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A potential method for shape optimization problems

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This paper is concerned with a potential method in a class of shape optimization problems. Potential representations are used to derive characterizations of the first variations of solutions of boundary value problems when the domains on which the boundary value problems are defined are varied. In view of these characterizations, the first order necessary condition is derived for each of the shape optimization problems. A counterexample is given to show that the potential method is not omnipotent.

1. Introduction

In this paper, we shall study a potential method for deriving a boundary value problem that defines a function called the first order variation of the solution of the original boundary value problem. This variation of the solution will play an important role for deriving the first–order necessary optimality condition for shape optimization problems. Shape optimization problems are problems, Fujii (1986A) — Fujii (1990), in each of which an objective functional, depending on (the shape of) a domain through the solution of a boundary value problem defined on the domain, must be minimized or maximized with respect to the domain. Pironneau (1973,1974) systematically studied the minimum drag problems, typical shape optimization problems, in both Stokes flows and Navier–Stokes flows. Zolesio (1981) developed his material derivative method for sensitivity analysis of shape optimization problems. Sokołowski and Zolesio (1985) applied the material derivative method to sensitivity analysis of an elastic–plastic problem. Goto and the present author, Goto, Fujii, Muramatsu (1987,1990), studied shape optimization problems with a Neumann problem as a constraint. They gave the second-order necessary conditions for optimality using Taylor expansion to get variations of the solution. In this paper, we shall deal with the same problems using a different device, a potential method.

In section 2, we shall give the problem statement and the way for deriving the first-order necessary optimality condition. In section 3, the potential method will be explained. In this paper, we shall confine ourselves to Neumann problems as a constraint.

2. Problem and first–order necessary condition

Let $\mathbf{R}^n$ be n–dimensional Euclidean space. Let Ω be a domain in $\mathbf{R}^n$ ($n \geq 3$); let $\Gamma \equiv \partial\Omega$ be its sufficiently smooth boundary. Let sufficiently smooth functions $k(x), f(x), h(x)$, and $\tau(x)$ be defined in $\mathbf{R}^n$. Let a sufficiently smooth function $g(x,u)$ be defined in $\mathbf{R}^n \times \mathbf{R}$. Let us consider the following boundary value problem (Neumann problem):

$$\Delta u(x) - k(x)u(x) = f(x) \qquad (x \in \Omega), \tag{1}$$

$$\frac{\partial u}{\partial n}(x) = \tau(x) \qquad (x \in \Gamma), \tag{2}$$

where we assume that $k(x) \geq 0$ ($k(x) \not\equiv 0$). Here, Δ is the Laplacian operator defined by

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and $\partial/\partial n$ denotes the directional differentiation along the outward normal $\vec{n}$. This boundary value problem admits, Courant (1962), a unique solution. We introduce as an objective functional the functional of solution u of the above boundary value problem,

$$J(\Omega; u) = \int_\Omega g(x,u)\,dx. \tag{3}$$

We require that the domain Ω satisfy

$$\int_\Omega h(x)dx = \kappa(\text{const.}); \tag{4}$$

this constraint is a generalization of the requisition of constant volume.

PROBLEM STATEMENT

Our problem is to find a domain Ω that minimizes the functional (3) of u, the solution of (1) and (2), under the constraint (4).

In general, it is very difficult to find an analytical solution for this problem. Therefore, we shall look for necessary conditions for optimality, which help us to find an optimal domain numerically. In what follows, we assume that this optimization problem has a solution Ω. Then, the following question arises. What conditions must Ω and corresponding u satisfy? In order to answer the question, we begin with the definition of the first variation $\delta^{(1)}J$ of $J(\Omega; u)$. Let $\rho(x)$, $(x \in \Gamma)$ be a given sufficiently smooth function defined on boundary Γ. Let ϵ be an arbitrary positive number. We consider at each point on Γ the normal and plot on it the segment $\epsilon\rho(x)$, so that positive values of $\epsilon\rho(x)$ lie on the outward normal $\vec{n}$. If ϵ is small enough, the end points of the segments form a smooth closed surface which encloses a new domain; hereafter, the surface will be denoted by Γ_ϵ, and Ω_ϵ stands for the new domain. We shall sometimes use the notation $\delta n = \epsilon\rho$; δn will be called the boundary variation. If we substitute Ω_ϵ for Ω in boundary value problem (1) and (2), we get a new boundary value problem. Let u_ϵ be its solution. For newly obtained Ω_ϵ and u_ϵ, the objective functional is given by

$$J(\Omega_\epsilon; u_\epsilon) = \int_{\Omega_\epsilon} g(x, u_\epsilon)\, dx. \tag{5}$$

