Control and Cybernetics

vol. **23** (1994) No. 3

Structural sensitivity analysis and optimal design of frames with respect to joint location

by

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In the present paper the structural sensitivity and optimization of linear elastic frames with elastic hinges is discussed. The sensitivity analysis is carried out with respect to hinge location, stiffness and prestress. The sensitivity operators for structural response for load combined with initial distortions, forced vibrations, eigenvibrations and buckling are derived using unified variational approach. Illustrative numerical examples are presented.

1. Introduction

Structural engineers often design internal hinges or semi rigid joints in beam or frame structures. The reason is that these connections can facilitate manufacturing and erection process and therefore diminish the cost of the structure. For example the hinges are often designed in prefabricated reinforced concrete frames, whereas the semi rigid joints are widely used in steel frames. The second reason for designing flexible connections is that they can improve the static or dynamic response of a structure by decreasing its stiffness and increasing compliance. Lower stiffness is particularly advantageous when high distortions are induced in a structure, e.g. due to temperature changes or imposed displacements, Garstecki, Mróz (1987).

Computer analysis and experimental study of structures with semi rigid connections and connections with gaps have recently focused great attention, Gawęcki (1992). Application of optimization formalism to finding of the best position and stiffness parameters of joints can remarkably improve the structural response, Garstecki (1988), Garstecki, Thermann (1992). The effects of optimization of joints can be similar to the effects of optimization of supports since the joints can be considered as internal constraints connecting substructures, whereas supports play the role of boundary constraints. The problem of optimal position and stiffness of supports has been discussed in a number of papers e.g. Åkesson, Olhoff (1988), Garstecki, Mróz (1987), Mróz, Rozvany (1975), Olhoff, Åkesson (1991), which demonstrated the practical applicability of derived sensitivity operators and optimality conditions.

In the present paper the structural sensitivity of linear elastic frames with elastic hinges is discussed. The sensitivity operators with respect to variations of hinge position, stiffness and prestress are derived. For better understanding only simple types of hinges will be considered, where the discontinuity of the slope of deflection line occurs, however, the theory can easily be generalized for joints in which discontinuities of other kinematic fields occur, for example the discontinuity of displacement field or torsion angle. The formulation can also be extended to plates, Dems, Mróz (1992).

Section 2 is devoted to structures subject to static load and to initial distortions. Forced vibrations are considered in Section 3. In Sections 4 and 5 the sensitivity of eigenfrequency and buckling load are discussed, respectively. Generalization of the theory is discussed in Section 6. Section 7 presents numerical examples which illustrate the application of the derived formulae. Section 8 contains concluding remarks.

2. Structures subject to load and initial distortion

Consider a frame or beam structure, illustrated schematically in Fig.1a. The beam is rigidly supported at the point a and elastically supported at the point b with stiffness coefficient K_b. Let R_b denotes the reactions, whereas u denotes transverse displacement. For brevity we will use the simple beam theory, hence M and k denote the bending moment and curvature $k = -u''$, respectively. The generalization to bending combined with longitudinal strain or to Timoshenko beam theory is straightforward. In the latter case M and k should be replaced by vectors of generalized stress and strain, $\boldsymbol{M} = \{M, T\}$ and $\boldsymbol{k} = \{k, v\}$, where T and v denote shear force and average shear strain, respectively.

Assume that there is an elastic hinge at an unspecified point $x = x_z$, where the discontinuity of the slope of deflection line occurs with a step κ_z

$$\kappa_z = u'_{z-} - u'_{z+} \tag{1}$$

which for brevity will be called hinge rotation.

