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On a class of insensitive control problems

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A class of insensitive control problems for the state equation involving a non-linear operator with a small parameter is considered. The conditions for the problem solvability are discussed.

1. Introduction

We consider the problem of an insensitive control for the equation of state involving a non-linear operator with a small parameter. The notion of an insensitive control was firstly introduced by J.-L. Lions (see Lions 1989, Lions 1988, Lions 1990). Some classes of insensitive control problems were studied in Agoshkov 1993, Agoshkov, Ipatova 1993. We follow the approach in Agoshkov 1993 and extend the results of Agoshkov 1993 to the problem under study using some results obtained in Agoshkov, Marchuk 1993 for data assimilation problems. The theorem on the problem solvability is proved.

2. Statement of the problem. Basic assumptions

Let H and X be real separable Hilbert spaces such that $X \hookrightarrow H$; H^*, X^* are adjoints of H, X. Introduce into consideration $L_2(0,T;\ H)$, $Y = L_2(0,T;\ X)$, $L_2(0,T;\ X^*)$ — spaces of abstract functions $f(t)$ with values in H, X, X^*, respectively; and the space

$$W = W(0,T) = \{f \in L_2(0,T;\ X) : \frac{df}{dt} \in L_2(0,T;\ X^*)\}\ ,$$

$$\|f\|_W = (\|\frac{df}{dt}\|^2_{L_2(0,T;\ X^*)} + \|f\|^2_{L_2(0,T;\ X)})^{1/2}\ .$$

By $C^0([0,T];H)$ we denote the Banach space of continuous functions $[0,T] \mapsto H$ with the norm

$$\|f\|_{C^0([0,T];H)} = \max_{t\in[0,T]} \|f\|_H \ .$$

We'll assume that

$$H \equiv H^* \ , \qquad X^* \equiv X^{-1} \ , \qquad (\cdot,\cdot)_{L_2(0,T;H)} = (\cdot,\cdot) \ ,$$

$$L_2(0,T;H) \equiv L_2^*(0,T;H) \equiv L_2(0,T;H^*) \ .$$

Let $a(t;\varphi,\psi)$ be a bilinear form defined for all $t \in [0,T]$ and for any $\varphi,\psi \in X$, and satisfying the inequalities :

$$|a(t;\varphi,\psi)| \le c_1\|\varphi\|_X \ \|\psi\|_X \ , \quad c_1 = const > 0 \ , \tag{2.1}$$

$$c_2\|\varphi\|_X^2 \le a(t;\varphi,\psi) \ , \quad c_2 = const > 0 \ , \quad \forall\, t \in [0,T] \ , \quad \forall\, \varphi,\psi \in X \ . \tag{2.2}$$

By $A(t) \in \mathcal{L}(Y,Y^*)$ we denote the operator generated by this form :

$$(A(t)\varphi,\psi)_H = a(t;\varphi,\psi) \qquad \forall\, \varphi,\psi \in X \ . \tag{2.3}$$

Consider the following evolutional problem

$$\begin{cases} \frac{d\varphi}{dt} + A(t)\varphi + \varepsilon F(\varphi) = f(t) \ , \quad t \in (0,T) \\ \varphi(0) = V \end{cases} \tag{2.4}$$

where $f \in Y^*$, $V \in H$, $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$ is a small parameter, $\varepsilon_0 \in \mathbb{R}^1$, $F(\varphi)$ is an analytic operator, $F : W \mapsto Y^*$.

Consider the functional of $\varphi \in W$ of the form :

$$S(\varphi,V) = \frac{\alpha}{2}\|\varphi\|_{Y^k}^2 + \frac{\beta}{2}\|V\|_H^2 + \frac{1}{2}\sum_{i=1}^{N}\int_0^T \alpha_i(\varphi_i - c_i)^2 dt + (\varphi,g_0), \tag{2.5}$$

where

$$k = 0,1 \ , \quad Y^0 \equiv L_2(0,T;H) \ , \quad Y^1 \equiv Y = L_2(0,T;X) \ ,$$
$$\alpha,\ \beta,\ \alpha_i = const \ge 0 \ , \quad c_i = c_i(t) \in L_2(0,T) \ ; \quad \varphi_i = (\varphi,p_i)_H \ ,$$
$$p_i \in L_\infty(0,T;X^*) \ , \quad i = \overline{1,N} \ , \quad n \in \mathbb{N} \ , \quad g_0 \in Y^* \ , \quad V \in H \ .$$

Let us set ourselves the problem of an insensitive control (Lions 1989, Lions 1988, Lions 1990, Agoshkov 1993) :

for given $f, g_0 \in Y^*$ find a pair of functions $(\varphi,V) \in W \times H$ such that (2.4) is satisfied and

$$\frac{dS}{d\tau}(\tilde{\varphi}, V + \tau V_1)_{|\tau=0} = 0 \qquad \forall\, V_1 \in H \ , \tag{2.6}$$

where $\tilde{\varphi}$ is the solution of (2.4) for $\tilde{\varphi}(0) = V + \tau V_1$, $\tau \in \mathbb{R}^1$.

By $F'(\varphi) \in \mathcal{L}(W, Y^*)$ we denote the Frechet derivative of the operator F at the point $\varphi \in W$, and $F'^*(\varphi) \in \mathcal{L}(Y, W^*)$ is the operator adjoint to $F'(\varphi)$. Later, we'll assume the following restrictions on the operator F to be imposed (see Agoshkov, Marchuk 1993) :

$$\begin{aligned} &\|F(\varphi)\|_{Y^*} \le c_1 , \quad \|F'(\varphi)h\|_{Y^*} \le c_2 \|h\|_W , \\ &\|F'^*(\varphi)\, q\|_{W^*} \le c_3 \|q\|_Y , \quad \forall\, h \in W , \ q \in Y , \\ &\forall\, \varphi \in \bar{S}(\varphi_0, R_0) = \{\varphi \in W : \|\varphi - \varphi_0\|_W \le R_0\} , \end{aligned} \tag{2.7}$$

where φ_0 is the solution of (2.4) for $\varepsilon = 0$, $c_i = c_i(R_0) = const > 0$, $i = 1, 2, 3$, $R_0 > 0$.

3. Equivalent formulation of the problem. The control equation

Assuming $\tilde{\varphi}$ be sufficiently regular in τ we can write the derivative $\frac{dS}{d\tau}$ from (2.6) in the form

$$\begin{aligned} \tfrac{dS}{d\tau}(\tilde{\varphi}, V + \tau V_1)_{|\tau=0} \ = \ & (V_1, \beta V)_H + (\varphi_1, \alpha \Lambda_Y^k \varphi + \\ & + \textstyle\sum_{i=1}^N \alpha_i [(\varphi, p_i)_H - c_i]\, p_i + g_0), \end{aligned} \tag{3.1}$$

where φ is the solution of (2.4), and φ_1 is the solution of the following problem :

$$\begin{cases} \frac{d\varphi_1}{dt} + A(t)\varphi_1 + \varepsilon F'(\varphi)\varphi_1 = 0 , \quad t \in (0, T) \\ \varphi_1(0) = V_1 . \end{cases}$$

Here Λ_Y is the canonical isomorphism from Y into Y^* defined by the formula $(\Lambda_Y \varphi, \psi) = (\varphi, \psi)_Y$, $k = 0, 1$ is its power.

Thus, the insensitive control problem can be reformulated as follows :

for given $f, g_0 \in Y^*$ find the functions (φ, φ_1, V) such that

$$\begin{cases} \frac{d\varphi}{dt} + A(t)\varphi + \varepsilon F(\varphi) = f , \quad t \in (0, T) \\ \varphi(0) = V , \end{cases} \tag{3.2}$$

$$(V_1, \beta V)_H + (\varphi_1, \alpha \Lambda_Y^k \varphi + \sum_{i=1}^N \alpha_i [(\varphi, p_i)_H - c_i] p_i + g_0) = 0 \ \ \forall V_1 \in H, \tag{3.3}$$

$$\begin{cases} \frac{d\varphi_1}{dt} + A(t)\varphi_1 + \varepsilon F'(\varphi)\varphi_1 = 0 , \quad t \in (0, T) \\ \varphi_1(0) = V_1 . \end{cases} \tag{3.4}$$

Denote by P the expression

$$P = \alpha \Lambda_Y^k \varphi + \sum_{i=1}^N \alpha_i [(\varphi, p_i)_H - c_i]\, p_i + g_0 , \tag{3.5}$$

and consider the adjoint problem :

$$\begin{cases} -\frac{dq}{dt} + A^*(t)\, q + \varepsilon F'^*(\varphi)\, q = P\,, \quad t \in (0,T) \\ q(T) = 0\,. \end{cases} \tag{3.6}$$

Due to the conjugacy relation, Marchuk, Agoshkov, Shutyaev 1991

$$(\varphi_1, P) = (V_1, q(0))_H\,, \tag{3.7}$$

(2.3) gives

$$(V_1, \beta V)_H + (V_1, q(0))_H = 0 \quad \forall\, V_1 \in H\,,$$

or

$$\beta V + q(0) = 0. \tag{3.8}$$

Then, the problem (3.2)–(3.4) can be written in the form :

find the functions φ, q, V such that

$$\begin{cases} \frac{d\varphi}{dt} + A(t)\varphi + \varepsilon F(\varphi) = f\,, \quad t \in (0,T) \\ \varphi(0) = V\,, \end{cases} \tag{3.9}$$

$$\begin{cases} -\frac{dq}{dt} + A^*(t)\, q + \varepsilon F'^*(\varphi)\, q = K\varphi + g\,, \quad t \in (0,T) \\ q(T) = 0\,, \end{cases} \tag{3.10}$$

$$\beta V + q(0) = 0\,, \tag{3.11}$$

where

$$K\varphi = \alpha \Lambda_Y^k \varphi + K_0\, \varphi\,, \qquad K_0\, \varphi = \sum_{i=1}^{N} \alpha_i (\varphi_i, p_i)_H\, p_i\,,$$

$$g = g_0 - \sum_{i=1}^{N} \alpha_i\, c_i\, p_i\,.$$

To study the solvability of the system (3.9)–(3.11) and to formulate a numerical algorithm, following the approach in Agoshkov, Marchuk 1993, we obtain an equation for the control V.

If the problems (3.9), (3.10) are solvable their solutions can be represented as

$$\varphi = G_1(f - \varepsilon F(\varphi)) + G_0 V\,, \tag{3.12}$$

$$q = G_1^{(\tau)}(g + K\varphi) - \varepsilon\, G_1^{(\tau)} F'^*(\varphi)\, q\,, \tag{3.13}$$

where $G_1 \in \mathcal{L}(Y^*, W)$, $G_0 \in \mathcal{L}(H, W)$, $G_1^{(\tau)} \in \mathcal{L}(Y^*, W)$.

Then for $\beta V + q(0)$ we have

$$\beta V + q(0) = MV + F_0 - \varepsilon F_1(\varphi, q)\ , \tag{3.14}$$

where

$$\begin{aligned} M &= T_0\, G_1^{(\tau)} K G_0 + \beta I\ , \quad I\varphi = \varphi\ , \quad T_0\varphi = \varphi(0)\ , \\ F_0 &= T_0\, G_1^{(\tau)}(g + KG_1 f)\ , \\ F_1(\varphi, q) &= T_0\, G_1^{(\tau)}(F'^{*}(\varphi)\, q + KG_1 F(\varphi))\ . \end{aligned}$$

From (3.14) we come to the control equation :

$$MV + F_0 - \varepsilon F_1 = 0\ . \tag{3.15}$$

To investigate the solvability of the problem (3.9)–(3.11), as was done in Agoshkov, Marchuk 1993, we consider first the unperturbed problem (3.15) for $\varepsilon = 0$, and use then the small parameter method.

4. Solvability of the insensitive control problem

First we consider the unperturbated problem (3.15) for $\varepsilon = 0$:

$$MV_0 + F_0 = 0. \tag{4.1}$$

This equation is equivalent to the following system :

$$\begin{cases} \frac{d\varphi_0}{dt} + A(t)\varphi_0 = f\ , \quad t \in (0, T) \\ \varphi_0(0) = V_0 \end{cases} \tag{4.2}$$

$$\begin{cases} -\frac{dq_0}{dt} + A^{*}(t)\, q_0 = K\varphi_0 + g\ , \quad t \in (0, T) \\ q_0(T) = 0 \end{cases} \tag{4.3}$$

$$\beta V_0 + q_0(0) = 0. \tag{4.4}$$

The properties of the operator M for $\beta > 0$ have been studied in Agoshkov, Marchuk 1993. We'll consider the case when β may be equal to zero.

Let $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$. The following assertion holds true :

LEMMA 4.1 *The operator M operates in H with the definition domain $\mathcal{D}(M) = H$, it is continuous, self-adjoint and positive operator :*

$$(M\rho, \rho)_H > 0 \quad \forall\, \rho \in H,\ \rho \neq 0. \tag{4.5}$$

PROOF. Let $\rho \in H$ and $\varphi = G_0\, \rho$. Then

$$M\rho = T_0 G_1^{(\tau)} K\varphi + \beta\rho\ .$$

Since Lions, Magenes 1968

$$\|\varphi\|_W \leq c\,\|\rho\|_H\,, \quad c = const > 0\,,$$

and similarly for $q = G_1^{(\tau)} K\varphi$

$$\|q\|_W \leq c\,\|K\varphi\|_{Y^*}\,, \quad c = const > 0\,,$$

and by definition of K,

$$\|K\varphi\|_{Y^*} \leq c\,(\alpha + \sum_{i=1}^{N} \alpha_i \sup_{t\in[0,T]} \|p_i\|_{X^*}^2)\|\varphi\|_Y \leq c\|\varphi\|_W\,,$$

then

$$\|q\|_W \leq c\,\|\rho\|_H\,, \quad c = const > 0$$

Due to the continuous imbedding of W into $C^0([0,T];H)$, Lions, Magenes 1968,

$$\|T_0\, q\|_H = \|q(0)\|_H \leq c\,\|q\|_W\,,$$

therefore for $M\rho = T_0\, q + \beta\rho$ we have

$$\|M\rho\|_H \leq \|q(0)\|_H + \beta\|\rho\|_H \leq c\,\|\rho\|_H\,,$$

that yields the continuity of the operator M. The selfadjointness of M is obvious, and the positiveness follows from the relations :

$$\begin{aligned}(M\rho,\rho)_H &= (T_0 G_1^{(\tau)} K G_0\,\rho,\rho)_H + \beta(\rho,\rho)_H = \\ &= (K\varphi,\varphi) + \beta(\rho,\rho)_H = \\ &= \alpha(\Lambda_Y^k\varphi,\varphi) + \textstyle\sum_{i=1}^N \int_0^T \alpha_i(\varphi,p_i)_H^2\,dt + \beta(\rho,\rho)_H \geq \\ &\geq \alpha(\Lambda_Y^k\varphi,\varphi) + \beta(\rho,\rho)_H\,.\end{aligned} \tag{4.6}$$

Thus, for $\alpha + \beta > 0$ we arrive at the inequality (4.5). Lemma is proven.

In the case $\alpha \geq 0$, $\beta > 0$ from the estimate (4.6) we deduce the positive definiteness of the operator M. This property has been proved earlier in Agoshkov, Marchuk 1993.

If $\beta = 0$ then in some cases the operator M still remains to be positive definite.

The following lemma is true.

LEMMA 4.2 *Let $k = 1$, $\alpha > 0$, $\beta \geq 0$ then the operator $M : H \mapsto H$ is positive definite :*

$$(M\rho,\rho)_H \geq c\,(\rho,\rho)_H \quad \forall\,\rho \in H\,, \quad c = const > 0. \tag{4.7}$$

PROOF. The inequality (4.6) for $k = 1$ gives

$$(M\rho, \rho)_H \geq \alpha \|\varphi\|_Y^2 + \beta \|\rho\|_H^2 \ . \tag{4.8}$$

For $\varphi = G_0 \rho$ the following estimate holds true :

$$\|\varphi\|_Y \geq c\|\rho\|_H \ , \quad c = const > 0. \tag{4.9}$$

Indeed, since φ satisfies the problem

$$\begin{cases} \frac{d\varphi}{dt} + A\varphi = 0 \ , & t \in (0,T) \\ \varphi(0) = \rho \end{cases}$$

then

$$\int_0^T (A\varphi, \varphi)_H \, dt + \frac{1}{2}\|\varphi(T)\|_H^2 = \frac{1}{2}\|\rho\|_H^2 \ .$$

Hence, taking into account the continuity of the operator $A^* : Y \mapsto Y^*$ (see (2.1)–(2.2)), we obtain

$$\|\rho\|_H^2 \leq 2c_1\|\varphi\|_Y^2 + \|\varphi(T)\|_H^2 \ . \tag{4.10}$$

Due to the continuous imbedding of W into $C^0([0,T]; H)$,

$$\|\varphi(T)\|_H \leq c \, \|\varphi\|_W \ , \quad c = const > 0 \ , \tag{4.11}$$

and the equation $\frac{d\varphi}{dt} = -A\varphi$, we deduce

$$\|\varphi\|_W^2 = \|\varphi\|_Y^2 + \|\frac{d\varphi}{dt}\|_{Y^*}^2 = \|\varphi\|_Y^2 + \|A\varphi\|_{Y^*}^2 \leq \|\varphi\|_Y^2 + c_1^2\|\varphi\|_Y^2 \ . \tag{4.12}$$

From (4.10)–(4.11) we come to the estimate (4.9). And (4.8), (4.9) yield the inequality

$$(M\rho, \rho)_H \geq (\alpha c + \beta)\|\rho\|_H^2 \ , \quad c = const > 0 \ ,$$

which assures the positive definiteness of M in the case $\alpha > 0$, $\beta \geq 0$. Lemma is proven.

It follows from lemmas 4.1, 4.2 that for all cases when $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$ the following statements are valid :

COROLLARY 4.1 *The equation $MV_0 + F_0 = 0$ is uniquely and densely solvable.*

COROLLARY 4.2 *The operator M^{-1} does exist (from H into H) with the definition domain $\mathcal{D}(M^{-1})$ dense in H.*

COROLLARY 4.3 *If $\alpha \geq 0$, $\beta > 0$ or $k = 1$, $\alpha > 0$, $\beta \geq 0$ then the equation $MV_0 + F_0 = 0$ is correctly and everywhere solvable.*

Using the corollary 4.3 we deduce, similarly to Agoshkov, Marchuk 1993, the following theorem :

THEOREM 4.1 *For $\alpha \geq 0$, $\beta > 0$ or $k = 1$, $\alpha > 0$, $\beta \geq 0$ the problem (4.2)-(4.4) has a unique solution $(\varphi_0, q_0, V_0) \in W \times W \times H$.*

If $k = 0$, $\alpha > 0$, $\beta \geq 0$ then the operator M^{-1} can be unbounded. Let us give an example of a rather wide class of specific cases when M^{-1} must necessarily be unbounded. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piece-wise smooth boundary, $2 \leq n \leq 4$, and

$$H = L_2(\Omega), \ \ X = \overset{0}{W}{}_2^1(\Omega), \ \ Y = L_2(0,T; \overset{0}{W}{}_2^1(\Omega)), \ \ Y^0 = L_2((0,T) \times \Omega).$$

Define the operator $A \in \mathcal{L}(Y, Y^*)$ by the bilinear form :

$$(A\varphi, \psi)_H = a(t; \varphi, \psi) \quad \forall\, \varphi, \psi \in \overset{0}{W}{}_2^1(\Omega)$$

where

$$a(t;\varphi,\psi) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + \sum_{i=1}^n a_i \frac{\partial\varphi}{\partial x_i}\psi + a\varphi\psi\Big)\, dx \ ,$$

$$a(t,x),\ a_{ij}(t,x),\ a_i(t,x) \in L_\infty((0,T)\times\Omega), \quad i,j = \overline{1,n}, \quad x \in \Omega,$$

$$a(t,x) \geq 0, \ \ \sum_{i=1}^n \frac{\partial a_i}{\partial x_i} = 0, \ \ \sum_{i,j=1}^n a_{ij}\lambda_i\lambda_j \geq \gamma \sum_{i=1}^n \lambda_i^2 \ \ \forall\, \lambda_i \in \mathbb{R}^1, \ \ \gamma = const > 0.$$

Let $\beta = 0$, $k = 0$, $\alpha_i = 0$, $i = \overline{1,N}$, $\alpha > 0$, $g_0 \in Y^0$.

LEMMA 4.3 *The operator $M : H \mapsto H$ is completely continuous.*

PROOF. Let us prove that the operator M maps a bounded set in H into a compact set. Consider the functions $\rho \in H$ such that $||\rho||_H \leq c_0$, $c_0 = const > 0$. Let $\varphi = G_0\, \rho$, then $M\rho = T_0 G_1^{(\tau)}\varphi$. For φ the following estimate holds true :

$$||\varphi||_W \leq c\, ||\rho||_H \ , \quad c = const > 0.$$

The function $q = G_1^{(\tau)}\varphi$ satisfies the problem

$$\begin{cases} -\frac{dq}{dt} + A^* q = \alpha\varphi + g_0 \ , & t \in (0,T) \\ q(T) = 0. & \end{cases}$$

Since $g_0 \in Y^0 = L_2((0,T)\times\Omega)$, $\varphi \in W \subset Y^0$, the last problem has a rather regular solution q (see Ladyzhenskaya, Uraltseva, Solonnikov 1967) such that

$$||q(0)||_X \leq c\, ||\varphi||_{Y^0} + c_1 ||g_0||_{Y^0} \leq c\, ||\varphi||_W + c_1 ||g_0||_{Y^0} \ .$$

Thus, the functions $M\rho = T_0 q = q(0)$ are uniformly bounded :

$$||M\rho||_X \leq c||\rho||_H + c_1||g_0||_{Y^0} \leq c_0\, c + c_1||g_0||_{Y^0}\ .$$

Therefore, the operator M maps a bounded set from H into a set bounded in the norm of X. The latter set in its turn is compact in H, Michailov 1976. Lemma is proven.

COROLLARY 4.4 *The operator $M^{-1} : H \mapsto H$ is unbounded.*

Hence, using the corollary 4.2 we deduce

COROLLARY 4.5 *The value $\lambda = 0$ is a point of the continuous spectrum of the operator M.*

So in the example considered the equation $MV_0 + F_0 = 0$ is uniquely solvable but it is not correctly solvable. Moreover, this equation is not solvable for all functions $F_0 \in H$, it has a solution only for functions F_0 from a dense set $\mathcal{D}(M^{-1})$ in H. With the equivalent formulation (4.2)–(4.4) we can obtain necessary conditions for the problem solvability.

The original nonlinear system (3.9)–(3.11), as we have seen, is equivalent to the equation

$$MV + F_0 - \varepsilon F_1(\varphi, q) = 0. \tag{4.13}$$

Using the arguments of Agoshkov, Marchuk 1993 and the corollary 4.3 we come to the following statement.

THEOREM 4.2 *Let $\beta > 0$, $\alpha \geq 0$ or $\beta \geq 0$, $\alpha > 0$, $k = 1$; $f, g_0 \in Y^*$. Then there exists $\varepsilon_0 > 0$ such that $\forall\ \varepsilon \in [-\varepsilon_0, \varepsilon_0]$ the problem (3.9)–(3.11) has a unique solution $(\varphi, q, V) \in W \times W \times H$ analytic in ε :*

$$\varphi = \varphi_0 + \sum_{i=1}^{\infty} \varepsilon^i \varphi_i\ , \quad q = q_0 + \sum_{i=1}^{\infty} \varepsilon^i q_i\ , \quad V = V_0 + \sum_{i=1}^{\infty} \varepsilon^i V_i\ ,$$

where $\{\varphi_i\}$, $\{q_i\}$, $\{V_i\}$ can be computed by the small parameter method.

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