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Well posedness of optimal control problems

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A notion of well posedness for optimal control problems requires existence and uniqueness of the optimal control, and strong convergence of every asymptotically minimizing sequence of control laws. Using a unifying abstract approach, well posedness is shown to be intimately related to the differentiability properties of the value function. Results of Fleming are thereby extended.

Introduction

We consider the global optimization problem (X, J), to minimize the extended real-valued function

$$J : X \to (-\infty, \infty\,]$$

over the given convergence space X. In the applications to optimal control problems we shall consider, X is a subset of a given real normed space equipped with the strong convergence.

In order to deal with a suitable notion of well posedness of (X, J), we shall embed the given problem in a smoothly parametrized family $[X, I(\cdot, p)]$ of minimization problems. Here p is a parameter belonging to a given Banach space, as well as the parameter value $p^\star$ to which (X, J) corresponds, i.e.

$$I(u, p^\star) = J(u) \text{ for all } u.$$

Thus we consider small perturbations of (X, J) corresponding to the parameters p close to $p^\star$. The definition of well posedness of (X, J) requires existence and

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uniqueness of the minimizer $u^\star$ and, for any sequence $p_n \to p^\star$, convergence to $u^\star$ of every asymptotically minimizing sequence u_n corresponding to p_n, i.e. every sequence $u_n \in X$ such that

$$I(u_n, p_n) - V(p_n) \to 0 \text{ as } n \to +\infty,$$

where V denotes the value function

$$V(p) = inf\{I(u,p) : u \in X\}.$$

This definition requires Tikhonov well posedness of (X, J) (see Tikhonov 1966) and a form of Hadamard well posedness, since we impose the stable behavior of the unique minimizer $u^\star$ under small perturbations of $p^\star$. In a sense, $u^\star$ is a continuous function of p at $p^\star$. Thus, well posedness of (X, J) is partly intrinsic to the given optimization problem, and partly depending on the choice of the embedding.

The main purpose of this paper is to apply a characterization of the above notion of well posedness, obtained in Zolezzi, Well posedness ..., section 2, to deterministic optimal control problems.

Let us describe (in a informal way) a model application to optimal control problems.

Consider the integral performance

$$\int_0^T f[y(s), u(s)]\, ds \tag{1}$$

to be minimized subject to the state equation

$$\dot{y}(s) = g[y(s), u(s)] \text{ a.e. in } [0, T] \tag{2}$$

with initial condition

$$y(0) = x^\star \tag{3}$$

and control constraint

$$u(s) \in U \text{ a.e. in } [0, T]. \tag{4}$$

Here the state variable $y \in R^N$ and the control variable $u \in R^M$.

We embed the above optimal control problem following a modification of the dynamic programming method (see Fleming, Rishel 1975). We replace the initial condition by

$$y(0) = p, \ p \text{ close to } x^\star. \tag{5}$$

The significant parameter is now p, while $p^* = x^*$ defines the original (unperturbed) problem. Here X is the set of the admissible (open loop) control laws equipped with the strong convergence of L^1. If

$$y(u,p)$$

denotes the state associated to the control u and the parameter p, i.e. the solution of the initial value problem

$$\dot{y} = g(y,u) \text{ a.e. in } [0,T], \quad y(0) = p$$

then

$$I(u,p) = \int_0^T f[y(u,p),u]\, ds$$

fits the above abstract model.

Roughly speaking, for a given embedding of problem (1),...,(4), well posedness means existence and uniqueness of the optimal control, and strong convergence to it in $L^1([0,T])$ of every asymptotically minimizing sequence corresponding to convergent perturbations of the relevant parameter (chosen by the embedding). Hence well posedness implies a robust behavior of the optimal control under small perturbations, in particular automatic convergence of every numerical optimization method which constructs minimizing sequences, even in presence of small changes of problem's data.

In section 1 we summarize the main abstract results of Zolezzi, Well posedness, ..., which exploit the differentiability properties of the value function.

In section 2 we apply the abstract results of section 1 to obtain well posedness criteria for optimal control problems monitored by ordinary differential equations with unconstrained terminal point. The nonsmooth behavior of the value function is shown to be related to ill posedness. These results extend known criteria for Tikhonov well posedness of free end point problems of optimal control in Fleming, Rishel 1975 and Fleming, Soner 1993.