Let us assume that the structure is subject to load p (Fig.1a) and that initial distortion k^i is induced. Note that in general k^i is kinematically inadmissible, therefore elastic strain k^e occurs so that the total strain k is admissible, where

$$k = k^i + k^e, \qquad M = M^e = D\,k^e \tag{2}$$

Here D denotes the bending stiffness coefficient. In the case of simple beam

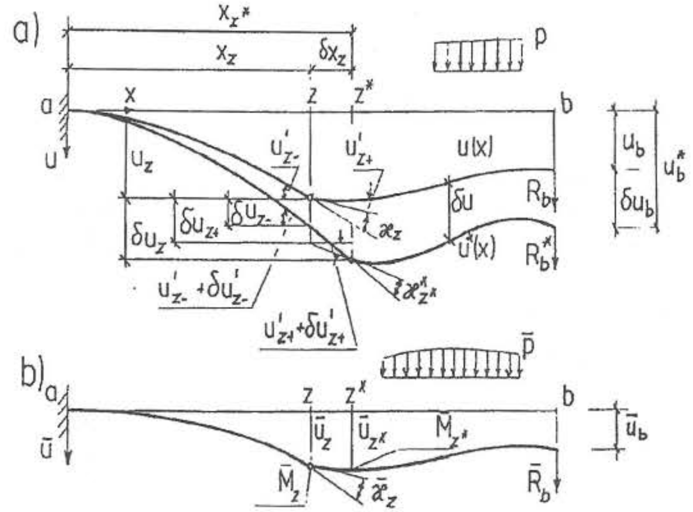

Figure 1. Elastic beam. a) Original structure: displacement field and its variation. b) Adjoint structure.

theory $D = EI$. In the case when M and k represent vectors of generalized stress and strain then D represents a cross sectional stiffness matrix.

As a measure of structural response the following functional will be assumed

$$G = \int_L F(u, M)\, dx \tag{3}$$

where F is an arbitrary Gateaux differentiable function of displacements u and stress M and the integration is performed over the length of all structural elements. The functional (3) can play the role of the objective function or constraint.

Our aim is to find the sensitivity derivatives of the functional (3) with respect to small variations of the scalar control parameters specifying the joint at x_z. Figure 1a also shows the structure for which x_z , κ_z and M_z take the new values x^*_{z*} , κ^*_{z*} and M^*_{z*} , whereas the external load p and initial distortion k^i remain unchanged. All kinematic and static fields of this structure are indicated by the superscript asterisk.

The variation of (3) takes the form

$$\delta G = \int_L \left(\frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial M} \delta M \right)\, dx \tag{4}$$

where the variations $\delta u = u^* - u$ and $\delta M = M^* - M$ (Fig.1a) are implicit functions of variations δx_z, $\delta \kappa_z$ and δM_z. In order to transform (4) to an explicit form we introduce an adjoint problem. Assume that the adjoint structure, which is shown in Fig.1b, is similar to the primary structure and is subject to the fictitious load $\overline{p}$ and initial distortion $\overline{k}^i$

$$\overline{p}(x,t) = \frac{\partial F}{\partial u}, \qquad \overline{k}^i = \frac{\partial F}{\partial M} \tag{5}$$

The respective kinematic and static fields of the adjoint structure will be indicated by bar. The virtual work theorem with the use of forces from the adjoint problem (Fig.1b) and variations of the primary kinematic fields (Fig.1a), yields

$$\begin{aligned} &\int_L (\overline{p}\,\delta u - \overline{M}\,\delta k)\,dx + \overline{M}_z\,\kappa_z - \overline{M}_{z*}\,\kappa^*_{z*} - \overline{M}_z(k^*_{z-} - k_{z+})\,\delta x_z \\ &\quad + \overline{R}_b\,\delta u_b = 0 \end{aligned} \tag{6}$$

Conversely, using adjoint kinematic fields and variations of primary static fields we obtain

$$\int_L (-\delta M\,\overline{k})\,dx - (M^*_z - M_z)\overline{\kappa}_z - (M^*_z - M_z)\,\overline{\kappa}_{z+}\,\delta x_z + \delta R_b\,\overline{u}_b = 0 \tag{7}$$

