Control and Cybernetics

vol. **23** (1994) No. 3

Sufficient conditions for optimality in shape optimizations

by

Nobuo Fujii

Osaka Sangyo University
3–1–1, Nakagaito
Daito–shi, Osaka
574 Japan

This paper deals with a set of sufficient conditions for optimality in shape optimization problems. First, a set of necessary conditions for optimality is derived. Then, a class of domains is introduced in order that isolated local minimums be defined. A set of sufficient conditions is derived for the isolated local minimum; an outline of the proof is given together with an illustrative example.

1. Introduction

The purpose of this paper is to give a set of sufficient conditions to shape optimization problems. These problems are optimization problems in each of which an objective function, depending on a domain through the solution of a boundary value problem defined on the domain, must be minimized or maximized with respect to the domain. The typical examples in elasticity will be found in the paper of Dems and Mróz (1984) (see also Masanao, Fujii, 1992A, 1992B). Céa (1981) has enumerated various shape optimization problems in engineering. Pironneau (1973,1974) systematically studied the minimum drag problems in fluid mechanics. He gave the first–order necessary optimality conditions for both Stokes flows and Navier–Stokes flows. On the other hand, Zolesio (1981) proposed the so–called material derivative method for sensitivity analysis of shape optimization problems. This method is highly sophisticated and widely used, Sokołowski, Zolesio (1992). In fact, Dems and Mróz (1984) used the method for their problems. The present author, Fujii (1986A)—Fujii (1986C), has independently developed a more heuristic and intuitive method. The present author and his student Goto, Fujii (1990) have proposed a numerical method for shape optimization problems in the case of Neumann problem. They have given a complete, so–called 'Hessian' representation for the objective functionals; the dimension of the space is two, though. The present author and his students, Goto, Fujii, Muramatsu (1990), have studied second-order necessary conditions also for shape optimization problems with a Neumann problem as a constraint.

In this paper, we shall give a set of sufficient conditions for optimality to shape optimization problems. In the next section, we shall give the problem formulation and the second-order necessary conditions of Kuhn–Tucker type, which have been thoroughly studied in Fujii (1990). In Section 3, we shall introduce a 'distance between two domains' into a set of domains. Thus, we introduce a 'topology' into a set of domains. Then, we shall give a set of sufficient conditions of second order.

2. Second–order necessary conditions

Let a domain Ω in two–dimensional Euclidean space $\mathbf{R}^2$ be bounded; coordinates of points of the space are denoted by $x = (x_1, x_2)$. Its boundary is denoted by $\Gamma(= \partial\Omega)$ and is assumed to be a differentiable manifold of class C^6. Let us assume that sufficiently smooth functions $k(x)(\geq 0)$, $f(x)$ and a constant κ be given in a sufficiently large domain. As a typical boundary value problem in shape (domain) optimization problems, consider the following boundary value problem (Dirichlet problem).

$$\Delta u(x) - k(x)u(x) = f(x) \qquad (x \in \Omega), \tag{1}$$

$$u(x) = \kappa(\text{const.}) \qquad (x \in \Gamma), \tag{2}$$

where Δ stands for the Laplacian operator and is defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

It is well known Gilbarg, Trudinger (1983) that the boundary value problem admits a unique solution $u(x)$ which is smooth enough. Let us introduce a functional $J(\Omega; u)$ of the solution $u(x)$ by

$$J(\Omega; u) \equiv \int_\Omega g(x, u(x))\, dx. \tag{3}$$

Here, $g(x, u)$ is a sufficiently smooth function of x and u.

Our problem is to find a domain Ω which minimizes (or maximizes) this functional. Here, the domain Ω must satisfy a constraint

$$I(\Omega) \equiv \int_\Omega h(x)\, dx = c\,(\text{const.}). \tag{4}$$

This constraint is a generalization of the requisition of constant area. In general, it is difficult analytically to find such a domain Ω. Hence, we shall confine ourselves to finding necessary conditions for optimality. As in the case of usual extremum problems, these necessary conditions will be helpful in finding an optimal domain. In fact, we can get numerical methods, Goto, Fujii (1990), as by-products. According to familiar ways of calculus of variation, we shall begin with the calculation of the variation of functional $J(\Omega; u)$. Since boundary Γ is smooth, we can introduce arclengths s to Γ. Let $\rho(s)$ be a sufficiently smooth function of s. Let ϵ be an arbitrary number. We consider at each point on Γ the normal and plot on it the segment $\epsilon\rho(s)$, so that positive values of $\epsilon\rho(s)$ lie on the outward normal $\vec{n} = (n_1, n_2)$. If $|\epsilon|$ is small enough, the endpoints of the segments form a smooth closed curve which encloses a new domain; hereafter, the curve will be denoted by Γ_ϵ, Ω_ϵ standing for the new domain. We shall sometimes use notation $\rho(x)$ when x lies on Γ and notation δn which is defined by

$$\delta n = \epsilon\rho. \tag{5}$$

We say that Γ_ϵ approaches Γ in the sense that each point on Γ_ϵ approaches the corresponding point on Γ, when $\epsilon \to 0$. In the same sense, we say that Ω_ϵ approaches Ω. Now, let us consider the following boundary value problem on Ω_ϵ:

$$\varDelta u_\epsilon(x) - k(x)u_\epsilon(x) = f(x) \qquad (x \in \Omega_\epsilon), \tag{6}$$

$$u_\epsilon(x) = \kappa\,(\text{const.}) \qquad (x \in \Gamma_\epsilon). \tag{7}$$

