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Explicit lamination parameters for three–dimensional shape optimization

by

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This paper is concerned with finding the most rigid microstructure of a perforated composite material capable of sustaining a given stress. The context is that of three–dimensional linear elasticity, and the motivation comes from a problem of optimal shape design (see Allaire, Kohn, 1993A, where the two–dimensional case is investigated). This question is equivalent to obtaining an optimal bound on the complementary energy for a composite obtained by micro–perforation of an isotropic material. An explicit formula for this bound is given, and for each value of the stress an associated optimal microstructure is exhibited in the class of so–called rank–three laminates. This result is the key of the three-dimensional numerical algorithm for shape optimization proposed in Allaire, Francfort (1994).

Keywords. Optimal bounds, composite materials, finite–rank laminates, shape optimization.

1. Introduction

This paper can be considered as a continuation of a previous joint work with R.V. Kohn (Allaire, Kohn, 1993B), which was concerned with optimal bounds on the effective behavior of composite materials obtained by mixing two linearly isotropic elastic components. Such bounds are of paramount importance for studying problems of optimal design, both from the theoretical and numerical point of view (for details about the link between the theory of composite materials and optimal design, see e.g. Bendsoe, Kikuchi, 1988; Kohn, Strang, 1986; Lurie, Cherkaev, Fedorov, 1982; Murat, Tartar, 1985). As the motivation of Allaire, Kohn (1993B) was to explore the theory of composite materials in great generality, the results presented therein are not completely explicit. Rather, the obtained optimal bounds are presented as the maximum/minimum value of some finite–dimensional concave/convex optimization problem. In the particular case of bounds on one energy in dimension two, explicit formulas for upper and lower bounds were obtained in the subsequent paper Allaire, Kohn (1993C). These explicit formulas in Allaire, Kohn (1993C) are the main ingredient in our work Allaire, Kohn (1993A) on two–dimensional shape optimization, and yield new numerical algorithms for computing optimal designs (see Allaire, Kohn, 1993A; Allaire, Francfort, 1993; Jog, Haber, Bendsoe, 1993). To generalize these algorithms to the three–dimensional case, explicit formulas for optimal bounds are thus required in dimension three, too. The purpose of the present paper is to furnish such crucial explicit formulas, which will be applied to optimal shape design in Allaire, Francfort (1994). However, since overwhelming and tedious calculations arise in 3–D, we restrict ourselves to what is strictly necessary for shape optimization, namely a lower bound on the complementary energy of a composite obtained by micro–perforation of a linearly isotropic elastic material (i.e. a mixture of a single material with void).

The remainder of this introduction is devoted to establishing notations and reviewing basic facts about composite materials (for details and references see Allaire, Kohn, 1993B; Avellaneda, 1987; Francfort, Murat, 1986; Hashin, Shtrikman, 1963; Milton, 1990). Consider two isotropic linearly elastic components, with Hooke's laws denoted by A_1 and A_2, finely mixed in proportions θ_1 and θ_2 respectively ($\theta_1 + \theta_2 = 1$). This fine mixture is a composite material whose macroscopic behavior is described by a linear effective Hooke's law A^* (not necessarily isotropic). Of course, A^* depends on the components' properties and on the particular arrangement of the mixture. Suppose that the components A_1 and A_2 are given with fixed proportions θ_1 and θ_2, but that their microstructure is free or unknown. The celebrated G–closure problem is then to determine the set of all possible values of A^* obtained by varying the microstructure. Unfortunately, this problem is still open, and we merely have some partial knowledge of the boundary of this set. In particular, the Hashin–Shtrikman variational principle gives the extremal values of primal or dual energies, $< A^*\varepsilon, \varepsilon >$ or $< {A^*}^{-1}\sigma, \sigma >$, where ε and σ are given strain and stress respectively. These extremal values are called optimal bounds. Optimality means that there exist some special microstructures (not necessarily unique) for which the corresponding energy is precisely the value of the bound. One can always find such optimal microstructures in the class of so–called finite–rank sequential laminates. Let us describe briefly what is a rank–p sequential laminate. It is obtained by p successive laminations : in a first step the original components A_1 and A_2 are mixed in fine layers orthogonal to a fixed direction to produce a first composite $A^*(1)$; then $A^*(1)$ is again layered with A_2 in another direction to produce $A^*(2)$, and so on. Finally, the rank–p sequential laminate $A^*(p)$ is obtained by layering $A^*(p-1)$ with A_2. Remark that the first component A_1 is used only in the first step, thus being the core of $A^*(p)$ (in other words, a sequential laminate looks as "plate–like" inclusions of A_1 in a matrix of A_2). By varying the directions and proportions of lamination at each step, and reversing the role of A_1 and

A_2, one obtains a great variety of composite materials, including optimal ones. Finite–rank sequential laminates have a great advantage on their own : there exists an explicit formula for their Hooke's law (see Francfort, Murat, 1986; Tartar, 1985).

So far, we have considered composite materials obtained by mixing two non–degenerate components. In this paper, we treat only the case of so–called perforated composite materials obtained by mixing a single non–degenerate component with void (or holes). This is obtained in the limit of a very weak component A_1 whose moduli go to zero. To simplify the notations, the remaining component A_2 is simply denoted by A, with bulk and shear moduli κ and μ, i.e. for any symmetric matrix ξ

$$A\xi = 2\mu\xi + (\kappa - \frac{2\mu}{N})(tr\,\xi)I_2 \tag{1.1}$$

where I_2 is the identity matrix, and $N = 2, 3$ the spatial dimension. We also denote by λ a quantity which has the same sign as the Poisson's ratio of A

$$\lambda = \kappa - \frac{2\mu}{N}. \tag{1.2}$$

Equivalently, a perforated composite material A^* is obtained by micro–perforations of the original material A (the boundaries of the holes created this way being traction–free). The proportion θ of material A in the perforated composite A^* is also called its density. The first goal of this paper is to compute explicitly an optimal lower bound $f(\sigma, A, \theta)$ for the complementary energy of a perforated composite A^* of density θ, under a given stress σ

$$< {A^*}^{-1}\sigma, \sigma > \;\geq\; f(\sigma, A, \theta). \tag{1.3}$$

Recall that σ, A and θ are fixed, and that the bound (1.3) is obtained by varying the microstructure of A^*. The bound (1.3) is optimal, since by a matter of theory, Avellaneda (1987), there exists some special microstructure for which there is equality in (1.3). The second goal of this paper is to exhibit a finite–rank sequential laminate which saturates (1.3) : its parameters will be computed explicitly in terms of σ, A, and θ. In other words, this laminate is the most rigid perforated composite capable of sustaining the stress σ (note however that it is not unique).