Henceforth we will use the notation for the integrals $\int_L \ldots\, dx = \int_0^{z-} \ldots\, dx + \int_{z+}^{L} \ldots\, dx$ where the integration is performed with the extraction of the singular point $x = x_z$. The integrals on the infinitely small region $\delta x_z = x_{z+} - x_{z-}$ are represented in (6) and (7) by the terms standing with δx_z. Subtracting (6) from (7) we obtain

$$\begin{aligned} &\int_L (\overline{p}\,\delta u - \overline{M}\,\delta k + \overline{k}\,\delta M)\,dx = -\overline{M}_z\,\kappa_z + \overline{M}_{z*}\,\kappa^*_{z*} + \\ &\quad -(M^*_z - M_z)\overline{\kappa}_z + \overline{M}_z(k^*_{z-} - k_{z+})\,\delta x_z - (M^*_z - M_z)\,\overline{\kappa}_{z+}\,\delta x_z + \\ &\quad -\overline{R}_b\,\delta u_b + \delta R_b\,\overline{u}_b = 0 \end{aligned} \tag{8}$$

Let us first transform

$$-\overline{M}\,\delta k + \overline{k}\,\delta M = -\overline{M}\,\delta k + (\overline{k}^i + \overline{k}^e)\,\delta M = \overline{k}^i\,\delta M \tag{9}$$

because $\overline{M}\,\delta k^e = D\,\overline{k}^e\,\delta k^e = \overline{k}^e\,D\,\delta k^e = \overline{k}^e\,\delta M$. Introducing (9) into (8) and comparing it with (5) we see that the left-hand side of (8) is equal to δG in (4). Let us also observe that due to the same support stiffness K_b in primary and adjoint structures, the last two terms in (8) vanish, because $\overline{R}_b\,\delta u_b = K_b\overline{u}_b\,\delta u_b = \overline{u}_b\,\delta R_b$.
Since all variations are infinitely small we can write

$$\overline{M}_{z*} = \overline{M}_z + \overline{M}'_z\,\delta x, \qquad M^*_z - M_z = \delta M_z - M'_z\,\delta x \tag{10}$$

$$\delta\kappa_z = \kappa^*_{z*} - \kappa_z = \delta u'_{z-} - \delta u'_{z+} + (u''_{z-} - u''_{z+})\,\delta x_z = \delta u'_{z-} - \delta u'_{z+} \tag{11}$$

because $u''_{z-} - u''_{z+} = k_{z-} - k_{z+} = 0$. Using (9)–(11) and retaining only the terms linear with respect to variations we transform (8) to the form

$$\delta G = \overline{M}_z\, \delta\kappa_z - \overline{\kappa}_z\, \delta M_z + (\overline{M}'_z \kappa_z + M'_z \overline{\kappa}_z)\, \delta x_z \tag{12}$$

Note that the variations $\delta\kappa_z$ and δM_z are mutually dependent since they have to satisfy the physical law of the joint. Let us consider elastic hinge with stiffness coefficient K_z and initial distortion κ_z^i. Denoting by κ_z the total hinge rotation and by κ_z^e its elastic part we can write

$$\kappa_z = \kappa_z^i + \kappa_z^e, \qquad M_z = M_z^e = K_z\, \kappa_z^e \tag{13}$$

Hence

$$\delta M_z = \delta K_z\, \kappa_z^e + K_z\, (\delta\kappa_z - \delta\kappa_z^i) \tag{14}$$

Assume that the initial distortion in the hinge of the adjoint structure (Fig.1b) is equal to zero, $\overline{\kappa}_z^i = 0$, and the hinge stiffness $\overline{K}_z = K_z$. Therefore

$$\overline{M}_z = K_z\, \overline{\kappa}_z^e = K_z\, \overline{\kappa}_z \tag{15}$$

Substituting (13)–(15) into (12) we can express the variation of G explicitly in terms of variations of location δx_z, stiffness δK_z and initial distortion of the hinge $\delta\kappa_z^i$

$$\delta G = \overline{M}_z\, \delta\kappa_z^i - \overline{\kappa}_z\, \delta K_z\, \kappa_z^e + (\overline{M}'_z\, \kappa_z + M'_z\, \overline{\kappa}_z)\, \delta x_z \tag{16}$$

Here κ_z and $\overline{\kappa}_z$ denote the total hinge rotations in the primary and adjoint structures, κ_z^e denotes the elastic part of κ_z according to (13), $\overline{M}_z$ denotes the adjoint bending moment in elastic hinge, and M'_z denotes the shear force.