In section 3 we compare the well posedness criteria, obtained in this paper, with some known results. Moreover we present some examples.

Links between Tikhonov well posedness and differentiability of the value function are known in convex optimization (Asplund, Rockafellar 1969), best approximation problems (Fitzpatrick 1980) and problems in the calculus of variations (Fleming 1969). As shown in Zolezzi, Well posedness ..., the results for free problems, summarized in section 1, can be considered as a common extension of them. Well posedness in the calculus of variations is obtained in Zolezzi, Well posedness, ..., using the same abstract approach.

The approach considered here can be applied to various embeddings of the given optimal control problem, as those listed in section 3 : not only perturbations of the initial point (as described above), but also perturbations of the dynamics (see Clarke 1986 and Clarke, Loewen 1986) and time delays (see Clarke, Wolenski 1991). Of course, parametric problems of general type, as some of those studied in Malanowski 1987, fit the approach presented here.

A short survey of some results related to this paper is contained in Zolezzi, Well posed ... and Zolezzi 1991. For a survey of well posedness in scalar optimization see Dontchev, Zolezzi 1993.

1. Abstract results

Throughout this section :

X is a fixed convergence space (as defined e.g. in Kuratowski 1958);
P is a given real Banach space; $p^\star$ is a fixed point of P ;
L is a closed ball in P of center $p^\star$ and positive radius ;

$$J : X \to (-\infty, +\infty\,]\,,\quad I : X \times L \to (-\infty, +\infty\,]$$

are proper extended real-valued functions.

For every $p \in L$ we consider the problem (p), which is denoted by

$$[\,X,\ I(\cdot, p)\,],$$

to minimize (globally) $I(u,p)$ subject to $u \in X$, assuming that

$$I(u, p^\star) = J(u) \text{ for all } u \in X.$$

The (optimal) value function is defined by

$$V(p) = \inf \{\, I(u,p) : u \in X\}\,,\quad p \in L.$$

The problem (X, J), to minimize $J(x)$ subject to $x \in X$, is called here well posed (with respect to the embedding defined by I) iff

$V(p) > -\infty$ for every $p \in L$ and

there exists a unique minimizer (6)

$u^\star = \arg\min\,(X, J)$;

for every sequence $p_n \to p^\star$, every sequence $u_n \in X$ such that (7)

$I(u_n, p_n) - V(p_n) \to 0$ as $n \to +\infty$

obeys $u_n \to u^\star$ in X.

Sequences u_n as in (7) will be referred to as asymptotically minimizing, corresponding to the sequence p_n. For simple notation we write

arg min (p) instead of arg min $[X, I(\cdot, p)]$

(possibly empty).

REMARKS.

(a) If problem $(p^\star)$, i.e. (X, J), is well posed, then it is Tikhonov well posed (simply take $p_n = p^\star$ in (7)).

(b) Let the convergence in X be induced by a topology. Write
$u \in \varepsilon - arg\ min(p)$ iff $V(p) > -\infty$, $u \in X$ and $I(u,p) \leq V(p) + \varepsilon$.
Then, as easily checked, (X, J) is well posed iff $V(p) > -\infty$ for all $p \in L$, (6) holds and the multifunction
$(\varepsilon,\ p) \rightrightarrows \varepsilon - arg\ min(p)$
is upper semicontinuous at $(0, p^\star)$. Thus, the definition of well posedness includes a form of continuous dependence of $u^\star$ on p.

We shall work with mildly smooth embeddings, according to the following assumption :

for every $u \in X$, $I(u, \cdot)$ is Gâteaux differentiable on int L
with continuous gradient at arg min $(p^\star) \times \{p^\star\}$. (8)

The following two theorems, proved in Zolezzi, Well posedness ..., in a slightly more general form, provide necessary and sufficient conditions for well posedness.

THEOREM 1 *(X, J) is well posed if condition (8) is fulfilled and the following assumptions hold :*

V is finite and upper semicontinuous on L, Gâteaux differentiable
on int$L$ with continuous gradient at $p^\star$; (9)

I is lower semicontinuous on $X \times L$, and $\nabla I(\cdot, p^\star)$
is one–to–one on arg min$(p^\star)$; (10)

for any sequence $p_n \to p^\star$, every asymptotically minimizing sequence u_n,
such that $\nabla I(u_n, p_n)$ converges strongly in $p^\star$, (11)
has a convergent subsequence.