Note that the boundary value problem (6) and (7) admits a unique solution $u_\epsilon(x)$, Gilbarg, Trudinger (1983). If we try to calculate the variation of objective functional $J(\Omega; u)$, the following questions arise: Do there exist functions $\phi(x)$ and $\psi(x)$ that satisfy

$$u_\epsilon(x) - u(x) = \epsilon\phi(x) + \epsilon^2\psi(x) + o(\epsilon^2) \quad (x \in \Omega \cap \Omega_\epsilon), \tag{8}$$

the expansion with respect to ϵ? If they exist, what are they? Here and hereafter, $o(\epsilon^2)$ denotes quantities such that $o(\epsilon^2) \to 0$ as $\epsilon \to 0$. There are a great deal of theory for the dependence of the solution on the boundary data in the case of fixed domain Ω. However, as far as the author knows, there is quite a limited theory, Garabedian, Schiffer (1953), for the solutions of partial differential equations in the case of variable domains. For the sake of simplicity, we shall call $\phi(x)$ the first variation of the solution and $\psi(x)$, the second variation. Let us define ϕ_ϵ by

$$\phi_\epsilon(x) \equiv \frac{1}{\epsilon}(u_\epsilon(x) - u(x)) \qquad (x \in \Omega \cap \Omega_\epsilon). \tag{9}$$

Let $\epsilon \to 0$. Then, for an arbitrary subsequence of $\{\phi_\epsilon\}$, there exist Fujii (1990) a function $\phi(x)$ and a subsequence (still denoted by $\{\phi_\epsilon\}$) such that

$$\phi_\epsilon \to \phi,\ \frac{\partial\phi_\epsilon}{\partial x_i} \to \frac{\partial\phi}{\partial x_i},\ \frac{\partial^2\phi_\epsilon}{\partial x_i\partial x_j} \to \frac{\partial^2\phi}{\partial x_i\partial x_j} \quad (i,j = 1,2) \tag{10}$$

uniformly on every compact (bounded and closed) subdomain of Ω. Furthermore, function $\phi(x)$ is shown in Fujii (1990) to be the solution of the following boundary value problem:

$$\Delta\phi(x) - k(x)\phi(x) = 0 \qquad (x \in \Omega), \tag{11}$$

$$\phi(x) = -\frac{\partial u}{\partial n}\rho(x) \qquad (x \in \Gamma), \tag{12}$$

where $\partial/(\partial n)$ denotes the directional differentiation along the outward normal $\vec{n}$. Note that boundary value problem (11), (12) unambiguously determines a function $\phi(x)$ on Ω. This means that function $\phi(x)$ does not depend on the choice of the subsequence of $\{\phi_\epsilon\}$ at all. In other words, we see that the entire sequence $\{\phi_\epsilon\}$ converges uniformly on every compact subdomain of Ω like (10). Let us define another sequence $\{\psi_\epsilon\}$ by

$$\psi_\epsilon(x) \equiv \frac{1}{\epsilon^2}(u_\epsilon(x) - u(x) - \epsilon\phi(x)). \tag{13}$$

We obtain the same result for $\{\psi_\epsilon\}$ as for $\{\phi_\epsilon\}$. Namely, we obtain the following theorem.

THEOREM 2.1 *Sequence $\{\phi_\epsilon\}$ and its derivatives converge uniformly on every compact subdomain of Ω such as (10), where function $\phi(x)$ is the solution of (11), (12). Also sequence $\{\psi_\epsilon\}$ and its derivatives converge uniformly on every compact subdomain of Ω such as*

$$\psi_\epsilon \to \psi,\ \frac{\partial\psi_\epsilon}{\partial x_i} \to \frac{\partial\psi}{\partial x_i},\ \frac{\partial^2\psi_\epsilon}{\partial x_i\partial x_j} \to \frac{\partial^2\psi}{\partial x_i\partial x_j} \quad (i,j=1,2), \tag{14}$$

where, $\psi(x)$ is the solution of

$$\Delta\psi(x) - k(x)\psi(x) = 0 \qquad (x \in \Omega), \tag{15}$$

$$\psi(x) = -\frac{\partial u}{\partial n}\sigma(x) - \frac{\partial\phi}{\partial n}\rho(x) - \frac{1}{2}\frac{\partial^2 u}{\partial n^2}\rho^2(x) \qquad (x \in \Gamma). \tag{16}$$

Note that the boundary value problem (15), (16) admits Gilbarg, Trudinger (1983) a unique solution $\psi(x)$.