The formulae (12)–(16) can be applied to various particular problems of optimal design and active control of structures. In the classic case when the structure is subject to loads only and the functional G coincides with the total complementary energy, then according to (5) $\overline{p} = 0$, whereas the initial distortion is equal to the total primary strain $\overline{k}^i = k$ which is kinematically admissible. Therefore $\overline{k}^e = 0$, $\overline{M} = 0$ and $\overline{M}_z = 0$, hence the variation (16) reduces to

$$\delta G = -\kappa_z\, \delta K_z\, \kappa_z + M'_z\, \kappa_z\, \delta x_z = -\kappa_z\, \delta K_z\, \kappa_z + T_z\, \kappa_z\, \delta x_z \tag{17}$$

From (17) it follows that extremum condition for δG with respect to δx_z reads $T_z \kappa_z = 0$. Minimum compliance will be for $\kappa_z = 0$ and maximum for $T_z = 0$.

In the following particular types of joints will be discussed.

Perfectly rigid joint with initial distortion.

Subject to sensitivity analysis is hinge location x_z and initial distortion κ_z^i, whereas the elastic parts of rotations vanish, $\kappa_z^e = \overline{\kappa}_z^e = 0$. Hence the variation (16) takes the form

$$\delta G = \overline{M}_z \, \delta\kappa_z^i + \overline{M}_z' \, \kappa_z^i \, \delta x_z \tag{18}$$

Ideal hinge.

In this case bending moments vanish, $M_z = \overline{M}_z = 0$, and the variation (16) reduces to

$$\delta G = (\overline{M}_z' \, \kappa_z + M_z' \, \overline{\kappa}_z) \, \delta x_z = (\overline{T}_z \, \kappa_z + T_z \, \overline{\kappa}_z) \, \delta x_z \tag{19}$$

Cost of joint taken into consideration.

Let us express the cost of joint as a function of generalized stress in the joint $f = f(M_z)$, Rozvany, Mróz (1975). Consider a functional

$$G = \int_L F(u,M)\,dx + f(M_z) = \int_L [F(u,M) + f(M_z)\,\delta(x - x_z)]\,dx \tag{20}$$

where $\delta(x - x_z)$ denotes Dirac's pseudo function. The initial distortion imposed on the adjoint structure will be equal to

$$\overline{k}^i = \frac{\partial F}{\partial M} + \frac{\partial f}{\partial M_z}\,\delta(x - x_z) \tag{21}$$

We notice that to allow for the cost of joint the concentrated initial distortion must be superimposed on the initial distortion and load, which were specified by (5).
The derived formulae for sensitivity derivatives remain valid.

3. Forced vibrations

Consider again a frame type structure, illustrated schematically in Fig.1a. Let the structure be subjected to arbitrary load $p(x,t)$ with vanishing initial distortions. Small vibrations without damping are allowed for. We restrict the sensitivity analysis to the following functional

$$G = \int_{t_0}^{t_1} \int_L F(u)\,dx\,dt \tag{22}$$

with specified time interval (t_0, t_1). In a special case the function G can represent the displacement at a prescribed point x_0, then $F(u) = u(x)\,\delta(x - x_0)$, with