Let $o(\epsilon)$ denote quantities such that $o(\epsilon)/\epsilon \to 0 (\epsilon \to 0)$. Let us define the first variation $\delta^{(1)}J$ by

$$J(\Omega_\epsilon; u_\epsilon) - J(\Omega; u) = \epsilon\, \delta^{(1)}J + o(\epsilon). \tag{6}$$

In order to calculate this variation $\delta^{(1)}J$, we have to clarify the first variation ϕ of u defined by

$$u_\epsilon(x) - u(x) = \epsilon\phi(x) + o(\epsilon). \tag{7}$$

To this end, we need a lemma.

Let us consider a function $w(x)$ which is twice continuously differentiable in Ω and continuous on $\Omega \cup \Gamma$ (i.e., $w \in C(\overline{\Omega}) \cap C^2(\Omega)$), where Ω is a bounded domain with smooth boundary Γ in n-dimensional space $\mathbf{R}^n$ $(n \geq 2)$. It is well known, Courant (1962), that w has the following potential representation:

$$\begin{aligned} w(x) \;=\; & -\int_\Gamma w(y)\, \frac{\partial}{\partial n_y} U(x, y)\, d\Gamma_y + \int_\Gamma \frac{\partial w(y)}{\partial n_y} U(x, y)\, d\Gamma_y \\ & -\int_\Omega \Delta_y w(y)\, U(x, y)\, dy \qquad\qquad (x \in \Omega). \end{aligned} \tag{8}$$

Here, subscript y denotes the operation with respect to y. Function $U(x, y)$ is a fundamental solution of Δ and is given by

$$U(x,y) = \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{|x-y|} & (n=2), \\ \dfrac{1}{n(n-2)\omega_n |x-y|^{n-2}} & (n \geq 3), \end{cases} \tag{9}$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. The first term of the r.h.s. of (8) is called a double layer potential with density $w(x)$; it is harmonic everywhere except on boundary Γ. The second term of the r.h.s. of (8) is called a single layer potential with density $(\partial w)/(\partial n)$. Hereafter, let $F(x)$ denote the single layer potential with density $\alpha(x)$. Function $F(x)$ is continuous throughout $\mathbf{R}^n$. It is also harmonic everywhere except on boundary Γ. More precisely, it satisfies

$$\Delta F(x) = 0 \qquad (x \notin \Gamma). \tag{10}$$

On the other hand, its derivatives are discontinuous on Γ. In particular, we know that the following relation

$$\frac{\partial F}{\partial n}_+ - \frac{\partial F}{\partial n}_- = \omega_n \alpha(x) \qquad (x \in \Gamma) \tag{11}$$

is true, Courant (1962), for directional differentiation along $\vec{n}$, where $(\partial F/\partial n)_+$ denotes the limit from the outside and $(\partial F/\partial n)_-$, from the inside. The third term of the r.h.s. of (8) is a volume (Newtonian) potential with density Δw. Let $G(x)$ be a volume potential with density $\beta(x)$. It is well known, Courant (1962), that

$$\Delta G(x) = -\beta(x) \qquad (x \in \Omega) \tag{12}$$

holds.

According to conventions, we shall call $u(x)\,(x \in \Gamma)$ Dirichlet data and $\partial u/\partial n(x)\,(x \in \Gamma)$, Neumann data of u. Let $w(x)$ be a function whose Neumann data are not known. Let its potential representation be given by

$$\begin{aligned} w(x) \;=\; & -\int_\Gamma w(y) \frac{\partial}{\partial n_y} U(x,y)\, d\Gamma_y + \int_\Gamma \alpha(y)\, U(x,y)\, d\Gamma_y \\ & - \int_\Omega \Delta_y w(y)\, U(x,y)\, dy \qquad (x \in \Omega). \end{aligned} \tag{13}$$

Then the following question arises. Are the Neumann data of $w(x)$ $\alpha(x)$? In other words : does the relation $\partial w(x)/\partial n = \alpha(x)\,(x \in \Gamma)$ hold? As an answer to this question, we have the following lemma:

LEMMA 2.1 *Let $n \geq 3$. If a function $w(x)$ is represented by (13), then*

$$\frac{\partial w}{\partial n}(x) = \alpha(x) \qquad (x \in \Gamma) \tag{14}$$

holds.