THEOREM 2 *V is Fréchet differentiable at $p^\star$ provided that condition (8) holds and (X, J) is well posed.*

2. Optimal control

We apply theorems 1 and 2 to the following optimal control problem. Minimize (1) subject to the state equation (2) and the constraints (3) and (4).

Here $f = f(x,u) : R^N \times R^M \to R$ and $g = g(x,u) : R^N \times R^M \to R^N$ are given functions, moreover $x^\star \in R^N$ and $T > 0$ are fixed; U is a nonempty subset of R^M. Every state variable y is absolutely continuous on $[0,T]$.

Throughout this section we assume the following conditions :

$$\begin{aligned} &f,\ g,\ f_x,\ g_x \text{ are continuous in } R^N \times U;\ U \text{ is compact :}\\ &\text{there exist constants } a,\ b \text{ such that}\\ &\quad | g(x,u) | \le a + b \mid x \mid \text{ everywhere.} \end{aligned} \tag{12}$$

The space X of admissible controls is defined by

$$u \in X \text{ iff } u \in L^\infty([0,T]) \text{ and } (4) \text{ is fulfilled ;}$$

X is equipped with the strong convergence of $L^1([0,T])$.

For any optimal trajectory $(u^\star, y^\star)$ we consider the Hamiltonian function

$$H(s,u) = q(s)'\, g(y^\star(s),u) - f(y^\star(s),u)$$

where the corresponding adjoint state q is defined by

$$\dot{q}(s) = -g_x[y^\star(s),u^\star(s)]'q(s) + f_x[y^\star(s),u^\star(s)],\ q(T) = 0.$$

Among the several embeddings of problem (1), (2), (3), (4) to which the abstact results of section 1 can be applied, we select a modification of the one associated with the dynamic programming approach (as described in the introduction), in order to characterize the corresponding notion of well posedness.

We consider the embedding defined by (1), (2), (4) and

$$y(0) = p \tag{13}$$

for points $p \in R^N$ close to $x^\star = p^\star$. The initial time $t = 0$ is held fixed, only the initial state is perturbed. The corresponding embedding is defined by

$$I(u,p) = \int_0^T f(y(u,p),u)\, ds$$

where, for $u \in X$ and $p \in R^N$, $y(u,p)$ denotes the unique solution in $[0,T]$ to (2) and (13).

Let us say that the lower closure property holds if for every sequence $p_n \to p^\star$ and for every asymptotically minimizing sequence u_n corresponding to p_n, if for some subsequence

$$y(u_n, p_n) \to y^\star \text{ weakly in } W^{1,1}(0,T),$$

then $y^\star$ is an optimal state for the original problem $(p^\star)$.

REMARK. A sufficient condition for lower closure is that (12) holds and the Cesari sets

$$Q(y) = \{(z, g(y,u)) \in R^{N+1} : z \geq f(y,u),\ u \in U\}$$

are convex for every $y \in R^N$ (see Cesari 1983).

THEOREM 3 *Problem (1), (2), (3), (4) is well posed if we assume (12), and the following conditions hold :*

$$V \textit{ is Gâteaux differentiable near } p^\star \textit{ with continuous gradient at } p^\star \ ; \tag{14}$$

for any optimal trajectory of problem $(p^\star)$ *and for a.e.t,*
there exists a unique maximizer of $H(t,\cdot)$ *on* U ; (15)

for any pair of optimal controls u_1, u_2 *of problem* $(p^\star)$
with corresponding adjoint states q_1, q_2*, the equality* (16)
$q_1(0) = q_2(0)$ *implies* $u_1 = u_2$;

the lower closure property holds. (17)

PROOF. We check the assumptions of theorem 1, having fixed a closed ball L around $x^\star$, on which V is Gâteaux differentiable, according to (14).
Condition 8. Smoothness of f and g implied by (12) guarantees as well known that $I(u,\cdot)$ is Gâteaux differentiable everywhere.
For each $u \in X$, and for every $h \in R^N$

$$\nabla I(u,p)'\, h = \int_0^T f_x(y(u,p),u)'\, w\, dt \tag{18}$$

where

$$\dot{w} = g_x(y(u,p),u)'\, w, \quad w(0) = h. \tag{19}$$

Routine calculations (based on (12)) show that ∇I is continuous at every point of $X \times R^N$.