$\delta(x)$ denoting the Dirac pseudo-function. More general behavioral functionals as those proposed in Haug, Choi, Komkov (1986) can easily be introduced.
We will derive the sensitivity derivatives of (22) with respect to variations of hinge position δx_z and hinge stiffness δK_z. For the sake of completeness we will allow for variations of the distributed cross section area δA and stiffness δD , too.
Figure 1a again illustrates the structure for which the above control parameters take new values x_{z*} , K_z^* and A^* , whereas the external load remains unchanged. Assuming the Gateaux differentiability of F, the variation of (22) takes the form

$$\delta G = \int_{t_0}^{t_1} \int_L \frac{\partial F}{\partial u} \delta u \, dx \, dt \tag{23}$$

where the variation $\delta u = u^* - u$ (Fig.1a) is an implicit function of variations δx_z, δK_z, δA. In order to transform (23) to an explicit form we introduce again the adjoint structure, shown in Fig.1b. It is similar to the primary structure and is subject to the fictitious load

$$\overline{p}(x,t) = \frac{\partial F}{\partial u} \tag{24}$$

The virtual work theorem with the use of forces from the adjoint problem (Fig.1b) and variations of the primary kinematic fields (Fig.1a), yields

$$\int_{t_0}^{t_1} \left\{ \int_L (\overline{p}\, \delta u - \rho A \ddot{\overline{u}}\, \delta u - \overline{M}\, \delta k) \, dx \; - \overline{M}_z\, \kappa_z - \overline{M}_{z*}\, \kappa_{z*}^* + \right.$$
$$\left. - \overline{M}_z (k_{z-}^* - k_{z+})\, \delta x_z + \overline{R}_b\, \delta u_b \right\} \, dt = 0 \tag{25}$$

Conversely, using adjoint kinematic fields and variations of primary dynamic fields we obtain

$$\int_{t_0}^{t_1} \left\{ \int_L \left[-\rho (A^* \ddot{u}^* - A\ddot{u})\, u - \delta M\, \overline{k} \right] \, dx - (M_z^* - M_z)\, \overline{\kappa}_z + \right.$$
$$\left. + (M_z^* - M_z) \overline{k}_{z+}\, \delta x_z + \delta R_b\, \overline{u}_b \right\} \, dt = 0 \tag{26}$$

where $\ddot{u} = \partial^2 u / \partial t^2$ and $k = -u'' = -\partial^2 u / \partial x^2$. The physical relations for the primary and adjoint structures are

$$M = Dk, \qquad \overline{M} = D\overline{k} \tag{27}$$

and the respective relations for the hinge are represented by

$$M_z = K_z\, \kappa_z, \qquad \overline{M}_z = K_z\, \overline{\kappa}_z \tag{28}$$

For small variations we can write

$$\delta M = \frac{\partial D}{\partial A} \delta A\, k + D\, \delta k \tag{29}$$

$$\delta M_z = M^*_{z*} - M_z = \delta K_z\, \kappa_z + K_z\, \delta\kappa_z \tag{30}$$

$$A^* \ddot{u}^* - A\ddot{u} = \delta A\, \ddot{u} + A\, \delta\ddot{u} \tag{31}$$

Let us subtract (25) from (26) with the introduction of (10), (11), (29)–(31). Retaining only the terms linear with respect to variations we obtain

$$\begin{aligned}\int_{t_0}^{t_1} \int_L \overline{p}\, \delta u\, dx\, dt = \\ \int_{t_0}^{t_1} \Big\{ \int_L \Big[\rho A \ddot{\overline{u}}\, \delta u - \rho A\, \delta\ddot{u}\, \overline{u} - \rho\, \delta A \ddot{u}\, \overline{u} - \frac{\partial D}{\partial A}\, \delta A\, k\overline{k} \Big]\, dx + \\ + (\overline{M}'_z\, \kappa_z + M'_z\, \overline{\kappa}_z)\, \delta x_z - \kappa_z\, \overline{\kappa}_z\, \delta K_z \Big\}\, dt\end{aligned} \tag{32}$$

The integrands containing $\ddot{u}$ and $\delta\ddot{u}$ can be integrated twice by parts with respect to time assuming the following initial conditions for the primary structure

$$u(x, t_0) = u_0(x), \qquad \dot{u}(x, t_0) = v_0(x) \tag{33}$$

and the terminal conditions for the adjoint structure, Haug, Choi, