)$, and hence — Ψ being polynomial — the set $\mathcal{A}_\sigma$ is algebraic in $\mathbb{R}^K$.

Step 2: The homogeneity property for $\mathcal{A}_\sigma$.

We will show that the following homogeneity property (cf. Gibson at al. 1976) holds:

For any two $\overline{A}, \overline{B}$ in $\mathcal{A}_\sigma$ a strata preserving local diffeomorphism, say p, exists from a $\mathbb{R}^K$–neighborhood of $\overline{A}$ onto a $\mathbb{R}^K$–neighborhood of $\overline{B}$ such that $p(\overline{A}) = \overline{B}$.

Firstly, we observe that for any orthogonal $n \times n$-matrix Q, the map $A \longmapsto Q^T A Q$ is a diffeomorphism from $\mathcal{A}_n$ onto $\mathcal{A}_n$ which preserves the eigenvalue distribution. So, we may assume (no loss of generality) that $\overline{A}$ and $\overline{B}$ are diagonal matrices, each with k distinct eigenvalues — say $\overline{\alpha_j}$ resp. $\overline{\beta}_j$ — of multiplicity m_j, $j = 1, 2, \cdots, k$.

A moment of reflection shows that there always exists a real polynomial p of degree k, with $p(\overline{\alpha}_j) = \overline{\beta}_j$ and $p'(\overline{\alpha}_j) \neq 0, j = 1, \cdots, k$. Choose such a polynomial, say $p(x) = a_k x^k + \cdots + a_1 x + a_0$. Then we define the smooth map $p : \mathcal{A}_n \to \mathcal{A}_n$ as follows

$$p(A) = a_k A^k + \cdots + a_1 A + a_0 I_n,$$

where I_n stands for the $n \times n$–unit matrix. Obviously, we have $p(\overline{A}) = \overline{B}$. Moreover, by a straightforward calculation we find that the Jacobian matrix $\nabla p(\overline{A})$ of p at $\overline{A}$ is a $K \times K$–diagonal matrix with diagonal elements $p'(\overline{\alpha}_j)$, $(p(\overline{\alpha}_j) - p(\overline{\alpha}_i))/(\overline{\alpha}_j - \overline{\alpha}_i)$, $i, j = 1, \cdots k, i < j$, which by construction are all non–zero. So, the map p is a local diffeomorphism around $\overline{A}$.

It remains to prove that p preserves — locally around $\overline{A}$ — the strata $\mathcal{A}_\sigma$. To this aim, we observe that given any m-fold eigenvalue α for $A \in \mathcal{A}_n$, then $p(\alpha)$ is an eigenvalue for $p(A)$ with multiplicity $\geq m$. Moreover, since the numbers $p'(\overline{\alpha}_j)$ are non–vanishing, for any A sufficiently close (in the Euclidean sense) to $\overline{A}$ it follows that $p(\alpha)$ is a m–fold eigenvalue for $p(A)$.

Step 3: Combination of the preceding steps.

Here we use standard arguments from stratification theory (cf. Gibson at al. 1976). The set $\mathcal{A}_\sigma$ being semi–algebraic in $\mathbb{R}^K$, there always exists an $\overline{A} \in \mathcal{A}_\sigma$ and a $\mathbb{R}^K$–neighborhood Ω of $\overline{A}$ such that $\Omega \cap \mathcal{A}_\sigma$ is diffeomorphic with an open Euclidean ball. By the homogeneity property it follows that $\mathcal{A}_\sigma$ is a smooth manifold.

Now, the sets $\mathcal{A}_\sigma$ being semi–algebraic smooth manifolds, a classical result by Whitney (cf. Gibson at al. 1976) yields for given $\mathcal{A}_\sigma$, $\mathcal{A}_{\sigma'}$ the existence of an $\overline{A} \in \mathcal{A}_\sigma$ such that $\mathcal{A}_{\sigma'}$ is Whitney Regular over $\mathcal{A}_\sigma$ at $\overline{A}$. Due to the local nature of the concept of Whitney Regular Stratification by the homogeneity property (Step 2) the stratum $\mathcal{A}_{\sigma'}$ is Whitney Regular over $\mathcal{A}_\sigma$ at any $A \in \mathcal{A}_\sigma$.