These two goals were achieved in the paper Allaire, Kohn (1993A) for the dimension $N = 2$, and in the particular case of zero Poisson's ratio (or $\lambda = 0$) for the dimension $N = 3$. Here, we generalize these results in dimension $N = 3$ for any material A having positive Poisson's ratio (or $\lambda \geq 0$). After this work has been completed, we learned from R.V. Kohn that the same computation has been done by L. Gibiansky and A. Cherkaev, Gibiansky, Cherkaev (1987). Their result is presented in a slightly different form and coincides with ours. Unfortunately, their work is in Russian and has not been published yet (although an English translation is in preparation). Anyway, we hope that the present paper will provide these results in a clear and simple way for practical applications in shape optimization.

The work reported here is a sequel of a long and fruitful collaboration with R.V. Kohn and G. Francfort; it is a pleasure for me to acknowledge their help and friendship.

2. Presentation of the main results

We begin this section by recalling some previous results concerning composite materials. At first, we give a form of the layering formula of Tartar–Francfort–Murat specialized to the case at hand.

PROPOSITION 2.1 *Let A^* be a rank–p sequential laminate of material A around a core of void, in proportion θ and $(1-\theta)$ respectively, with lamination directions $(e_i)_{1\leq i\leq p}$ and lamination parameters $(m_i)_{1\leq i\leq p}$ satisfying $0 \leq m_i \leq 1$ and $\sum_{i=1}^{p} m_i = 1$ (these parameters are related to the proportion of material A at each step of the lamination process, see Francfort, Murat, 1986 for details). Then the Hooke's law A^* is given by*

$$(1-\theta)\left[A^{*^{-1}} - A^{-1}\right]^{-1} = \theta \sum_{i=1}^{p} m_i \, f_A^c(e_i) \tag{2.1}$$

where $f_A^c(e_i)$ is a fourth order tensor (a degenerate Hooke's law) defined, for any symmetric matrix ξ, by the quadratic form

$$\begin{aligned} < f_A^c(e_i)\xi, \xi > \; = \; < A\xi, \xi > \; & -\tfrac{1}{\mu}\left[|A\xi e_i|^2 - < A\xi e_i, e_i >^2\right] - \\ & - \tfrac{1}{2\mu+\lambda} < A\xi e_i, e_i >^2 \; . \end{aligned} \tag{2.2}$$

PROPOSITION 2.2 *In space dimension N, the optimal lower bound on complementary energy*

$$< A^{*^{-1}}\sigma, \sigma > \; \geq \; f(\sigma, A, \theta) \tag{2.3}$$

is achieved by a sequential laminate of rank N (at most), whose directions of lamination coincide with the eigendirections of the stress σ.

All the above results are classical. For example, Proposition 2.1 is nothing but a combination of formulas (6.11), (6.18), and (7.6) in Allaire, Kohn (1993B), while Proposition 2.2 is a direct consequence of Remark 3.7 and formula (7.6) in Allaire, Kohn (1993B).

REMARK 2.3 *Proposition 2.2 is our starting point for the computation of the lower bound (2.3). Indeed, to calculate its value $f(\sigma, A, \theta)$ it is sufficient to minimize $< A^{*^{-1}}\sigma, \sigma >$ among all possible rank–N laminates A^* with lamination directions corresponding to the eigendirections of σ. In view of formula (2.1) for A^*, this minimization takes place over the parameters $(m_i)_{1\leq i\leq N}$ satisfying*

$0 \leq m_i \leq 1$ and $\sum_{i=1}^{N} m_i = 1$. The good news is that it involves only one degree of freedom in 2–D, and two in 3–D. The bad news is that the inversion of the lamination formula (2.1) yields $< A^{*^{-1}}\sigma, \sigma >$ as an awful function of the $(m_i)_{1\leq i\leq N}$ (see Proposition 3.1). This inversion is the focus of section 3, while the minimization is accomplished in section 4.

For the sake of comparison, we recall the bound in 2–D as obtained in Allaire, Kohn (1993A).

THEOREM 2.4 *In two dimensions, the bound (2.3) takes the form*

$$< A^{*^{-1}}\sigma, \sigma > \; \geq \; < A^{-1}\sigma, \sigma > + \frac{(\kappa+\mu)(1-\theta)}{4\kappa\mu\theta} \left(|\sigma_1| + |\sigma_2|\right)^2 \tag{2.4}$$

where σ_1 and σ_2 are the eigenvalues of the stress σ (a two–by–two matrix in 2–D). Furthermore, the associated optimal rank–2 sequential laminate is characterized by its parameters

$$m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|}, \qquad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}. \tag{2.5}$$

We also recall the bound in 3–D when the material has zero Poisson's ratio (see Allaire, Kohn, 1993A).

THEOREM 2.5 *In three dimensions, assume the material satisfies $\lambda = 0$. Then, labeling the eigenvalues of σ in such a way that $|\sigma_1| \leq |\sigma_2| \leq |\sigma_3|$, the bound (2.3) takes the form*

$$< A^{*^{-1}}\sigma, \sigma > \; \geq \; < A^{-1}\sigma, \sigma > + \frac{(1-\theta)}{4\mu\theta} \left(|\sigma_1| + |\sigma_2| + |\sigma_3|\right)^2 \tag{2.6a}$$

if $|\sigma_3| \leq |\sigma_1| + |\sigma_2|$, and

$$< A^{*^{-1}}\sigma, \sigma > \; \geq \; < A^{-1}\sigma, \sigma > + \frac{(1-\theta)}{2\mu\theta} \left((|\sigma_1| + |\sigma_2|)^2 + |\sigma_3|^2\right) \tag{2.6b}$$

if $|\sigma_3| \geq |\sigma_1| + |\sigma_2|$.
Furthermore, optimality in the first regime (2.6a) is achieved by a rank–3 sequential laminate with parameters

$$m_1 = \frac{|\sigma_3| + |\sigma_2| - |\sigma_1|}{|\sigma_1| + |\sigma_2| + |\sigma_3|}, \quad m_2 = \frac{|\sigma_1| - |\sigma_2| + |\sigma_3|}{|\sigma_1| + |\sigma_2| + |\sigma_3|}, \quad m_3 = \frac{|\sigma_1| + |\sigma_2| - |\sigma_3|}{|\sigma_1| + |\sigma_2| + |\sigma_3|} \tag{2.7a}$$

while optimality in the second regime (2.6b) is achieved by a rank–2 sequential laminate with parameters

$$m_1 = \frac{|\sigma_2|}{|\sigma_1| + |\sigma_2|}, \quad m_2 = \frac{|\sigma_1|}{|\sigma_1| + |\sigma_2|}, \quad m_3 = 0\,. \tag{2.7b}$$

We now turn to the main result proved in this paper, which holds for any material having positive Poisson's ratio.

THEOREM 2.6 *In three dimensions, assume the material A satisfies $\lambda \geq 0$. Then, the bound (2.3) takes the form*

$$< A^{*^{-1}}\sigma, \sigma > \; \geq \; < A^{-1}\sigma, \sigma > + \frac{(1-\theta)}{2\mu\theta} g(A, \sigma) \tag{2.8}$$

where, labeling the eigenvalues of σ so that $\sigma_1 \leq \sigma_2 \leq \sigma_3$, $g(A, \sigma)$ is defined by
(A) if $0 \leq \sigma_1 \leq \sigma_2 \leq \sigma_3$

$$g(A, \sigma) = \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} (\sigma_1 + \sigma_2 + \sigma_3)^2 \quad if \quad \sigma_3 \leq \sigma_1 + \sigma_2 \tag{2.9Aa}$$

$$g(A, \sigma) = (\sigma_1 + \sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \quad if \quad \sigma_3 \geq \sigma_1 + \sigma_2 \tag{2.9Ab}$$

(B) if $\sigma_1 \leq 0 \leq \sigma_2 \leq \sigma_3$

$$g(A, \sigma) = \frac{2\mu+\lambda}{2(2\mu+3\lambda)} \left(\sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda}\sigma_1\right)^2 \quad if \quad \begin{cases} \sigma_3 + \sigma_2 \geq -\frac{\mu}{\mu+\lambda}\sigma_1 \\ \sigma_3 - \sigma_2 \leq -\frac{\mu}{\mu+\lambda}\sigma_1 \end{cases} \tag{2.9Ba}$$

$$g(A, \sigma) = (\sigma_3 + \sigma_2)^2 + \sigma_1^2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \quad if \quad \sigma_3 + \sigma_2 \leq -\frac{\mu}{\mu + \lambda}\sigma_1 \tag{2.9Bb}$$

$$g(A, \sigma) = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{2\mu}{\mu + \lambda}\sigma_1\sigma_2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \quad if \quad \sigma_3 - \sigma_2 \geq -\frac{\mu}{\mu + \lambda}\sigma_1 \tag{2.9Bc}$$

(C) the other cases are obtained from (A) and (B) by symmetry, changing σ in $-\sigma$.
Furthermore, optimality in the regime (2.9Aa) is achieved by a rank–3 sequential laminate with parameters

$$m_1 = \frac{\sigma_3 + \sigma_2 - \sigma_1}{\sigma_1 + \sigma_2 + \sigma_3}, \; m_2 = \frac{\sigma_1 - \sigma_2 + \sigma_3}{\sigma_1 + \sigma_2 + \sigma_3}, \; m_3 = \frac{\sigma_1 + \sigma_2 - \sigma_3}{\sigma_1 + \sigma_2 + \sigma_3} \tag{2.10Aa}$$

in the regime (2.9Ab) it is achieved by a rank–2 sequential laminate with parameters

$$m_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad m_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}, \quad m_3 = 0 , \tag{2.10Ab}$$

in the regime (2.9Ba) it is achieved by a rank–3 sequential laminate with parameters

$$m_1 = \frac{\sigma_3 + \sigma_2 + \frac{\mu}{\mu+\lambda}\sigma_1}{\sigma_3 + \sigma_2 - \frac{\mu+2\lambda}{\mu+\lambda}\sigma_1}, \quad m_2 = \frac{\mu+\lambda}{\mu}\frac{\sigma_3 - \sigma_2 - \frac{\mu}{\mu+\lambda}\sigma_1}{\sigma_3 + \sigma_2 - \frac{\mu+2\lambda}{\mu+\lambda}\sigma_1},$$

$$m_3 = -\frac{\mu+\lambda}{\mu}\frac{\sigma_3 - \sigma_2 + \frac{\mu}{\mu+\lambda}\sigma_1}{\sigma_3 + \sigma_2 - \frac{\mu+2\lambda}{\mu+\lambda}\sigma_1}, \tag{2.10Ba}$$

in the regime (2.9Bb) it is achieved by a rank–2 sequential laminate with parameters

$$m_1 = 0, \quad m_2 = \frac{\sigma_3}{\sigma_2 + \sigma_3}, \quad m_3 = \frac{\sigma_2}{\sigma_2 + \sigma_3}, \tag{2.10Bb}$$

in the regime (2.9Bc) it is achieved by a rank–2 sequential laminate with parameters

$$m_1 = \frac{\sigma_2}{\sigma_2 - \sigma_1}, \quad m_2 = \frac{-\sigma_1}{\sigma_2 - \sigma_1}, \quad m_3 = 0. \tag{2.10Bc}$$

REMARK 2.7 *Let us emphasize again that, by virtue of Proposition 2.2, the optimal sequential laminates described in Theorems 2.4, 2.5, 2.6 have lamination directions $(e_i)_{1\leq i\leq N}$ which coincide with the eigenvectors associated to the eigenvalues $(\sigma_i)_{1\leq i\leq N}$ of the stress σ. Remark that, on the contrary of its 2–D analogue, the 3–D bound has different regimes corresponding to optimal rank–3, or rank–2, sequential laminates. Physically, a rank–3 laminate is optimal if the three eigenvalues of the stress are of the same order of magnitude (this microstructure looks like isolated holes in a matrix of material). On the other hand, if one of the stress eigenvalues is large compared to the two other ones, then a rank–2 laminate is optimal (there is no lamination in the eigendirection of the dominating eigenvalue, and this microstructure looks like a matrix of material perforated by long pipes or channels (of holes) parallel to that eigendirection). Let us also remark that, of course, Theorem 2.5 is recovered by taking $\lambda = 0$ in Theorem 2.6.*

As already mentioned in the introduction, Theorem 2.6 has also been established by L. Gibiansky and A. Cherkaev in their unpublished work, Gibiansky, Cherkaev (1987). Their proof (based on comparison between translation bounds and ad–hoc sequential laminates) is different from ours, and in our opinion, less systematic.

From a practical point of view, and apart from the proof of Theorem 2.6, it remains to compute the Hooke's law of a rank–N sequential laminates, having mutually orthogonal lamination directions, from the lamination formula (2.1). In view of applications in shape optimization (see Allaire, Francfort, 1993 where a stress–based formulation is used), we need to compute $A^{*^{-1}}$ rather than A^* itself. This is the focus of the next section.

3. Hooke's law of an orthogonal rank–N sequential laminate

By definition, a rank–N sequential laminate is said to be orthogonal if its lamination directions $(e_i)_{1\leq i\leq N}$ form an orthonormal basis of $\mathbb{R}^N$. The purpose of this section is to invert the lamination formula (2.1), that is to compute $A^{*^{-1}}$, for such an orthogonal rank–N sequential laminate. Recall that its lamination parameters $(m_i)_{1\leq i\leq N}$ satisfy

$$0 \leq m_i < 1\,, \quad \sum_{i=1}^{N} m_i = 1. \tag{3.1}$$

Let us introduce new parameters $(\alpha_i)_{1\leq i\leq N}$ defined by

$$\alpha_i = \left(1 - \frac{2\mu m_i}{2\mu+\lambda}\right)^{-1}. \tag{3.2}$$

Throughout the remainder of this paper we assume that the material A has positive Poisson's ratio, i.e.

$$\lambda \geq 0\,. \tag{3.3}$$

Combining (3.1) and (3.3), it is easy to see that α_i is bounded by

$$1 \leq \alpha_i \leq \frac{2\mu+\lambda}{\lambda}.$$

PROPOSITION 3.1 *The inverse Hooke's law $A^{*^{-1}}$ of an orthogonal rank–N sequential laminate is defined by the following quadratic form*

$$< A^{*^{-1}}\sigma, \sigma > \; = \; < A^{-1}\sigma, \sigma > + \frac{1-\theta}{2\mu\theta} G(\alpha_i, \sigma) \tag{3.4}$$

with

$$G(\alpha_i,\sigma) = \sum_{i,j=1, i\neq j}^{N} \frac{\sigma_{ij}^2}{1-m_i-m_j} + \sum_{i=1}^{N} \alpha_i \sigma_{ii}^2 - \frac{\lambda}{N\kappa}\left(\sum_{i=1}^{N}\sigma_{ii}\right)^2 + \frac{\lambda}{N\kappa}\frac{\left(\sum_{i=1}^{N}(\alpha_i-1)\sigma_{ii}\right)^2}{1-\frac{\lambda}{N\kappa}\sum_{i=1}^{N}\alpha_i},$$

where σ_{ij} denotes the entries of a symmetric matrix σ in the orthonormal basis of lamination directions.

REMARK 3.2 *The quadratic form (3.4) defines a coercive Hooke's law A^* in dimension $N \geq 3$ as soon as none of the parameters m_i is zero, that is, if the material is effectively laminated in all the N directions e_i. (Indeed, $m_i > 0$ for $1 \leq i \leq N$ implies that $1 - m_i - m_j > 0$ for $1 \leq i, j \leq N$ and $i \neq j$). Thus, in three dimensions, an orthogonal rank-3 laminate is a realistic composite material. On the contrary, in two dimensions, we always have $1 - m_i - m_j = 0$! Thus, formula (3.4) is valid only for stresses σ which are diagonal in the basis of lamination directions (i.e. such that $\sigma_{ij} = 0$). In other words, in 2-D, an orthogonal rank-2 laminate cannot support a stress whose eigendirections are not aligned with the lamination directions. This fact has previously been recognized by many authors (see Allaire, Francfort, 1993; Jog, Haber, Bendsoe, 1993 for comments).*

REMARK 3.3 *In view of (3.1) and (3.3), it is easily checked that the denominator $1 - \lambda(N\kappa)^{-1}\sum_{i=1}^{N}\alpha_i$ in formula (3.4) is always positive. Furthermore, it can be equal to zero if, and only if, it corresponds to a rank-one laminate (i.e. all m_i but one equal to zero). In other words, an orthogonal laminate of rank at least 2 can support any stress which is aligned with its lamination directions (in any spatial dimension), while a rank-one laminate can support only stresses orthogonal to its single lamination direction.*

PROOF. The starting point is the lamination formula (2.1) which gives for any symmetric matrix ε

$$(1-\theta)\left[A^{*^{-1}} - A^{-1}\right]^{-1}\varepsilon = \theta\sum_{i=1}^{N} m_i\, f_A^c(e_i)\,\varepsilon\,. \tag{3.5}$$

Let us define a matrix σ by

$$\sigma = \sum_{i=1}^{N} m_i\, f_A^c(e_i)\,\varepsilon\,. \tag{3.6}$$

With this definition (3.5) becomes

$$A^{*^{-1}}\sigma = A^{-1}\sigma + \frac{1-\theta}{\theta}\varepsilon\,. \tag{3.7}$$

Thus, it remains to compute ε in terms of σ. The degenerate Hooke's law $f_A^c(e_i)$ is defined by (2.2). This yields

$$\sigma = A\,\varepsilon - \frac{1}{\mu}A\sum_{i=1}^{N} m_i\left[(A\,\varepsilon e_i)\otimes e_i - \frac{\mu+\lambda}{2\mu+\lambda} < A\,\varepsilon e_i, e_i > e_i\otimes e_i\right]\,, \tag{3.8}$$

where $\otimes$ denotes the symmetrized tensor product of two vectors, i.e. $(u\otimes v)_{ij} = 1/2(u_i v_j + u_j v_i)$. Since $A\,\varepsilon = 2\mu\varepsilon + \lambda(tr\varepsilon)I_2$, formula (3.8) can be developed as

$$\sigma = 2\mu \sum_{i,j=1,i\neq j}^{N} (1-m_i-m_j)\varepsilon_{ij} e_i \otimes e_j$$

$$+2\mu \sum_{i=1}^{N} \left[(1-\frac{2\mu m_i}{2\mu+\lambda})\varepsilon_{ii} + \frac{\lambda}{2\mu+\lambda}(tr\varepsilon)(1-m_i) \right. \tag{3.9}$$

$$\left. -\frac{\lambda}{2\mu+\lambda}(\sum_{j=1}^{N} m_j\varepsilon_{jj}) \right] e_i \otimes e_i\,.$$

From formula (3.9), inverting the off–diagonal terms is easy

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu(1-m_i-m_j)} \qquad if \quad i \neq j\,. \tag{3.10}$$

Using definition (3.2) of the parameters α_i, the diagonal terms are solutions of a $N \times N$ linear system

$$\varepsilon_{ii} + \frac{\lambda}{2\mu+\lambda}(tr\varepsilon)(1-m_i)\alpha_i - \frac{\lambda}{2\mu+\lambda}(\sum_{i=1}^{N} m_j\varepsilon_{jj})\alpha_i = \frac{\alpha_i\sigma_{ii}}{2\mu}\,. \tag{3.11}$$

To invert system (3.11), we compute $tr\varepsilon$ and $\sum_{j=1}^{N} m_j\varepsilon_{jj}$ by summing adequatly weighted lines of (3.11). This gives the following simple two–by–two system

$$\left\{ \begin{array}{l} tr\varepsilon + \frac{\lambda}{2\mu+\lambda}(tr\varepsilon)\sum_{i=1}^{N}(1-m_i)\alpha_i - \frac{\lambda}{2\mu+\lambda}(\sum_{j=1}^{N} m_j\varepsilon_{jj})(\sum_{i=1}^{N}\alpha_i) = \\ \qquad \sum_{i=1}^{N}\frac{\alpha_i\sigma_{ii}}{2\mu} \\ \sum_{i=1}^{N} m_i\varepsilon_{ii} + \frac{\lambda}{2\mu+\lambda}(tr\varepsilon)\sum_{i=1}^{N}(1-m_i)m_i\alpha_i - \\ \qquad \frac{\lambda}{2\mu+\lambda}(\sum_{j=1}^{N} m_j\varepsilon_{jj})(\sum_{i=1}^{N} m_i\alpha_i) = \sum_{i=1}^{N}\frac{\alpha_i\sigma_{ii}m_i}{2\mu}\,. \end{array} \right.$$

A routine calculation leads to its solution

$$\left\{ \begin{array}{l} tr\varepsilon = \frac{1}{N\kappa}\sum_{i=1}^{N}\alpha_i\sigma_{ii} + \frac{2\mu\lambda}{N\kappa(2\mu+\lambda)}\frac{(\sum_{i=1}^{N}\alpha_i)(\sum_{i=1}^{N}\alpha_i m_i\sigma_{ii})}{N\kappa-\lambda\sum_{i=1}^{N}\alpha_i} \\ \sum_{i=1}^{N} m_i\varepsilon_{ii} = \frac{\lambda}{N\kappa(2\mu+\lambda)}\sum_{i=1}^{N}\alpha_i m_i\sigma_{ii} - \frac{\lambda}{N\kappa 2\mu}\sum_{i=1}^{N}\alpha_i\sigma_{ii} + \\ \qquad \frac{2\mu}{2\mu+\lambda}\frac{\sum_{i=1}^{N}\alpha_i m_i\sigma_{ii}}{N\kappa-\lambda\sum_{i=1}^{N}\alpha_i}\,, \end{array} \right. \tag{3.12}$$

which has been simplified with the help of the following identities

$$\sum_{i=1}^{N} m_i^2\alpha_i = \frac{2\mu+\lambda}{2\mu}\left(\sum_{i=1}^{N} m_i\alpha_i - 1\right),\ \sum_{i=1}^{N} m_i\alpha_i = \frac{2\mu+\lambda}{2\mu}\left(\sum_{i=1}^{N}\alpha_i - N\right).$$

Combining (3.11) and (3.12) gives the diagonal terms of ε in terms of those of σ. Finally, multiplying equation (3.7) by σ and replacing ε by its value in terms of σ yields the desired result (3.4).

4. Proof of Theorem 2.6

From Proposition 2.2, we know that the lower optimal bound $f(\sigma, A, \theta)$ on the complementary energy of a perforated composite of density θ is attained by a rank–N sequential laminate whose lamination directions coincide with the eigendirections of the stress σ. Proposition 3.1 gives the value $< A^{*^{-1}}\sigma, \sigma >$ of the complementary energy of such a laminate in terms of its lamination parameters. Therefore, to obtain the value of the lower bound, it is enough to minimize this quadratic form over these parameters. Using (3.4), this means that

$$f(\sigma, A, \theta) \;=\; < A^{-1}\sigma, \sigma > + \frac{1-\theta}{2\mu\theta} Min_{\alpha_i} G(\alpha_i, \sigma), \tag{4.1}$$

with

$$G(\alpha_i, \sigma) = \sum_{i=1}^{N} \alpha_i \sigma_i^2 - \frac{\lambda}{N\kappa}\left(\sum_{i=1}^{N} \sigma_i\right)^2 + \frac{\lambda}{N\kappa} \frac{\left(\sum_{i=1}^{N} (\alpha_i - 1)\sigma_i\right)^2}{1 - \frac{\lambda}{N\kappa}\sum_{i=1}^{N} \alpha_i} \tag{4.2}$$

where σ_i denotes the eigenvalues of σ. Remark that there is no contribution from the off–diagonal entries of σ since, by definition, σ is diagonal in the basis of the lamination directions. The minimization in (4.1) is subject to the constraints

$$1 \le \alpha_i \le \frac{2\mu + \lambda}{\lambda}, \tag{4.3}$$

which is equivalent to $0 \le m_i \le 1$, and

$$\sum_{i=1}^{N} \frac{1}{\alpha_i} = N - \frac{2\mu}{2\mu + \lambda}, \tag{4.4}$$

which comes from $\sum_{i=1}^{N} m_i = 1$.

Let us briefly explain our strategy for minimizing (4.2). First, by ignoring the constraint (4.3) (but not (4.4)), optimality conditions are easily obtained which yields the values of the optimal parameters α_i in terms of σ. In a second step, the constraint (4.3) will be tested for those optimal parameters, and according to the value of σ there will be two cases. If it is satisfied, then the minimum value of (4.2) is attained for a rank–N sequential laminate corresponding to those parameters; if not, then one of the α_i is set equal to 1 (i.e. $m_i = 0$), and (4.2) will be minimized over $N-1$ parameters only (corresponding to rank–$N-1$ sequential laminates). It won't be necessary to iterate this process (i.e. investigating lower and lower rank laminates) since the second step of this calculation will be completed only in the three-dimensional case (for the 2-D case see Allaire, Kohn, 1993A). Finally, the value of the bound (4.1) and the different regimes of optimal laminates will be deduced from these optimal parameters.

LEMMA 4.1 *Consider the minimization of (4.2) under the sole constraint (4.4). The optimal parameters α_i (if any) satisfy*

$$\alpha_i = \frac{C}{|\sigma_i + D|}, \tag{4.5}$$

where the constant C is given in terms of D by

$$C = \left(N - \frac{2\mu}{2\mu+\lambda}\right)^{-1} \sum_{i=1}^{N} |\sigma_i + D| \tag{4.6}$$

and D is solution of the piecewise linear equation

$$D + \frac{\lambda}{N\kappa} tr\sigma - \frac{(2\mu+\lambda)\lambda}{(2\mu+N\lambda)(2(N-1)\mu+N\lambda)} \left(\sum_{i=1}^{N} |\sigma_i + D|\right) \times \left(\sum_{i=1}^{N} \frac{\sigma_i + D}{|\sigma_i + D|}\right) = 0\,. \tag{4.7}$$

PROOF. The optimality condition, with the constraint that $\sum_{i=1}^{N} \alpha_i^{-1}$ is fixed, is nothing but

$$\frac{\partial G(\alpha_i, \sigma)}{\partial \alpha_k} = \frac{C^2}{\alpha_k^2}$$

for some positive constant C. Differentiating (4.2) gives

$$\frac{\partial G(\alpha_i, \sigma)}{\partial \alpha_k} = \left[\sigma_k + \frac{\lambda}{N\kappa} \frac{\sum_{i=1}^{N} (\alpha_i - 1)\sigma_i}{1 - \frac{\lambda}{N\kappa} \sum_{i=1}^{N} \alpha_i}\right]^2 .$$

This yields (4.5) with the following value of D

$$D = \frac{\lambda}{N\kappa} \frac{\sum_{i=1}^{N} (\alpha_i - 1)\sigma_i}{1 - \frac{\lambda}{N\kappa} \sum_{i=1}^{N} \alpha_i} .$$

The constraint (4.4) gives the value of C in terms of D, while equation (4.7) is obtained from the above formula for D by replacing α_i by its value (4.5). One can also check that for the optimal (α_i), defined by (4.5), the function $G(\alpha_i, \sigma)$ takes the value

$$\left(N - \frac{2\mu}{2\mu+\lambda}\right)^{-1} \left(\sum_{i=1}^N |\sigma_i + D|\right) \left(\sum_{i=1}^N \frac{\sigma_i + D}{|\sigma_i + D|}\sigma_i\right)$$
$$-D\sum_{i=1}^N \sigma_i - \frac{\lambda}{N\kappa}\left(\sum_{i=1}^N \sigma_i\right)^2 . \quad (4.8)$$

The next step is to solve equation (4.7) to compute the constant D. In the general case, this requires a formidable amount of computation; many different cases have to be investigated according to the sign of $\sigma_i + D$. For this reason, from now on we restrict ourselves to the dimension $N = 3$ (recall also that we assume $\lambda \geq 0$). In 3–D, labeling the eigenvalues of the stress σ such that

$$\sigma_1 \leq \sigma_2 \leq \sigma_3 , \quad (4.9)$$

there are two basic cases to investigate for solving (4.7) : the first one corresponds to $\sigma_1 + D \geq 0$, and the second one to $\sigma_2 + D \geq 0 \geq \sigma_1 + D$ (the two remaining cases $\sigma_3 + D \geq \sigma_2 + D$ and $0 \geq \sigma_3 + D$ are obtained from the previous ones by symmetry, changing σ to $-\sigma$).

(1) Assume $\sigma_1 + D \geq 0$.

Then, equation (4.7) reduces to

$$D + \frac{\lambda}{3\kappa} tr\sigma - \frac{(2\mu+\lambda)3\lambda}{(2\mu+3\lambda)(4\mu+3\lambda)}(tr\sigma + 3D) = 0 ,$$

which gives the following value for D

$$D = \frac{\lambda}{4\mu}(\sigma_1 + \sigma_2 + \sigma_3).$$

This yields

$$C = \frac{2\mu+\lambda}{4\mu} tr\sigma , \quad and \quad \alpha_i = \frac{(2\mu+\lambda)tr\sigma}{4\mu\sigma_i + \lambda tr\sigma} . \quad (4.10)$$

However, the constraint (4.3) on the lamination parameters is

$$1 \leq \alpha_3 \leq \alpha_1 \leq \frac{2\mu+\lambda}{\lambda} ,$$

which, using (4.10), is easily seen to be equivalent to

$$\sigma_1 \geq 0 , \quad and \quad \sigma_3 \leq \sigma_1 + \sigma_2 . \quad (4.11)$$

Remark that condition (4.11) automatically implies the assumption $\sigma_1+D\geq 0$. In view of (4.8), the extremal value of (4.2) corresponding to (4.10) is

$$G(\alpha_i,\sigma)=\frac{2\mu+\lambda}{2(2\mu+3\lambda)}(\sigma_1+\sigma_2+\sigma_3)^2\ . \tag{4.12}$$

Together with the admissibility condition (4.11), it is nothing else than regime (2.9Aa), (2.10Aa) in Theorem 2.6.

(2) Assume $\sigma_2+D\geq 0\geq\sigma_1+D$.

Then, equation (4.7) reduces to

$$D+\frac{\lambda}{3\kappa}tr\sigma-\frac{(2\mu+\lambda)\lambda}{(2\mu+3\lambda)(4\mu+3\lambda)}(\sigma_3+\sigma_2-\sigma_1+D)=0\ ,$$

which gives the following value for D

$$D=\frac{-\lambda}{4(\mu+\lambda)^2}\left((\mu+\lambda)(\sigma_3+\sigma_2)+(3\mu+2\lambda)\sigma_1\right)\ .$$

This yields

$$C=\frac{2\mu+\lambda}{4(\mu+\lambda)^2}\left((\mu+\lambda)(\sigma_3+\sigma_2)-(\mu+2\lambda)\sigma_1\right)\ ,$$

and

$$\left\{\begin{array}{lcl}\alpha_1&=&(2\mu+\lambda)\frac{(\mu+\lambda)(\sigma_3+\sigma_2)-(\mu+2\lambda)\sigma_1}{\lambda(\mu+\lambda)(\sigma_3+\sigma_2)-(4\mu^2+5\mu\lambda+2\lambda^2)\sigma_1}\\ \alpha_2&=&(2\mu+\lambda)\frac{(\mu+\lambda)(\sigma_3+\sigma_2)-(\mu+2\lambda)\sigma_1}{-\lambda(\mu+\lambda)\sigma_3+(\mu+\lambda)(4\mu+3\lambda)\sigma_2-\lambda(3\mu+2\lambda)\sigma_1}\\ \alpha_3&=&(2\mu+\lambda)\frac{(\mu+\lambda)(\sigma_3+\sigma_2)-(\mu+2\lambda)\sigma_1}{(\mu+\lambda)(4\mu+3\lambda)\sigma_3-\lambda(\mu+\lambda)\sigma_2-\lambda(3\mu+2\lambda)\sigma_1}\end{array}\right. \tag{4.13}$$

Now, the constraint (4.3) on the lamination parameters takes the form

$$1\leq\alpha_3\leq\alpha_2\leq\frac{2\mu+\lambda}{\lambda}\ ,\quad \mathit{and}\quad 1\leq\alpha_1\leq\frac{2\mu+\lambda}{\lambda}\ ,$$

which, combined with (4.13), and after a few lines of calculation (it helps to remark that the denominator of α_i, $i=1,2,3$ is positive), leads to

$$\left\{\begin{array}{l}\sigma_2\geq 0\geq\sigma_1\\ \sigma_3-\sigma_2\leq\frac{-\mu}{\mu+\lambda}\sigma_1\\ \sigma_3+\sigma_2\geq\frac{-\mu}{\mu+\lambda}\sigma_1\end{array}\right. \tag{4.14}$$

A tedious, but simple, computation shows that (4.14) automatically implies the assumption $\sigma_2+D\geq 0\geq\sigma_1+D$. In view of (4.8), the extremal value of (4.2) corresponding to (4.13) is

$$G(\alpha_i,\sigma)=\frac{2\mu+\lambda}{2(2\mu+3\lambda)}\left(\sigma_3+\sigma_2-\frac{\mu+2\lambda}{\mu+\lambda}\sigma_1\right)^2\ . \tag{4.15}$$

Together with the admissibility condition (4.14), it is nothing else than regime (2.9Ba), (2.10Ba) in Theorem 2.6.

If condition (4.11) ((4.14) resp.) is not satisfied in case (1) (case (2) resp.), it means that $G(\alpha_i, \sigma)$ does not attain its extrema inside the domain defined by the constraints (4.3), (4.4), but rather on the boundaries of that domain, which are made of rank–2 laminates. Let us consider the case of rank–2 laminates in the directions e_1 and e_2, i.e. $m_3 = 0$, (the other cases will be obtained by symmetry). Taking into account that $\alpha_3 = 1$, we now have to minimize the simplified expression of $G(\alpha_i, \sigma)$

$$\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \sigma_3^2 - \frac{\lambda}{3\kappa}\left(\sum_{i=1}^{3}\sigma_i\right)^2 + \frac{\lambda}{3\kappa}\frac{((\alpha_1-1)\sigma_1 + (\alpha_2-1)\sigma_2)^2}{1-\frac{\lambda}{3\kappa}(\alpha_1+\alpha_2+1)}$$

with the new constraints

$$1 \le \alpha_1 \le \frac{2\mu+\lambda}{\lambda}, \quad 1 \le \alpha_2 \le \frac{2\mu+\lambda}{\lambda}, \tag{4.16}$$

and

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{2(\mu+\lambda)}{2\mu+\lambda}. \tag{4.17}$$

The optimality conditions under the sole constraint (4.17) are still of the same type as in Lemma 4.1, and the optimal parameters are given by

$$\alpha_1 = \frac{A}{|\sigma_1 + B|}, \quad \alpha_2 = \frac{A}{|\sigma_2 + B|},$$

where the constant A is given in terms of B by

$$A = \frac{2\mu+\lambda}{2(\mu+\lambda)}\left(|\sigma_1+B| + |\sigma_2+B|\right)$$

and B is solution of the piecewise linear equation

$$B + \tfrac{\lambda}{2(\lambda+\mu)}(\sigma_1+\sigma_2) - \frac{(2\mu+\lambda)\lambda}{4(\mu+\lambda)^2}\left(|\sigma_1+B| + |\sigma_2+B|\right)\cdot \left(\frac{\sigma_1+B}{|\sigma_1+B|} + \frac{\sigma_2+B}{|\sigma_2+B|}\right) = 0. \tag{4.18}$$

The solution of (4.18) is similar to that of (4.7), but a lot simpler since the corresponding value of the optimal parameters always satisfy the remaining constraints (4.16). Keeping in mind the labeling convention (4.9), there are again two basic cases (the remaining ones being obtained by symmetry, changing σ in $-\sigma$).

(A) Assume $\sigma_1 + B \geq 0$.

Then, the solution of (4.18) is $B = \lambda(\sigma_1+\sigma_2)/(2\mu)$, and the corresponding parameters are

$$\alpha_1 = \frac{(2\mu+\lambda)(\sigma_1+\sigma_2)}{\lambda\sigma_2+(2\mu+\lambda)\sigma_1}\,, \quad \alpha_2 = \frac{(2\mu+\lambda)(\sigma_1+\sigma_2)}{(2\mu+\lambda)\sigma_2+\lambda\sigma_1}\,. \tag{4.19}$$

The constraint (4.16) is equivalent to

$$\sigma_1 \geq 0\,, \tag{4.20}$$

while the value of $G(\alpha_i, \sigma)$ is

$$G(\alpha_i,\sigma) = (\sigma_1+\sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu+3\lambda}(tr\sigma)^2\,. \tag{4.21}$$

(B) Assume $\sigma_2 + B \geq 0 \geq \sigma_1 + B$.

Then, the solution of (4.18) is $B = -\lambda(\sigma_1+\sigma_2)/(2\mu+2\lambda)$, and the corresponding parameters are

$$\alpha_1 = \frac{(2\mu+\lambda)(\sigma_2-\sigma_1)}{\lambda\sigma_2-(2\mu+\lambda)\sigma_1}\,, \quad \alpha_2 = \frac{(2\mu+\lambda)(\sigma_2-\sigma_1)}{(2\mu+\lambda)\sigma_2-\lambda\sigma_1}\,. \tag{4.21}$$

The constraint (4.16) is equivalent to

$$\sigma_2 \geq 0 \geq \sigma_1\,, \tag{4.22}$$

while the value of $G(\alpha_i, \sigma)$ is

$$G(\alpha_i,\sigma) = (\sigma_1^2+\sigma_2^2+\sigma_3^2) - \frac{2\mu}{\mu+\lambda}\sigma_1\sigma_2 - \frac{\lambda}{2\mu+3\lambda}(tr\sigma)^2\,. \tag{4.23}$$

The cases of rank–2 laminates in other directions (i.e. $m_1 = 0$ or $m_2 = 0$) are obtained by simply permuting the indices 1,2,3 in the above formulas. As a matter of fact, the extremal values (4.21) and (4.23) of $G(\alpha_i, \sigma)$ are minimum values, among rank–2 laminates, since, in the limit of rank–one laminates, $G(\alpha_i, \sigma)$ goes to infinity (see Remark 3.3). Furthermore, these minimum values are easily checked to be always larger than the extremal values (4.12) and (4.15) for rank–3 laminates, which are therefore minimum values themselves.

The proof of Theorem 2.6 can now be completed by simply seeking the best rank–2 laminates when the admissibility conditions for the existence of an optimal rank–3 laminate are not satisfied in cases (1) and (2) above. We can safely leave to the reader the task of comparing the different optimal rank–2 laminates (i.e. $m_1 = 0$, $m_2 = 0$, or $m_3 = 0$). We simply indicate the final result. If the compatibility condition (4.11) in case (1) is not satisfied, then the minimum of $G(\alpha_i, \sigma)$ is attained for a rank–2 laminate corresponding to case (A) above (this is regime (2.9Ab) and (2.10Ab) in Theorem 2.6). If the compatibility condition (4.14) in case (2) does not hold, then the minimum of $G(\alpha_i, \sigma)$ is attained for one of the following two rank–2 laminates : if $\sigma_3 - \sigma_2 \geq -\mu(\mu+\lambda)^{-1}\sigma_1$, case (B) above is optimal (this is regime (2.9Bc) and (2.10Bc) in Theorem 2.6), and if $\sigma_3 + \sigma_2 \leq -\mu(\mu+\lambda)^{-1}\sigma_1$, interchanging directions 1 and 3 in case (A) (i.e. $m_1 = 0$) gives the optimal result (this is regime (2.9Bb) and (2.10Bb) in Theorem 2.6).

Let us conclude by remarking that a rank–2 laminate is required when one of the eigenvalues of the stress is large compared to the two other ones, and that, in such a situation, there is no lamination in the eigendirection of the dominating eigenvalue.

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