REMARK 2.1 *If $n = 2$, then the corresponding assertion is false.*

The proof of Lemma 2.1 and a counterexample for $n = 2$ will be given in the next section. If we use Lemma 2.1, we can characterize the first variation ϕ of u under looser assumptions for derivatives of u and u_ϵ.

Let us confine ourselves to the case of $n = 3$ till the end of this section. We place the following assumption:

ASSUMPTION 2.1 *We assume that there exists a function ϕ for u and u_ϵ such that*

$$u_\epsilon(x) - u(x) = \epsilon\phi(x) + o(\epsilon) \qquad (x \in \Omega \cap \Omega_\epsilon), \tag{15}$$

hold. Here (15) agree with (7).

Under this assumption, we can discuss as in Fujii (1986B) and (1986C) using potential representations of u and u_ϵ to obtain the characterization for ϕ. In fact, we have the following expression for $\phi(x)$:

$$\begin{aligned}\phi(x) \;=\; & -\int_\Gamma \phi \frac{\partial}{\partial n_y} U(x,y) d\Gamma_y \\ & + \int_\Gamma \left\{ \left(-\frac{\partial^2 u}{\partial n^2}(y) + \frac{\partial \tau}{\partial n}(y) \right) \rho(y) \right. \\ & \qquad\qquad \left. + \operatorname{grad}_\Gamma \rho(y) \cdot \operatorname{grad}_\Gamma u(y) \right\} U(x,y) d\Gamma_y \\ & - \int_\Omega k(y)\phi(y) U(x,y) dy. \end{aligned} \tag{16}$$

Here, $\operatorname{grad}_\Gamma$ denotes the gradient operator on boundary Γ. In view of (12) and the fact that single layer and double layer potentials are harmonic in Ω, we can observe that

$$\Delta\phi(x) - k(x)\phi(x) = 0 \qquad (x \in \Omega) \tag{17}$$

holds. Applying Lemma 2.1 to (16), we see that ϕ satisfies

$$\frac{\partial \phi}{\partial n}(x) = \left(-\frac{\partial^2 u}{\partial n^2}(x) + \frac{\partial \tau}{\partial n}(x) \right) \rho(x) + \operatorname{grad}_\Gamma \rho(x) \cdot \operatorname{grad}_\Gamma u(x) \, (x \in \Gamma). \tag{18}$$

Thus, we see that ϕ is the solution of boundary value problem (17), (18).

Using these equations, we can obtain an expression for the first variation $\delta^{(1)}J$ of $J(\Omega; u)$ as

$$\delta^{(1)} J = \int_\Gamma \rho \left[g(x,u) + p(x) \left\{ \frac{\partial^2 u}{\partial n^2} - \frac{\partial \tau}{\partial n} \right\} + \operatorname{div}_\Gamma \big(p(x) \operatorname{grad}_\Gamma u\big) \right] d\Gamma. \tag{19}$$

Here, div_Γ stands for the divergence operator on boundary Γ. The new function $p(x)$ is called the adjoint variable and is defined by

$$\Delta p(x) - k(x)p(x) = \frac{\partial g}{\partial u}(x) \qquad (x \in \Omega), \tag{20}$$

$$\frac{\partial p}{\partial n}(x) = 0 \qquad (x \in \Gamma). \tag{21}$$

On the other hand, Ω_ϵ and Ω must satisfy (4). This means that ρ must satisfy

$$\int_\Gamma \rho(x)h(x)\,d\Gamma = 0. \tag{22}$$

Since, Ω is the solution of the optimization problem, $\delta^{(1)}J$ represented by (19) must be 0 for all ρ that satisfy (22) (condition of stationarity). Hence, we get (see Goto, Fujii, Muramatsu, 1987, 1990)

THEOREM 2.1 *If domain Ω is an optimal domain, for the corresponding solution u of (1), (2) and solution p of (20), (21), there exists a constant λ such that*

$$g(x,u) + p(x)\left\{\frac{\partial^2 u}{\partial n^2}(x) - \frac{\partial \tau}{\partial n}\right\} + \mathrm{div}_\Gamma\,(p(x)\,\mathrm{grad}_\Gamma u(x)) = \lambda h(x) \quad (x \in \Gamma) \tag{23}$$

holds.

We have given an outline of deriving the first-order necessary optimality condition.

3. Proof of Lemma 2.1

Lemma 2.1 played an essential role in the former section. Lemma 2.1 is a classical result; however, the author does not know the statement as well as the proof in any other article. In this section, we give the proof and a counterexample for the case of 2-dimensional space.

For the time being, let $n \geq 3$. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ.

LEMMA 3.1 *Let a function $F(x)$ $(x \in \mathbf{R}^n)$ be defined by*

$$F(x) \equiv \int_\Gamma \alpha(y)\,U(x,y)\,d\Gamma_y. \tag{24}$$

Then, if

$$F(x) = 0 \qquad (x \in \Omega) \tag{25}$$

holds,

$$F(x) = 0 \qquad (x \in \mathbf{R}^n) \tag{26}$$

holds.

PROOF. Formula (24) means that $F(x)$ is a single layer potential with density $\alpha(x)$ on Γ. As is pointed out in the former section, $F(x)$ is continuous in the entire space. Thus, from (25), we see that

$$F(x) = 0 \qquad (x \in \Gamma). \tag{27}$$

Choose a point x_0 in the interior of Ω and fix it. Let us consider a ball of radius R_0 with its center at x_0 such that the ball contains Ω in its interior. Let ϵ be an arbitrary positive number. Let us consider a sphere S of radius R with its center at x_0 such that $R > R_0$ and

$$(R - R_0)^{n-2} > \frac{1}{\epsilon n(n-2)\omega_n} \int_\Gamma |\alpha(x)|\, d\Gamma \tag{28}$$

hold. Let Ω' be the domain surrounded by S and by Γ. Let us estimate $F(x)$ at x on S . From the representation (9) for $U(x, y)$, we obtain the estimate

$$\begin{aligned} |F(x)| &= \left| \int_\Gamma \alpha(y) U(x,y)\, d\Gamma_y \right| \\ &\leq \int_\Gamma |\alpha(y)| \frac{1}{n(n-2)\omega_n |x-y|^{n-2}}\, d\Gamma_y \\ &\leq \frac{1}{n(n-2)\omega_n (R-R_0)^{n-2}} \int_\Gamma |\alpha(y)|\, d\Gamma_y, \end{aligned} \tag{29}$$

using $|x - y| \geq (R - R_0)$. In view of (28), we get the estimate

$$|F(x)| < \epsilon \qquad (x \in S). \tag{30}$$

As is noted in the former section, $F(x)$ is harmonic in Ω'. Hence, the maximum principle, Courant (1962), is applicable to the modulus of $F(x)$. Namely, the maximum of the modulus of $F(x)$ in $\Omega' \cup \Gamma \cup S$ is attained on Γ or on S. Thus, from (27) and (30), we can see that the estimate

$$|F(x)| < \epsilon \qquad (x \in \Omega') \tag{31}$$

is valid. Since ϵ is arbitrary and R is arbitrary, provided that it is large enough, we finally obtain

$$F(x) = 0 \qquad (x \in \mathbf{R}^n - \Omega). \tag{32}$$

Therefore, from (25), (27), and (32), we observe that (26) holds. ■

LEMMA 3.2 *If the identity*

$$F(x) \equiv \int_\Gamma \alpha(y) U(x,y)\, d\Gamma_y = 0 \qquad (x \in \Omega) \tag{33}$$

holds, we have

$$\alpha(x) = 0 \qquad (x \in \Gamma). \tag{34}$$

PROOF. From Lemma 3.1 and (33), we see that

$$F(x) = 0 \qquad (x \in \mathbf{R}^n) \tag{35}$$

holds. In particular,

$$\frac{\partial F}{\partial n_+} = \frac{\partial F}{\partial n_-} = \frac{\partial F}{\partial n} \equiv 0 \qquad (x \in \Gamma)$$

holds. Function $F(x)$ has been a single layer potential with density $\alpha(x)$. Hence, from (11), we can see that (34) holds. ■

Now, we are in a position to prove Lemma 2.1.

PROOF OF LEMMA 2.1

There are two expressions (8) and (13) for $w(x)$. If we subtract both sides of (13) from both sides of (8), we get

$$\int_\Gamma \left(\frac{\partial w}{\partial n_y}(y) - \alpha(y) \right) U(x,y)\, d\Gamma_y \equiv 0 \qquad (x \in \Omega). \tag{36}$$

If we apply Lemma 3.2 to this expression, we obtain

$$\frac{\partial w}{\partial n}(x) = \alpha(x) \qquad (x \in \Gamma). \tag{37}$$

This is nothing but the conclusion (14) of Lemma 2.1. ■

As for 2-dimensional spaces, the statement corresponding to Lemma 2.1 is not valid. Let us give a counterexample showing this fact. Let Γ be the unit circle with its center at the origin of $\mathbf{R}^2$. Let Ω be the unit disk surrounded by Γ. Let k be a constant. Let us define a function $G(x)$ by

$$G(x) \equiv \int_\Gamma k\, U(x,y)\, d\Gamma_y = \frac{1}{2\pi} \int_\Gamma k \log \frac{1}{|x-y|}\, d\Gamma_y. \tag{38}$$

Let us show that $G(x) = 0$ $(x \in \Omega)$. Since $|y| = 1$ $(y \in \Gamma)$, we see that

$$G(0) = \frac{1}{2\pi} \int_\Gamma k \log \frac{1}{|y|}\, d\Gamma_y = 0. \tag{39}$$

On the other hand, from its definition, it is obvious that $G(x)$ remains invariant with respect to rotation around the origin. Therefore, it is a function of only $r = |x|$; we can write $G(r)$ instead of $G(x)$. Since $G(r)$ is harmonic in Ω, we can apply the mean value theorem, Courant (1962), to $G(r)$. More precisely, for every circle S_R with radius R $(0 < R \le 1)$,

$$0 = G(0) = \frac{1}{2\pi R} \int_{S_R} G(R)\, dS_R, = G(R) \tag{40}$$

Since R is arbitrary, we have shown that

$$G(x) = \int_\Gamma k\, U(x,y)\, d\Gamma_y = 0 \qquad (x \in \Omega) \tag{41}$$

holds. A sufficiently smooth function $u(x)$ has its potential representation:

$$\begin{aligned} u(x) = & - \int_\Gamma u(y) \frac{\partial}{\partial n_y} U(x,y)\, d\Gamma_y + \int_\Gamma \frac{\partial u(y)}{\partial n_y} U(x,y)\, d\Gamma_y \\ & - \int_\Omega \Delta_y u(y)\, U(x,y)\, dy \qquad (x \in \Omega). \end{aligned} \tag{42}$$

Adding both sides of (41) to both sides of (42), we obtain

$$\begin{aligned} u(x) = & - \int_\Gamma u(y) \frac{\partial}{\partial n_y} U(x,y)\, d\Gamma_y + \int_\Gamma \left(\frac{\partial u(y)}{\partial n_y} + k \right) U(x,y)\, d\Gamma_y \\ & - \int_\Omega \Delta_y u(y)\, U(x,y)\, dy \qquad (x \in \Omega). \end{aligned} \tag{43}$$

If we apply to the above the assertion corresponding to Lemma 2.1, for any k, we get

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial n} + k \qquad (x \in \Gamma). \tag{44}$$

This is a contradiction. Therefore, Lemma 2.1 is not true for the case of $n = 2$.

4. Concluding remarks

In this paper, we gave an outline of deriving the first-order necessary condition for a class of shape optimization problems. In each of these problems, a Neumann problem has been the constraint. We also showed that potential representations for solutions of Neumann problems play an important role. Because of the existence of a counterexample for $n = 2$, we did not adopt potential representations in Goto, Fujii, Muramatsu (1987) and (1990); however, we had to assume the smoothnesses of the higher derivatives of the solutions and their variations. If we use potential representations, we have only to assume the smoothnesses of the lower derivatives. This fact is an advantage in developing the theory of shape optimization problems; we can much reduce the efforts of showing the smoothnesses of the derivatives. However, the counterexample for $n = 2$ shows that the potential representations are not omnipotent. Furthermore, the counterexample tell us that characteristics of boundary value problems depend not only on the type but also on the dimension of the space considered.

Deriving the first-order necessary optimality condition produces numerical methods as by-products. These numerical methods are very important for engineering. As for mathematical problems, existence problems as well as sufficient optimality conditions (see Fujii, 1994) are left.

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