We proceed by proving the statement on codim $\mathcal{A}_\sigma$. This proof will provide an elementary (independent) proof of the proposition that $\mathcal{A}_\sigma, \sigma \in \mathcal{S}$ is a smooth manifold in $\mathcal{A}_n$. In the sequel we will use the following fact (cf. Jongen et al.

1983, 1986):
Let A be a symmetric $n \times n$ matrix of the form

$$A = \begin{pmatrix} D & C \\ C^T & B \end{pmatrix}, \quad D \text{ a } k \times k-, \ B \text{ a regular } (n-k) \times (n-k)\text{–matrix.}$$

Then

$$\operatorname{rank} A = n - k \Leftrightarrow D = CB^{-1}C^T. \tag{1}$$

Now, for any fixed $\sigma = \{m_1, \cdots, m_k\}$ we consider the diagonal matrix $\overline{D} \in \mathcal{A}_\sigma$ with m_j–fold eigenvalues $\overline{\lambda}_j, j = 1, \cdots, k$. We put $A = \overline{D} + S$, $S \in \mathcal{A}_n$, and decompose A as follows:

$$A = \begin{pmatrix} D_1 & & & C \\ & D_2 & & \\ & & \ddots & \\ C^T & & & D_k \end{pmatrix}, \quad D_j \ m_j \times m_j\text{–matrices}, \ j = 1, \ldots, k, \tag{2}$$

where for $A = \overline{D}$ we have $D_j = \overline{\lambda}_j I_{m_j}$ and $C = 0$. If A is sufficiently close to $\overline{D}$ (i.e. the Euclidean norm $\| S \|$ of S is small) then mutually disjoint $\mathbb{R}$–neighborhoods Ω_j of $\overline{\lambda}_j$ exist, such that in each Ω_j there lie m_j eigenvalues of A (counted with multiplicities). Selecting in any Ω_j one eigenvalue λ_j of A we have for all j, $j = 1, \cdots, k$:

$$\lambda_j \text{ is an } m_j\text{–fold eigenvalue of } A \Leftrightarrow \operatorname{rank}(A - \lambda_j I_n) = n - m_j. \tag{3}$$

By using appropriate permutation matrices E_j, the matrix $E_j^T A E_j$ has the form

$$E_j^T A E_j = \begin{pmatrix} D_j & C_j \\ C_j^T & B_j \end{pmatrix},$$

with D_j as in (2), B_j an $(n-m_j)\times(n-m_j)$–matrix. Since $\operatorname{rank} E_j^T(A-\lambda_j I_n)E_j = \operatorname{rank}(A - \lambda_j I_n)$, by using the above result (1) we find that for A near $\overline{D}$ the conditions in (3) are equivalent with the conditions

$$F^j(A, \lambda_j) = 0, \quad j = 1, \cdots, k,$$

where

$$F^j(A, \lambda) = (D_j - \lambda I_{m_j}) - C_j(B_j - \lambda I_{n-m_j})^{-1} C_j^T. \tag{4}$$

By symmetry (of A), $F^j = 0$ represents a system of $(m_j + 1)m_j/2$ equations $F_i^j = 0$, $i = 1, \cdots, (m_j + 1)m_j/2$, each component F_i^j corresponding to an element of ("the upper part") of D_j. By a straightforward calculation it is not difficult to see, that the following holds: for any fixed $\lambda \in \mathbb{R}$, all partial derivatives with respect to (the K elements of) A,

$$\partial_A F_\rho^j(\overline{D}, \lambda), \quad \begin{matrix} j = 1, \ldots, k, \\ \rho = 1, \ldots, \frac{(m_j+1)m_j}{2} \end{matrix} \quad \text{are linearly independent.} \tag{5}$$

Now, for any $j = 1, \cdots, k$ we will apply the Implicit Function Theorem to the equation $F_1^j(A, \lambda) = 0$ (corresponding to the element d_{11}^j of D_j). To this aim, we note that $\partial_\lambda F_1^j(\overline{D}, \lambda) = -1$ (cf. (4)), where ∂_λ stands for the partial derivative with respect to λ. So, in a neighborhood $\overline{U}$ of $\overline{D}$, the relation $F_1^j(A, \lambda) = 0$ defines a (unique) C^∞-function $\lambda_j : \overline{U} \to \Omega_j$, such that $F_1^j(A, \lambda_j(A)) = 0$, $\lambda_j(\overline{D}) = \overline{\lambda}_j$ and

$$\nabla \lambda_j(\overline{D}) = \partial_A F_1^j(\overline{D}, \overline{\lambda}_j). \tag{6}$$

By substituting the functions $\lambda_j(A)$ into (4) we finally arrive at the equations

$$f_i^j(A) := F_i^j(A, \lambda_j(A)) = 0, \quad \begin{array}{l} j = 1, \ldots, k, \\ i = 2, \ldots, \frac{(m_j+1)m_j}{2}, \end{array} \tag{7}$$

with $\sum_{j=1}^k \left(\frac{(m_j+1)m_j}{2} - 1\right)$ functions $f_i^j \in C^\infty(\overline{U}, \mathbb{R})$ ($\overline{U}$ a sufficiently small neighborhood of $\overline{D}$). By construction we have shown that for any A in a neighborhood $\overline{U}$ of $\overline{D}$ the following holds:

A has k distinct eigenvalues λ_j with multiplicities m_j
iff the equations (7) are satisfied.

To complete the proof, it remains to show that all gradients $\nabla f_i^j(\overline{D})$, $j = 1, \ldots, k$, $i = 2, \ldots, \frac{(m_j+1)m_j}{2}$ are linearly independent. This follows immediately from (cf. (6))

$$\begin{aligned} \nabla f_i^j(\overline{D}) &= \nabla F_i^j(A, \lambda_j(A))_{|_{A=\overline{D}}} \\ &= \partial_A F_i^j(\overline{D}, \overline{\lambda}_j) + \partial_\lambda F_i^j(\overline{D}, \overline{\lambda}_j) \cdot \nabla \lambda_j(\overline{D}) \\ &= \partial_A F_i^j(\overline{D}, \overline{\lambda}_j) + \partial_\lambda F_i^j(\overline{D}, \overline{\lambda}_j) \cdot \partial_A F_1^j(\overline{D}, \overline{\lambda}_j) \end{aligned}$$

and using the linear independence of the vectors $\partial_A F_\rho^j(\overline{D}, \overline{\lambda}_j)$ (cf. 5)). ■

4. Corollaries

Apparently the set of all families of real symmetric $n \times n$–matrices, smoothly (C^∞) depending on a parameter $t \in \mathbb{R}$, can be identified with the set $C^\infty(\mathbb{R}, \mathbb{R}^K)$. For each natural number $r \geq 0$, the latter set can be endowed with the so–called strong C^r–topology, denoted by C_s^r (cf. Hirsch 1976). In fact, the C_s^r–topology is generated by allowing perturbations of the components of the maps and the derivatives (up to order r) of these components, which are controlled by continuous, strictly positive functions $\varepsilon : \mathbb{R} \to \mathbb{R}$ (rather than by positive constants); note that the infimum of $\varepsilon(\cdot)$ over $\mathbb{R}$ might be zero.

Now, we can formulate our Stability and Approximation Property.

STABILITY AND APPROXIMATION PROPERTY: *The subset $\mathcal{F}_n$ of $C^\infty(\mathbb{R}, \mathbb{R}^K)$, representing all one–parameter families of matrices $A(t), t \in \mathbb{R}$, such that*

$$\sigma(A(t)) = \{1, 1, \cdots, 1\}, \quad \textit{for all } t \in \mathbb{R} \quad \textit{(i.e. all eigenvalues simple)},$$

is C_s^r-dense for all $r \geq 0$ and C_s^r-open for all $r \geq 1$.

The main tool (apart from our Stratification Theorem) needed for the proof of this Stability and Approximation Property is the following version of Thom's Transversality Theorem (cf. Hirsch 1976 or Jongen et al. 1983, 1986).

THEOREM. (R. Thom) *Given any closed Whitney Regular stratified subset $\mathcal{V}$ of $\mathbb{R}^K$, and let $\pitchfork\mathcal{V}$ denote the subset of $C^\infty(\mathbb{R}, \mathbb{R}^K)$ containing all functions which are transversal to each stratum of $\mathcal{V}$. Then the set $\pitchfork\mathcal{V}$ is C_s^r-dense for all $r \geq 0$ and C_s^r-open for all $r \geq 1$.*

What we mean by "transversal", will be explained in the actual situation of our stratification $\{\mathcal{A}_\sigma\}_{\sigma \in \mathcal{S}}$. Given $f \in C^\infty(\mathbb{R}, \mathbb{R}^K)$, let $\nabla f(t)$ be the derivative of f at t. Then, we say that f is transversal to a stratum $\mathcal{A}_\sigma$ (notation: $f \pitchfork \mathcal{A}_\sigma$) if for each $t \in \mathbb{R}$ we have:

Either $f(t) \notin \mathcal{A}_\sigma$
or $f(t) \in \mathcal{A}_\sigma$ in which case $\nabla f(t)[\mathbb{R}] + T_{f(t)}\mathcal{A}_\sigma = \mathbb{R}^K$.

Note that in particular this means that, whenever $f \pitchfork \mathcal{A}_\sigma$ and codim $\mathcal{A}_\sigma \geq 2$, then the intersection $f(\mathbb{R}) \cap \mathcal{A}_\sigma$ is empty (cf. Jongen et al. 1983, 1986). Now the verification of the Stability and Approximation Property is straightforward.

PROOF. (of the Stability and Approximation Property)
For any $\sigma \neq \{1, 1, \cdots, 1\}$ we have codim $\mathcal{A}_\sigma \geq 2$. This follows from the explicit expression for codim $\mathcal{A}_\sigma$ in the Stratification Theorem. Hence, $\mathcal{F}_n$ is just the set $\pitchfork\mathcal{A}_n$. Now, Thom's theorem together with our stratification result yields the assertion. ∎

In particular, the approximation property above implies that given a family $A(t), t \in \mathbb{R}$, then by a suitable, arbitrarily small perturbation (in C_s^r) we obtain a family $\hat{A}(t)$ with only simple eigenvalues for all $t \in \mathbb{R}$. We give an instructive example.

EXAMPLE *Consider the families*

$$A(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \textit{ and the perturbation } \hat{A}(t) = \begin{pmatrix} t & \delta(t) \\ \delta(t) & -t \end{pmatrix},\ t \in \mathbb{R},$$

of matrices in A_2. Then $A(0)$ has a double eigenvalue. However, for any strictly positive function $\delta \in C^\infty(\mathbb{R}, \mathbb{R})$ the perturbation $\hat{A}(t)$ has simple eigenvalues for all $t \in \mathbb{R}$.

We proceed with analyzing the critical points for the one–parameter optimization problems $\mathcal{P}(t)$ as given in Section 1. The pair $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ is called a critical point (c.point) for $\mathcal{P}(t)$ if x fulfils the following Necessary Optimality Conditions (so-called Karush–Kuhn–Tucker (KKT) equation, see e.g. Guddat et al. 1990).

$$(KKT)\left\{\begin{array}{ll} A(t)x-\lambda x &= 0 \quad , \quad \text{some} \quad \lambda \in \mathbb{R}, \\ \frac{1}{2}x^T x - 1 &= 0. \end{array}\right.$$

The set of all c.points for the family $\mathcal{P}(t), t \in \mathbb{R}$, is called the critical set. The c.point $(\overline{x}, \overline{t})$ is called non-degenerate if at $(\overline{x}, \overline{t})$ and with the corresponding multiplier $\overline{\lambda}$ the Jacobian matrix

$$\begin{pmatrix} A(t)-\lambda I_n & -x \\ x^T & 0 \end{pmatrix}$$

of KKT w.r.t. x and λ is non-singular; otherwise $(\overline{x}, \overline{t})$ is called a degenerate c.point. One easily sees that this non-degeneracy condition at $(\overline{x}, \overline{t})$ is equivalent with:

(ND) The $(n-1) \times (n-1)$ matrix $V_{\overline{x}}^T \cdot (A(\overline{t}) - \overline{\lambda} I_n) \cdot V_{\overline{x}}$ is non-singular,

where $V_{\overline{x}}$ is any $n \times (n-1)$-matrix for which the columns span the orthogonal complement in $\mathbb{R}^n$ of $\overline{x}$.

Let $(\overline{x}, \overline{t})$ be a non-degenerate c-point for $\mathcal{P}(t)$. Then, locally around $(\overline{x}, \overline{t})$, the critical set can be smoothly parametrized by t (due to the Implicit Function Theorem). Moreover, since the non-degenerate c-points for $\mathcal{P}(t)$ are just the so-called Type 1 singularities (in the sense of Jongen, Jonker and Twilt, cf. Jongen et al. 1986) we have: any c-point, sufficiently close to $(\overline{x}, \overline{t})$, is non-degenerate and has the same quadratic Morse (co-)index as $(\overline{x}, \overline{t})$. From these observations it follows that in the case where $\mathcal{P}(t)$, $t \in \mathbb{R}$, attains merely non-degenerate c-points, the connected components of the critical set can be smoothly parametrized as $(x(t), t)$, where t traverses the whole $\mathbb{R}$; moreover, along each component of the critical set, the local structures of the optimization problems $\mathcal{P}(t)$ remain constant (up to diffeomorphisms).

In the case where $(\overline{x}, \overline{t}, \overline{\lambda})$ fulfils KKT, of course $\overline{x}$ is an eigenvector for $A(\overline{t})$ with eigenvalue $\overline{\lambda}$. By choosing the matrix $V_{\overline{x}}$ in (ND) such that the columns are the eigenvectors of $A(\overline{t})$ orthogonal to $\overline{x}$, we find that $(\overline{x}, \overline{t})$ is non-degenerate iff $\overline{\lambda}$ is a simple eigenvalue of $A(\overline{t})$. Since each $\mathcal{P}(t)$ is represented by the matrix $A(t)$, we therefore have the following direct consequence of our Stability and Approximation Property:

Generic structure of the critical set of $\mathcal{P}(t)$
The subset $\mathcal{F}_n$, representing all optimization problems $\mathcal{P}(\cdot)$ such that $\mathcal{P}(t)$ has only non-degenerate critical points for all $t \in \mathbb{R}$, is C_s^r-dense for all $r \geq 0$, and C_s^r-open for all $r \geq 1$.

For $\mathcal{P}(\cdot) \in \mathcal{F}_n$ the critical set consists of connected components which are diffeomorphic to $\mathbb{R}$; moreover along each component of the critical set the local structure of $\mathcal{P}(t)$ remains constant (up to diffeomorphisms). In particular this means that whenever $\overline{x}$ is a local minimizer for $\mathcal{P}(\overline{t})$, the component of the critical set through $(\overline{x}, \overline{t})$ consists of points $(x(t), t)$ where $x(t)$ is a local minimizer for $\mathcal{P}(t)$, all $t \in \mathbb{R}$.

5. Final remark

We consider the one–parameter families $\mathcal{R}(t)$, $t \in \mathbb{R}$, of optimization problems

$$\mathcal{R}(t) \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A(t) x + b^T(t) x \quad \text{subject to} \quad \frac{1}{2} x^T x - 1 = 0,$$

as introduced in Section 1. (Non–)degenerate critical point for $\mathcal{R}(t)$ are defined as in the case of $\mathcal{P}(t)$, see Section 4. In the present section, we will show that — in contradistinction with the case $\mathcal{P}(t)$ — the occurrence of degenerate c.points for $\mathcal{R}(t)$ cannot be excluded, even not generically.

Firstly, we note that for any t the problem $\mathcal{R}(t)$ is given by the $n \times (n+1)$–matrix $[A(t)|b(t)]$, i.e. the matrix $A(t)$ augmented by $b(t)$ as $(n+1)^{th}$ column. Hence, the set of all problems $\mathcal{R}(t)$, $t \in \mathbb{R}$, can be identified with $C^\infty(\mathbb{R}, \mathbb{R}^{K+n})$. We endow this set with the C^r_s–topology, $r \geq 0$. Next we introduce the set $\mathcal{D} \subset \mathbb{R}^{K+n}$ which is defined as follows:
$[A|b] \in \mathcal{D}$ iff the following conditions hold for some pair $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}$:

$$Ax + b - \lambda x = 0 \tag{8}$$

$$\frac{1}{2} x^T x - 1 = 0 \tag{9}$$

$$\det \begin{pmatrix} A - \lambda I_n & -x \\ x^T & 0 \end{pmatrix} = 0 \tag{10}$$

Relations (8) and (9) are the (KKT) optimality conditions defining a critical point. Equation (10) is the condition for degeneracy of this point. Thus we have:

$$\begin{array}{l} \mathcal{R}(\overline{t}) \text{ has at least one} \\ \text{degenerate c. point} \end{array} \quad \Leftrightarrow \quad [A(\overline{t})|b(\overline{t})] \in \mathcal{D}.$$

Clearly, $\mathcal{D}$ is closed. Now, the following lemma holds.

LEMMA

a) $\mathcal{D}$ admits a Whitney Regular Stratification.

b) The dimension of $\mathcal{D}$ (in $\mathbb{R}^{K+n}$) is $K + n - 1$.

(For a definition of $\dim \mathcal{D}$, cf. Gibson at al. 1976, p.18.)

PROOF.

a) It is sufficient to show that $\mathcal{D}$ is (semi–)algebraic (see for example Gibson at al. 1976, page 20). To this aim, we consider the subset $\mathcal{V} \subset \mathbb{R}^K \times \mathbb{R}^n \times \mathbb{R}$ given by the triples (A, x, λ) which fulfil the above (polynomial) Equations (9) and (10). Apparently, $\mathcal{V}$ is (semi–)algebraic. Now, let p be the polynomial map $\mathbb{R}^K \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^K \times \mathbb{R}^n$ given by $p(A, x, \lambda) = (A, -Ax + \lambda x)$. Then, by the theorem of Tarski–Seidenberg (cf. Gibson at al. 1976) it follows that $\mathcal{D}(= p(\mathcal{V}))$ is semi–algebraic.

Note that this proof does not give an explicit description of the strata into which $\mathcal{D}$ is partitioned.

b) Let be given an element $[\overline{A}|\overline{b}] \in \mathcal{D}$ and suppose that $\overline{b} \neq 0$ and that with a solution $(\overline{x}, \overline{\lambda})$ of the equations (8)–(10) we have

$$\overline{\lambda} \notin \{\lambda_j \mid \lambda_j,\ j = 1, \ldots, n, \text{ are the eigenvalues of } \overline{A}\}.$$

Now, we consider (A, b, x, λ) near such a tuple $(\overline{A}, \overline{b}, \overline{x}, \overline{\lambda})$. By assumption, (8) leads to $x = -(A - \lambda I_n)^{-1} b$ and (9) becomes

$$f(A, b, \lambda) := \frac{1}{2} b^T (A - \lambda I_n)^{-2} b - 1 = 0. \tag{11}$$

It is not difficult to see that equation (10) reduces to $x^T (A - \lambda I_n)^{-1} x = 0$. Hence, (10) coincides with the condition

$$\partial_\lambda f(A, b, \lambda) = b^T (A - \lambda I_n)^{-3} b = 0.$$

Since by assumption $\overline{b} \neq 0$, it follows that $\partial_\lambda^2 f(\overline{A}, \overline{b}, \overline{\lambda}) = 3\overline{b}^T (\overline{A} - \overline{\lambda} I_n)^{-4} \overline{b} = 3\|(\overline{A} - \overline{\lambda} I_n)^{-2} \overline{b}\|^2 \neq 0$. By applying the Implicit Function Theorem, in a neighborhood $\mathcal{U}$ of $(\overline{A}, \overline{b})$ we can solve $\partial_\lambda f = 0$ for a function $\lambda(A, b)$ such that for all $(A, b) \in \mathcal{U}$ we have

$$g(A, b) := f(A, b, \lambda(A, b)) = 0 \quad \Rightarrow \quad [A|b] \in \mathcal{D}.$$

The gradient of g at $(\overline{A}, \overline{b})$ is not the zero–vector. This follows from the relation $\partial_b g(\overline{A}, \overline{b}) = \partial_b f(\overline{A}, \overline{b}, \overline{\lambda}) + \partial_\lambda f(\overline{A}, \overline{b}, \overline{\lambda}) \cdot \partial_b \lambda(\overline{A}, \overline{b}) = \partial_b f(\overline{A}, \overline{b}, \overline{\lambda}) = \overline{b}^T (\overline{A} - \overline{\lambda} I_n)^{-2} \neq 0$ where we have used that $\overline{b} \neq 0$ and $\partial_\lambda f(\overline{A}, \overline{b}, \overline{\lambda}) = 0$.

As soon as we have shown that there is at least one tuple (A, b, x, λ) which satisfies the requirements as demanded above, the Implicit Function Theorem yields the existence of a manifold $\mathcal{D}_1$ in $\mathcal{D}$ with $\dim \mathcal{D}_1 = K + n - 1$ (namely the solution set of $g = 0$ around $(\overline{A}, \overline{b})$). In fact, consider:

$$\hat{A} = \begin{pmatrix} 0 & & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \\ & & & & 1 \end{pmatrix}, \ \hat{b} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \ \hat{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ \hat{\lambda} = \frac{1}{2}.$$

The tuple $(\hat{A}, \hat{b}, \hat{x}, \hat{\lambda})$ fulfils all assumptions on $(\overline{A}, \overline{b}, \overline{\lambda})$. In addition, for the pair $(\hat{A}, \hat{b})$ we have that for all (and not for only one) solutions (x, λ) of Equations (8)–(10) it follows:

$$\lambda \notin \{\hat{\lambda}_j \mid \hat{\lambda}_j,\ j = 1, \ldots, n, \text{ are the eigenvalues of } \hat{A}\} \tag{12}$$

By a result in Gibson at al. 1976, p.19, the so-called regular points lie dense in $\mathcal{D}$. Thus, by a continuity argument and using (12), as well as the fact that x lies in a compact set, it follows, that in any neighborhood of $(\hat{A}, \hat{b})$ there is a regular point $(\overline{A}, \overline{b}) \in \mathcal{D}$ as above. Now, a moment of reflection shows that $\dim \mathcal{D} \geq K+n-1$. One can prove, that in any neighborhood of $(\overline{A}, \overline{b})$ there exist elements $(A, b) \notin \mathcal{D}$. Consequently we can conclude that $\dim \mathcal{D} = K + n - 1$. Together with a) this implies, that the stratum of maximal dimension in $\mathcal{D}$ has codimension 1. ∎

As for the one-parameter problems $\mathcal{P}(\cdot)$ in Section 4, from Thom's Theorem we get a genericity result for the problems $\mathcal{R}(\cdot)$.

COROLLARY *The subset* $\overline{\pitchfork}\mathcal{D}$ *of* $C^\infty(\mathbb{R}, \mathbb{R}^{K+n})$ *is* C^r_s*-dense for all* $r \geq 0$ *and* C^r_s*-open for all* $r \geq 1$.

The relevance of this corollary relies on the fact that for any $[A(\cdot)|b(\cdot)] \in \overline{\pitchfork}\mathcal{D}$ we have (cf. Hirsch 1976 or Jongen et al. 1983, 1986):

Either $[A(t)|b(t)] \notin \mathcal{D}$ for all $t \in \mathbb{R}$,
or $[A(t)|b(t)] \in \mathcal{D}$ iff t in some closed discrete subset of $\mathbb{R}$; for such t-values, $[A(t)|b(t)]$ hits $\mathcal{D}$ in the stratum of dimension $K + n - 1$.

In case of the first alternative, the corresponding problems $\mathcal{P}(t)$ do not exhibit non-degenerate c-points. In case of the second alternative, the problems $\mathcal{P}(t)$ do attain degenerate c-points for isolated t-values. One easily shows that there always exists a $[\tilde{A}(\cdot)|\tilde{b}(\cdot)] \in \overline{\pitchfork}\mathcal{D}$ for which the second alternative holds, and moreover that this remains the case under small C^r_s-perturbations of $[\tilde{A}(\cdot)|\tilde{b}(\cdot)]$ with $r \geq 1$..

We end up with a remark on related literature.

REMARK *The property on the structure of the critical set in Section 4 and the Corollary above are to be seen in connection with a result due to Jongen, Jonker and Twilt 1986, which states (among others):*
"Generically", the (generalized) critical points for smooth, one-parameter optimization problems in $\mathbb{R}^n$ *under finitely many (in-)equality constraints classify into five different types (among them non-degenerate critial points); in case of only equality constraints there are three types (among them non-degenerate critical points)."*

Now it is natural to ask for similar classifications in cases where we restrict ourselves to certain subclasses of the one-parameter problems mentioned *above. Accounts to this subject have been given by Pateva 1991 concerning Linear Optimization problems, and by Henn, Jonker and Twilt 1986, with respect to Quadratic Optimization problems.*

Note that in these latter cases the KKT–equations are linear in x, whereas this is not true for the problems $\mathcal{P}(t)$ and $\mathcal{R}(t)$ as analysed in the present paper.

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