We are now in a position to state second-order necessary conditions of Kuhn–Tucker type for optimality. In what follows we assume that Ω is an optimal domain. Let ϵ be a number. Let us introduce a new domain Ω_ϵ as before. It is easy to observe that for any smooth function $w(x)$ of class C^2 the following formula is valid up to the second order of ϵ:

$$\int_{\Omega_\epsilon} w(x)dx - \int_\Omega w(x)dx = \epsilon\int_\Gamma w(x)\rho(x)d\Gamma + \epsilon^2\int_\Gamma\left\{w(x)\sigma(x) + \frac{1}{2}\left(\frac{\partial h}{\partial n} + \frac{w(x)}{R}\right)\rho^2\right\}d\Gamma + o(\epsilon^2). \tag{17}$$

Here, R denotes the radius of curvature of Γ at $d\Gamma$ and is defined to be positive when the curve is concave to the domain. Domain Ω_ϵ should satisfy constraint (4). Then,

$$I(\Omega_\epsilon) = \int_{\Omega_\epsilon} h(x)\,dx = c \tag{18}$$

must hold. This expression, constraint (4), and formula (17) tell us that

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} h(x)\,dx - \int_{\Omega} h(x)\,dx \\ &= \epsilon \int_\Gamma h(x)\rho(x)\,d\Gamma + o(\epsilon) \end{aligned} \tag{19}$$

must hold. From this, we see that ρ must satisfy

$$\int_\Gamma h(x)\rho(x)\,d\Gamma = 0. \tag{20}$$

As to the objective functional, we obtain using (17),

$$J(\Omega_\epsilon; u_\epsilon) - J(\Omega; u) = \epsilon\delta^{(1)}J + \epsilon^2\delta^{(2)}J + o(\epsilon^2). \tag{21}$$

Here, $\delta^{(1)}J$ and $\delta^{(2)}J$ are given by the following expressions:

$$\delta^{(1)}J = \int_\Gamma g(x,u)\rho(x)\,d\Gamma + \int_\Omega \frac{\partial g}{\partial u}(x,u)\phi(x)\,dx; \tag{22}$$

$$\begin{aligned} \delta^{(2)}J &= +\int \frac{\partial g}{\partial u}(x,u)\phi(x)\rho(x)\,d\Gamma \\ &\quad + \frac{1}{2}\int_\Gamma \left\{ \frac{g(x,u)}{R} + \left(\frac{\partial g}{\partial x}(x,u) + \frac{\partial g}{\partial u}(x,u)\nabla u \right) \cdot \vec{n} \right\} \rho^2\,d\Gamma \\ &\quad + \int_\Omega \frac{\partial g}{\partial u}(x,u)\psi(x)\,dx + \frac{1}{2}\int_\Omega \frac{\partial^2 g}{\partial u^2}\phi^2(x)\,dx + o(\epsilon^2). \end{aligned} \tag{23}$$

Here, $\phi(x)$ is the solution of (11) and (12); $\psi(x)$, (15) and (16). Notation $\dfrac{\partial g}{\partial x}(x,u)$ stands for a vector-valued function defined by

$$\frac{\partial g}{\partial x}(x,u) = (\frac{\partial g}{\partial x_1}(x,u),\ \frac{\partial g}{\partial x_2}(x,u)). \tag{24}$$

Now, let us introduce a new function $p(x)$ as the solution of the following boundary value problem:

$$\Delta p(x) - k(x)p(x) = \frac{\partial g}{\partial u}(x, u(x)) \quad (x \in \Omega), \tag{25}$$

$$p(x) = 0 \quad (x \in \Gamma); \tag{26}$$

$p(x)$ being called the adjoint variable. Note that boundary value problem (25), (26) admits a unique solution. Multiplying both sides of (25) by $\phi(x)$, integrating by parts, and using (11), (12), we get

$$\int_\Omega \frac{\partial g}{\partial u}(x,u)\phi(x)\,dx = -\int_\Gamma \frac{\partial p}{\partial n}\frac{\partial u}{\partial n}\rho(x)\,d\Gamma. \tag{27}$$

Similarly, we obtain

$$\int_\Omega \frac{\partial g}{\partial u}(x,u)\psi(x)\,dx = -\int_\Gamma \frac{\partial p}{\partial n}\left(\frac{\partial \phi}{\partial n}\rho + \frac{1}{2}\frac{\partial^2 u}{\partial n^2}\rho^2\right)\,d\Gamma \tag{28}$$

Substitution of (27) into (22) yields

$$\delta^{(1)}J = \int_\Gamma \left\{ g(x,u) - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n}\right\}\rho(x)\,d\Gamma. \tag{29}$$

Substitution of (28) into (23) gives

$$\begin{aligned}\delta^{(2)}J \;&=\; \frac{1}{2}\int_\Gamma \left\{\frac{g(x,u)}{R} + \frac{\partial g}{\partial x}\cdot\vec{n} - \frac{\partial g}{\partial u}\frac{\partial u}{\partial n} - \frac{\partial^2 u}{\partial n^2}\frac{\partial p}{\partial n}\right\}\rho^2\,d\Gamma \\ &\quad - \int_\Gamma \frac{\partial p}{\partial n}\frac{\partial \phi}{\partial n}\rho\,d\Gamma + \frac{1}{2}\int_\Omega \frac{\partial^2 g}{\partial u^2}\phi^2\,dx.\end{aligned} \tag{30}$$

Since domain Ω is the optimal domain, $\delta^{(1)}J$ must vanish for all $\rho(x)$ that satisfy (20) (condition for stationarity). Simultaneously, we know that, also in (infinite dimensional) Hilbert space $\mathcal{H}$, $(S^\perp)^\perp = S$ is true provided that subspace S is finite dimensional. Putting $S = \{h\}$, we see that we can use the Lagrange multiplier rule in our case. If we define $\delta^{(1)}I$ and $\delta^{(2)}I$ by

$$I(\Omega_\epsilon) - I(\Omega) = \epsilon\delta^{(1)}I + \epsilon^2\delta^{(2)}I + o(\epsilon^2),$$

we have from (17) the following expression for $\delta^{(2)}I$:

$$\delta^{(2)}I = \frac{1}{2}\int_\Gamma \left(\frac{h}{R} + \frac{\partial h}{\partial n}\right)\rho^2\,d\Gamma. \tag{31}$$

Thus, we obtain the following theorem (see Fujii, 1990).

THEOREM 2.2 (Necessary conditions of Kuhn–Tucker type) *Necessary conditions that Ω attains a minimum for the optimization problem are that there exist a constant (Lagrange multiplier) λ such that*

$$g(x,u) - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n} - \lambda h(x) = 0 \quad (x \in \Gamma), \tag{32}$$

and that, for every $\rho(s)$ which satisfies (20),

$$\delta^{(2)}J - \lambda\delta^{(2)}I \geq 0 \tag{33}$$

holds. Here, $\delta^{(2)}J$ and $\delta^{(2)}I$ are given by (30) and (31), respectively.

3. Sufficient conditions

In the previous section, we have given a set of necessary conditions. In this section, we shall give a set of sufficient conditions for optimality. To this end, we introduce an appropriate topology to a class of domains. The class of domains is a class of domains each of which has a differentiable manifold of class C^6 as its boundary. Let Ω denote a domain in the class. Let Γ be its boundary. We can introduce an arclength s to Γ which is measured from $x_0 \in \Gamma$. Let $W^{m,p}(\Gamma)$ denote a space of functions of s whose distributional derivatives up to m-th order belong to $L^p(\Gamma)$. In what follows, we use abbreviations $W^{m,p}$ and L^p in place of $W^{m,p}(\Gamma)$, $L^p(\Gamma)$, respectively. The norm of $u \in W^{m,p}$ is defined by

$$\| u \|_{m,p} \equiv \left\{ \sum_{0 \le \alpha \le m} \| D^\alpha u \|_p^p \right\}^{1/p} \qquad (1 \le p < \infty), \tag{34}$$

where α stands for integers, D^α denotes α times differentiation with respect to s. and $\| \cdot \|_p$ stands for L^p norm. With this norm, $W^{m,p}$ is a Sobolev space.

Let Ω' be another domain which is 'close' to Ω. At each point on Γ, plot the normal. Let $\rho(s)$ be the length of the normal cut by Γ and $\Gamma' \equiv \partial\Omega'$, and is defined to be positive when Γ' lies in the exterior of Ω. Obviously, $\rho(\cdot)$ belongs to $W^{4,p}$. Let us define $\rho(\Gamma, \Gamma')$ by

$$\rho(\Gamma, \Gamma') \equiv \sup_{x_0 \in \Gamma} \|\rho(\cdot)\|_{4,p}. \tag{35}$$

Similarly, we can define $\rho(\Gamma', \Gamma)$ by

$$\rho(\Gamma', \Gamma) \equiv \sup_{x_0 \in \Gamma'} \|\rho'(\cdot)\|_{4,p}. \tag{36}$$

Thus, we can introduce a 'distance' between Ω and Ω' by

$$d(\Omega, \Omega') \equiv d(\Gamma, \Gamma') \equiv \max(\rho(\Gamma, \Gamma'), \rho(\Gamma', \Gamma)) \tag{37}$$

Thus, we can introduce a notion of ϵ-neighborhood $N_\epsilon(\Omega)$ of domain Ω by the following definition.

$$N_\epsilon(\Omega) \equiv \left\{ \Omega' \in C^6\text{class} | d(\Omega, \Omega') < \epsilon \right\}. \tag{38}$$

We are now in a position to give the notion of an isolated local minimum of the shape optimization problem considered.

DEFINITION 3.1 *We say that Ω attains an isolated local minimum if there exists a positive number ϵ such that for any domain Ω' $(\neq \Omega) \in N_\epsilon(\Omega)$*

$$J(\Omega'; u') > J(\Omega, U) \tag{39}$$

holds, where u' is the solution of the boundary value problem on Ω'.

Let Ω be an admissible domain; i.e., Ω is assumed to satisfy (4). Let $\{\Omega_n\}$ be a sequence of admissible domains such that $d(\Omega_n, \Omega) \to 0$. Let $\rho_n(s)$ be the length of the segment on the normal of Ω at s cut by Γ and Γ', where s stands for an appropriately introduced arclength on Γ. From the assumption,

$$||\rho_n(\cdot)||_{4,p} \to 0 \qquad (n \to \infty). \tag{40}$$

Let $\epsilon_n \equiv ||\rho_n(\cdot)||_{4,p}$ and $\xi_n(s) \equiv \epsilon_n^{-1}\rho_n(s)$, then

$$\epsilon_n \to 0(n \to \infty), \qquad ||\xi_n(\cdot)||_{4,p} = 1 \quad (n = 1, 2, \cdots). \tag{41}$$

Thus, $\{\xi_n\}$ is a norm bounded sequence in $W^{4,p}$. Let $C_B^3(\Gamma)$ be the space of functions whose derivatives up to the third order are bounded on Γ. As is well known, the norm of $u \in C_B^3(\Gamma)$ is defined by

$$||u; C_B^3(\Gamma)|| \equiv \max_{0 \le \alpha \le 3} \sup_{x \in \Gamma} |D^\alpha u(x)|.$$

By virtue of the Rellich–Kondrachov theorem, Adams (1975), we know that the imbedding $W^{4,p}(\Gamma) \to C_B^3(\Gamma)$ is compact. Therefore, there exists a subsequence of $\{\xi_n\}$, still denoted by $\{\xi_n\}$, such that $\xi_n \to \xi$ in $C_B^3(\Gamma)$. For the time being, we shall confine ourselves to this subsequence. Let u_n be a solution of the boundary value problem:

$$\begin{aligned} \Delta u_n(x) - k(x)u_n(x) &= f(x) \qquad (x \in \Omega_n), &(42)\\ u_n(x) &= \kappa \qquad (x \in \Gamma_n \equiv \partial\Omega_n). &(43) \end{aligned}$$

Let us define $\phi_n(x)$ by

$$\phi_n(x) \equiv \epsilon_n^{-1}(u_(x) - u(x)),$$

where, of course, $u(x)$ is the solution of (1) and (2). Then by the argument similar to ref. Fujii (1990), we can show that there exists a function $\phi \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that, as $n \to \infty$,

$$\phi_n \to \phi,\ \frac{\partial \phi_n}{\partial x_i} \to \frac{\partial \phi}{\partial x_i},\ \frac{\partial^2 \phi_n}{\partial x_i \partial x_j} \to \frac{\partial^2 \phi}{\partial x_i \partial x_j} \qquad (i, j = 1, 2) \tag{44}$$

uniformly on every compact subdomain of Ω. Hence, we see that

$$u_n - u = \epsilon_n \phi + o(\epsilon_n) \ \ (x \in \Omega \cap \Omega_n).$$

Here, $\phi(x)$ is the solution of the following boundary value problem.

$$\begin{aligned} \Delta\phi(x) - k(x)\phi(x) &= 0 \qquad (x \in \Omega), &(45)\\ \phi(x) &= -\frac{\partial u}{\partial n}\xi(x) \qquad (x \in \Gamma). &(46) \end{aligned}$$

When we define $\psi_n(x)$ by

$$\psi_n(x) \equiv \epsilon_n^{-2}(u_n(x) - u(x) - \epsilon\phi(x)),$$

we know that there exists a function such that, as $n \to \infty$,

$$\psi_n \to \psi,\ \frac{\partial \psi_n}{\partial x_i} \to \frac{\partial \psi}{\partial x_i},\ \frac{\partial^2 \psi_n}{\partial x_i \partial x_j} \to \frac{\partial^2 \psi}{\partial x_i \partial x_j} \qquad (i, j = 1, 2) \tag{47}$$

uniformly on every compact subdomain of Ω. In other words, we see that

$$u_n - u = \epsilon_n \phi + \epsilon_n^2 \psi + o(\epsilon_n) \ \ (x \in \Omega \cap \Omega_n).$$

Here, $\psi(x)$ is the solution of the boundary value problem:

$$\Delta\psi(x) - k(x)\psi(x) = 0 \qquad (x \in \Omega), \tag{48}$$

$$\psi(x) = -\frac{\partial \phi}{\partial n}\xi(x) - \frac{1}{2}\frac{\partial^2 u}{\partial n^2}\xi^2(x) \qquad (x \in \Gamma). \tag{49}$$

Let us define $\delta^{(2)} I(\rho(\cdot))$ and $\delta^{(2)} J(\rho(\cdot))$ by

$$\delta^{(2)} I(\rho(\cdot)) = \frac{1}{2}\int_\Gamma \left(\frac{h}{R} + \frac{\partial h}{\partial n}\right) \rho^2 \, d\Gamma, \tag{50}$$

and

$$\begin{aligned} \delta^{(2)} J(\rho(\cdot)) \ &= \ \frac{1}{2}\int_\Gamma \left\{ \frac{g(x,u)}{R} + \frac{\partial g}{\partial x}\cdot \vec{n} - \frac{\partial g}{\partial u}\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2} \right\} \rho^2 \, d\Gamma \\ &\quad - \int_\Gamma \frac{\partial p}{\partial n}\frac{\partial \phi}{\partial n}\rho \, d\Gamma + \frac{1}{2}\int_\Omega \frac{\partial^2 g}{\partial u^2}\phi^2 \, dx. \end{aligned} \tag{51}$$

Here, ϕ is the solution of (11), (12) corresponding to $\rho(\cdot)$. We can now give an outline of the proof to the following theorem, the main result of this paper.

THEOREM 3.1 *A domain Ω of class C^6 attains an isolated local minimum if there exists a constant λ such that*

$$g(x, u) - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n} - \lambda h(x) = 0 \quad (x \in \Gamma), \tag{52}$$

and if, for any $\rho(s)$ of class C^3 that satisfies (20) and does not vanish,

$$\delta^{(2)} J(\rho(\cdot)) - \lambda \delta^{(2)} I(\rho(\cdot)) > 0 \tag{53}$$

holds. Here, p is the solution of (25) and (26), and ϕ — the solution of (11) and (12).

OUTLINE OF THE PROOF. Let us suppose the contrary to the conclusion of the theorem. That is, Ω would not be an isolated local minimum. Then, there would exist a sequence $\{\Omega_n\}$ $(\Omega_n \neq \Omega)$ of domains of class C^6 such that $d(\Omega, \Omega_n) \to 0$ and

$$J(\Omega_n; u_n) \leq J(\Omega; u). \tag{54}$$

Let us choose a point x_0 on Γ and fix it. Let us measure arclengths s from x_0. Let $\rho_n(s)$ be the length of the normal segment cut by Γ and Γ_n; it is defined to be positive provided the segment lies on the outward normal. From the definition, $||\rho(\cdot)||_{4,p}$ would tend to zero because $d(\Omega, \Omega') \to 0$. In other words, ϵ_n would approach zero, where

$$\epsilon_n \equiv ||\rho_n(\cdot)||_{4,p}.$$

Note that $\epsilon_n \neq 0$. Let $\xi_n(\cdot) \in W^{4,p}(\Gamma)$ be defined by

$$\xi_n = \epsilon_n^{-1}\rho_n(s); \tag{55}$$

then, $||\xi_n(\cdot)||_{4,p} = 1$ for any n. That is, $\{\xi_n(\cdot)\}$ would be a norm–bounded sequence. Since the imbedding $W^{4,p}(\Gamma) \to C_B^3(\Gamma)$ is compact (the Rellich-Kondrachov theorem, Adams, 1975), we could extract a subsequence from $\{\xi_n(\cdot)\}$, still denoted by $\{\xi_n(\cdot)\}$, such that $\xi_n \to \xi$ in $C_B^3(\Gamma)$, where of course $\xi(s)$ would be a function in $C_B^3(\Gamma)$. Since Ω_n must satisfy (4), we would have

$$\begin{aligned} 0 &= \int_{\Omega_n} h(x)dx - \int_{\Omega} h(x)dx \\ &= \epsilon_n \int_\Gamma \xi_n h d\Gamma + \epsilon_n^2 \int_\Gamma \left(\frac{h}{R} + \frac{\partial h}{\partial n}\right)\xi_n^2 d\Gamma + o(\epsilon_n^2). \end{aligned} \tag{56}$$

From this expression, we would know, for any n,

$$\int_\Gamma \xi_n h d\Gamma = 0. \tag{57}$$

Therefore, we would know from the limiting processes that

$$\int_\Gamma \xi h d\Gamma = 0, \tag{58}$$

would be valid for $\xi(\cdot)$; i.e., $\xi(\cdot)$ would satisfy (20).

From (54), we would have the inequality:

$$0 \geq \epsilon_n \int_\Gamma \left\{g(x,u) - \frac{\partial p}{\partial n}\frac{\partial u}{\partial n}\right\}\xi_n d\Gamma + \epsilon_n^2 \delta^{(2)}J(\xi_n(\cdot)) + o(\epsilon_n^2). \tag{59}$$

Substituting (52) into the first term of the r.h.s. of (59) and using (56), we would immediately obtain

$$0 \geq \epsilon_n^2 \left(\delta^{(2)}J(\xi_n(\cdot)) - \lambda\delta^{(2)}I(\xi_n(\cdot))\right) + o(\epsilon_n^2). \tag{60}$$

Divide both sides of (60) by ϵ_n^2 and let $n \to \infty$, then we would get an inequality:

$$0 \geq \delta^{(2)}J(\xi(\cdot)) - \lambda\delta^{(2)}I(\xi(\cdot)). \tag{61}$$

This contradicts (53). The proof is thereby completed. ■

Let us give an illustrative example.

EXAMPLE 3.1 *Banichuk (1976) formulated a problem of maximum torsional rigidity as follows. Let the objective functional* $J(\Omega; u)$ *be defined by*

$$J(\Omega; u) = \int_\Omega (-u)\, dx. \tag{62}$$

Look for a domain Ω *that minimizes this objective functional, where the area of* Ω *remains* π *and* u *is the solution of the boundary value problem:*

$$\Delta u(x) = \ -1 \qquad (x \in \Omega), \tag{63}$$

$$u(x) = \ 0 \qquad (x \in \Gamma). \tag{64}$$

Banichuk showed that the optimal domain is a unit disk. Hereafter, let Ω *be the unit disk with its center at the origin. If we introduce the polar coordinate* (r, θ), u *is given by*

$$u = \frac{1}{4}(1 - r^2). \tag{65}$$

From (25) and (26), we see that p *is the solution of*

$$\Delta p \ = \ -1 \qquad (x \in \Omega), \tag{66}$$

$$u \ = \ 0 \qquad (x \in \Gamma). \tag{67}$$

This is the same boundary value problem as for u. *We see at once that* p *is explicitly given by*

$$p = \frac{1}{4}(1 - r^2). \tag{68}$$

Let a sufficiently smooth function ρ *be defined on* Γ. *Let a boundary variation* δn *of* Ω *be given by* $\delta n = \epsilon\rho$. *In order that this variation be admissible (the area of domains remain* π*),* ρ *must satisfy*

$$\int_\Gamma \rho\, d\Gamma = 0. \tag{69}$$

From (11) and (12), we see that the first variation ϕ *corresponding to* ρ *is given as the solution of the boundary value problem:*

$$\Delta\phi \ = \ 0 \qquad (x \in \Omega), \tag{70}$$

$$\phi \ = \ \frac{1}{2}\rho \qquad (x \in \Gamma). \tag{71}$$

In this case, constant $\lambda = -\dfrac{1}{4}$ *satisfies (32); i.e.,*

$$-\frac{1}{4} - \lambda = 0 \qquad (x \in \Gamma)$$

holds.

The second variation $\delta^{(2)}J$ of the objective functional, corresponding to ρ, is given by

$$\delta^{(2)}J = -\frac{1}{8}\int_\Gamma \rho^2\, d\Gamma + \frac{1}{2}\int_\Gamma \frac{\partial\phi}{\partial n}\rho\, d\Gamma. \tag{72}$$

From (31), we obtain

$$\delta^{(2)}I = \frac{1}{2}\int_\Gamma \rho^2\, d\Gamma. \tag{73}$$

From these relations, we have

$$\delta^{(2)}J - \lambda\delta^{(2)}I = \frac{1}{2}\int_\Gamma \frac{\partial\phi}{\partial n}\rho\, d\Gamma. \tag{74}$$

Using (70) and (71), we observe that

$$\begin{aligned}\frac{1}{2}\int_\Gamma \frac{\partial\phi}{\partial n}\rho\, d\Gamma &= \int_\Gamma \frac{\partial\phi}{\partial n}\phi\, d\Gamma \\ &= \int_\Omega |\nabla\phi|^2\ dx \geq 0.\end{aligned} \tag{75}$$

Here, ∇ stands for the gradient operator. More specifically, ∇ is defined by

$$\nabla\phi(x) = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right).$$

The equality in inequality (75) can occur only if $\nabla\phi \equiv 0$ in Ω. That is, $\phi \equiv$ const. should hold. In view of (71), this implies $\rho \equiv$ const. In order that (69) be valid, ρ should vanish. Hence,

$$\delta^{(2)}J - \lambda\delta^{(2)}I > 0$$

for any ρ that satisfies (69) and does not vanish. Thus, the unit disk Ω satisfies the sufficient conditions.

4. Concluding remarks

In this paper, a set of sufficient conditions for optimality in shape optimization problems is derived; to this end, a special class of domains is introduced. An illustrative example is given. Also a heuristic and intuitive method for deriving necessary conditions of shape optimization problems is surveyed.

Recently, Koski and his colleague, Koski, Silvennoinen (1990), obtained numerical Pareto–optimum solutions for multiobjective optimization problems which appear with piston crowns of engines. Kacimov (1991) has treated a problem of optimal shapes of trenches using conformal mappings. In Section 2, we assumed the existence of the optimal shape (domain). In mathematics, this existence problem is mainly interesting. As to a existence problem, Chenais

(1975) introduced a class of domains in order to show the existence of an optimal domain in domain identification problems. The present author, Fujii (1988) studied an existence problem of an optimal domain in shape optimization problems using Chenais' class of domains. However, this is not complete. Existence problems in shape optimization problems are very important and left to be solved.

Key issues of shape optimization problems are numerical methods. Haslinger and Neittaanmäki (1988) have proposed numerical methods with finite element methods for solutions of boundary value problems. Goto and the present author, Goto, Fujii (1990) proposed a Newton method for numerical solutions of a shape optimization problem.

References

ADAMS R.A. (1975) *Sobolev Spaces*, p. 144, New York, Academic Press.

BANICHUK N.V. (1976) Optimization of elastic bars in torsion, *International Journal of Solids and Structures*, vol. **12**, 275–286.

CEA J. (1981) Problems of shape optimal design, *Optimization of Distributed-Parameter Structures* (edited by E. J. Haug and J. Cea), Alphen aan den Rijn, Holland, Sijthhoff and Noordhoff.

CHENAIS D. (1975) On the existence of a solution in a domain identification problem, *Journal of Mathematical Analysis and Applications*, vol. **52**, 189–219.

DEMS K., MRÓZ K.Z. (1984) Variational approach by means of adjoint systems to structural optimization and sensitivity analysis, Part 2, *International Journal of Solids and Structures*, vol. **20**, 527–552.

FUJII N. (1986A) Necessary conditions for a domain optimization problem in elliptic boundary value problems, *SIAM J. Control and Optimization*, vol. **24**, 346–360.

FUJII N. (1986B) Domain optimization problems with a boundary value problem as a constraint, *Proceedings of the 4th IFAC Symposium on Control of Distributed Parameter Systems*, Los Angeles, 5–9.

FUJII N. (1986C) Second variation and its application in a domain optimization problem, *Proceedings of the 4th IFAC Symposium on Control of Distributed Parameter Systems*, Los Angeles, 431–436.

FUJII N. (1990) Second-order necessary conditions in a domain optimization problem, *J. Optimization Theory and Applications*, vol. **65**, No. 2, 223-244.

FUJII N. (1988) Lower semicontinuity in domain optimization problems, *Journal of Optimization Theory and Applications*, vol. **59**, 407–422.

GARABEDIAN P.R., SCHIFFER M. (1953) Convexity of domain functionals, *Journal d'Analyse Mathématique*, vol. **2**, 281–368.

GILBARG D., TRUDINGER N.S. (1983) *Elliptic Partial Differential Equations of Second Order*, p. 181, Berlin, Springer–Verlag.

GOTO Y., FUJII N. (1990) Second-order numerical method for domain optimization problems, *Journal of Optimization Theory and Applications*, vol. **67**, 533–550.

GOTO Y., FUJII N., MURAMATSU Y. (1990) Second-order necessary optimality conditions for domain optimization problems of the Neumann type, *Journal of Optimization Theory and Applications*, vol. **65**, No.3, 431–445.

HASLINGER J., NEITTAANMÄKI P. (1988) Finite Element Approximation for Shape Optimal Design, Chichester, J. Wiley & Sons.

KACIMOV A. (1991) Steady, two-dimensional flow of ground water to a trench, *Journal of Hydrology*, vol. **127**, 71–83.

KOSKI J., SILVENNOINEN R. (1990) Multicriteria design of ceramic piston crown, *Engineering Costs and Production Economics*, vol **20**, 175–189.

MASANAO T., FUJII N. (1992A) Second-order necessary conditions for domain optimization problems in elastic structures, Part 1, *Journal of Optimization Theory and Applications*, vol. **72**, 355–382.

MASANAO T., FUJII N. (1992B) Second-order necessary conditions for domain optimization problems in elastic structures, Part 2, *Journal of Optimization Theory and Applications*, vol. **72**, 383–401.

PIRONNEAU O. (1973) On optimum problems in Stokes flow, *Journal of Fluid Mechanics*, vol. **59**, 117–128.

PIRONNEAU O. (1974) On optimum design in fluid mechanics, *Journal of Fluid Mechanics*, vol. **64**, 97–110.

SOKOŁOWSKI J., ZOLESIO J.-P. (1992) *Introduction to Shape Optimization*, Berlin, Springer–Verlag.

ZOLESIO J.-P. (1981) The material derivative (or speed) method for shape optimization, *Optimization of Distributed-Parameter Structures* (edited by E. J. Haug and J. Cea), Alphen aan den Rijn, Holland, Sijthoff and Noordhoff.