Condition (9). We check the Lipschitz continuity of V on L. From (12) we see that $y(u,p)$ is pointwise equi–bounded as $u \in X$ and $p \in L$, hence V is finite on L. Given p, q in L and $\varepsilon > 0$, let $u \in X$ be such that

$$I(u,p) = \int_0^T f(y(u,p),u)\,dt \leq V(p) + \varepsilon,$$

and let $y = y(u,p)$, $z = y(u,q)$. Then by (12)

$$V(q) - V(p) \leq \int_0^T [f(z,u) - f(y,u)]\,ds + \varepsilon \leq (const.) \int_0^T \mid y - z \mid \,ds + \varepsilon.$$

Again by (12) and Gronwall's lemma

$$\mid y(t) - z(t) \mid \leq (const.) \mid p - q \mid,\ 0 \leq t \leq T,$$

hence

$$V(q) - V(p) \leq \varepsilon + (const.) \mid p - q \mid$$

yielding Lipschitz continuity, since ε is arbitrary.
Condition 10. Continuity of I on $X \times L$ has been shown in the proof of (8). We check injectivity of $\nabla I(\cdot, p^\star)$. Given u_1, u_2 in $arg\ min(p^\star)$, let q_1, q_2 be the corresponding adjoint states. Then

$$\nabla I(u_1, x^\star) = \nabla I(u_2, x^\star) \tag{20}$$

implies, by (18), for every $h \in R^N$

$$\int_0^T f_x(y_1,u_1)'\ w_1\,ds = \int_0^T f_x(y_2,u_2)'\ w_2\,ds$$

where $y_i = y(u_i, x^\star)$, and w_i solves (19) with $u = u_i$, $i = 1, 2$.
Remembering the definition of q_i we have (as well known)

$$\int_0^T w_i'\ f_x(y_i,u_i)\,ds = -h'\ q_i(0)$$

hence (20) is equivalent to $q_1(0) = q_2(0)$, thus $u_1 = u_2$ by (16).
Condition (11). We have $x_n \to x^\star$, $u_n \in X$ such that

$$\int_0^T f(y_n,u_n)\,ds - V(x_n) \to 0 \tag{21}$$

where $y_n = y(u_n, x_n)$. Since $f(\cdot,u)$, $g(\cdot,u)$ are equi–Lipschitz continuous on compact sets by (12), and $y_n(t)$, $u_n(t)$ are equi–bounded, we have

$$\mid g[y^\star(t),u_n(t)] - g[y_n(t),u_n(t)] \mid + \mid f[y^\star(t),u_n(t)] - f[y_n(t),u_n(t)] \mid \leq \\ \leq (const.) \mid y_n(t) - y^\star(t) \mid \tag{22}$$

for a.e.t $\in [0,T]$. Since $| y_n(t) | \leq const.$, for a subsequence we have

$$y_n \to y^\star \text{ weakly in } W^{1,1}(0,T)$$

and there exists $u^\star \in \arg\min (x^\star)$ such that $y^\star = y(u^\star, x^\star)$. Let H, q be the Hamiltonian and the adjoint state corresponding to the optimal trajectory $(u^\star, y^\star)$. We claim that

$$\int_0^T H(t, u_n)\, dt \to \int_0^T H(t, u^\star)\, dt \tag{23}$$

(for the same subsequence as before).

By (21) and continuity of V

$$\int_0^T f(y_n, u_n)\, dt \to \int_0^T f(y^\star, u^\star)\, dt,$$

hence by (22), to show (23) it suffices to prove that

$$\int_0^T q' \, [g(y_n, u_n) - g(y^\star, u^\star)]\, dt \to 0,$$

which amounts to

$$\int_0^T q' \, (\dot{y}_n - \dot{y}^\star)\, dt \to 0. \tag{24}$$

But (24) follows from the convergence of (the subsequence) y_n.
Thus (23) is proved. By the maximum principle and (15)

$$u^\star(t) = \arg\max\, [U, H(t, \cdot)], \; a.e.\ t.$$

Then by Zolezzi 1980, theorem 5.2, $u^\star$ maximizes the integral functional

$$u \to Q(u) = \int_0^T H(t, u)\, dt$$

over X, and (X, Q) is Tikhonov well posed. It follows by (23) that u_n is a maximizing sequence for (X, Q), hence $u_n \to u^\star$ in X, as required.
The assumptions of theorem 1 are thereby fulfilled, and well posedness of problem $(x^\star)$ follows. ■

As a corollary of theorem 2, we get

THEOREM 4 *If (12) holds and problem (1), (2), (3), (4) is well posed, then V is Fréchet differentiable at $x^\star$.*

PROOF. In the proof of theorem 3 we checked that (12) entails (8), hence theorem 3 can be applied. ■

The following proposition yields a sufficient condition for assumption (16). Consider a fixed ball Z in R^{2N} containing all points $(y(t), q(t))$, $0 \leq t \leq T$, for all optimal states y and corresponding adjoint states q, whose existence is guaranteed by (12). For given $A, B \in R^N$ write

$$\theta = (A, B) \in R^{2N} \text{ and } h(\theta, u) = B' g(A, u) - f(A, u).$$

Let D be the projection of Z on the first copy of R^N.

PROPOSITION 5 *Suppose that (12) holds. Then (16) is fulfilled if*

$$g,\ f_x,\ g_x \text{ are Lipschitz on } D \times U; \tag{25}$$

$$\text{for every } \theta \in Z \text{ there exists a unique} \tag{26}$$

$$u^\star(\theta) = arg\ max[U,\ h(\theta, \cdot)]$$

and $u^\star$ is Lipschitz on Z.

PROOF. Let u_i, q_i, $i = 1, 2$, be as in (16). Put $c = q_i(0)$. By (26) and the maximum principle

$$u_i(t) = u^\star[y_i(t), q_i(t)],\ i = 1, 2. \tag{27}$$

Hence (y_i, q_i), $i = 1, 2$, are solutions on $[0, T]$ to the following initial–value problem

$$\dot{y} = g[y, u^\star(y, q)],\quad \dot{q} = -h_x[y, u^\star(y, q)],\quad y(0) = x^\star,\quad q(0) = c. \tag{28}$$

By (25), (26) we see that (28) has uniqueness in the large, since the right–hand side is Lipschitz continuous. It follows that

$$y_1 = y_2\ ,\quad q_1 = q_2 \quad \text{a.e. in } [0, T],$$

hence $u_1 = u_2$ by (27) and (26). ■

REMARK. Extensions of theorems 3 and 4 may be obtained (by standard means) in the case when $f = f(t, x, u)$, $g = g(t, x, u)$, the control region U is unbounded, and the performance (1) is modified to

$$\int_0^T f[s, y(s), u(s)]\, ds + k[y(T)]$$

with a suitable function $k : R^N \to R$.

3. Remarks and examples

3.1. A comparison with some known results

Theorems 3 and 4 are extensions of the results in Fleming, Rishel 1975, ch. VI, section 9. There it is shown that Tikhonov well posedness follows if one assumes convexity of U and of $f(x,\cdot)$, affinity of $g(x,\cdot)$, uniqueness of the optimal control $u^\star$, and condition (15) for $u^\star$. (The dynamics g and the running cost f may depend on t, however only routine modifications of the proofs here are required to handle this case, as mentioned before). These assumptions imply conditions (16) and (17). In Fleming, Rishel 1975 the dynamic programming approach is considered, while here only the initial state is perturbed. Theorem 3 obtains well posedness (a stronger property than Tikhonov's) by assuming differentiability of the value function, which is a necessary condition too, as shown by theorem 4. More important, the results of this paper follow from the abstract approach outlined in section 1, which unifies several well posedness results, scattered in different fields of optimization theory.

3.2. Other embeddings

The same approach we followed for the embedding described in section 2 can be used to obtain well posedness of the given optimal control problem with respect to the following embeddings.

3.2.1. Dynamic programming

Given $t \in [0,T]$ and $x \in R^N$, replace the time interval $[0,T]$ by $[t,T]$ in (1), (2), (4), and (3) by $y(t) = x$. Then the relevant parameter is now $p = (t,x)$, and $p^\star = (0, x^\star)$. The differentiability properties of the value function $V = V(t,x)$ at a given point are relevant as far as the Hamilton–Jacobi–Bellman equation is concerned. See Fleming, Rishel 1975 and Fleming, Soner 1993. We get a more restrictive well posedness concept than that of section 2, and results quite similar to theorems 3 and 4.

3.2.2. Perturbations of the dynamics

Given $p \in L^2([0,T])$, replace the state equations (2) by

$$\dot{y} = g(y,u) + p \quad \text{a.e. in } [0,T].$$

Then the (infinite–dimensional) parameter is now p, and $p^\star = 0$. Results about the differentiability properties of this value function are in Clarke 1986 and Clarke, Loewen 1986. As an example, consider the nondifferentiable value function in example 1.2 of Moussaoui, Seeger 1992. By the corresponding version of theorem 4, such an optimal control problem is ill posed.

3.3. Examples

The following examples exhibit well (or ill) posed optimal control problems in the sense defined by the embedding treated in section 2.

(a) Let U be compact and convex, $x^\star$ arbitrary, $g(x,u) = A(x) + B(x)\ u$, with A, B continuously differentiable and fulfilling the linear growth condition in (12). Assume that there exists a unique optimal control, and let f be continuously differentiable with $f(x,\cdot)$ strictly convex (or, more generally, let (15) be fulfilled). Then, standard modifications of the proof given in Fleming, Rishel 1975, ch. VI, th. 9.1 show that the problem is well posed. (Hence, by theorem 4, the value function is Fréchet differentiable at $x^\star$).

(b) The linear regulator problem : let

$$g(x,u) = A\ x + B\ u\ ,\quad f(x,u) = x'\ P\ x + u'\ Q\ u\ ,\quad U = R^M$$

for suitable matrices $A, B, P \geq 0,\ Q > 0$. We define X to be the set of all $u \in L^\infty([0,T])$ such that (4) holds. The proof of theorem 3 can be modified in a standard way to handle this case as well (even if A, B, P, Q are time dependent : of course, a direct proof of well posedness is readily obtained). Here f is bounded from below, and every asymptotically minimizing sequence is bounded in $L^2([0,T])$. Since $Q > 0$, we have Tikhonov well posedness of $[R^M, H(t,\cdot)]$. As well known (see Fleming, Rishel 1975) there exists a unique optimal control (hence (16) is fulfilled), and the value function is continuously differentiable everywhere (even for the embedding of the dynamic programming type described in 3.2.1 above). We get well posedness from the modification of theorem 3 mentioned above. This is the simplest and best known well posed problem of optimal control.

(c) Assumption (15) cannot be removed in theorem 3. Consider

$$M = N = 1,\ \ g(x,u) = u,\ \ f(x,u) = x^2,\ \ x^\star = 0,\ \ T = 1,\ \ U = [-1,1].$$

This is an ill posed problem, since for the minimizing sequence of states

$$x_n(t) = sin\ (nt)/n\ ,\ \ 0 \leq t \leq 1,$$

the controls $\dot{x}_n$ do not converge strongly in $L^1([0,1])$. Here the Hamiltonian corresponding to the (unique) optimal pair $u^\star = 0$, $y^\star = 0$, is constant. Moreover the value function is

$$V(p) = \mid p \mid^3 /3 \ \text{ if } \ \mid p \mid < 1,$$

hence (14) holds, and (16), (17) are trivially fulfilled.

(d) Let

$$M = N = 1,\ g(x,u) = u\ x,\ f(x,u) = (u-1)\ x,\ T = 2,\ x^\star > 0,\ U = [0,1].$$

Here the value function turns out to be $V(p) = -p\ e$, and the unique optimal control is given by

$$u^\star(t) = 1 \ \text{ if } \ 0 \leq t \leq 1,\quad u^\star(t) = 0 \ \text{ if } \ 1 < t \leq 2.$$

By explicit calculations, it is easily checked that (15) holds. Of course (14) and (16) are fulfilled. Due to theorem 3, the given problem is well posed. This example is discussed in Clarke, Loewen 1986, section 4 (from a different point of view).

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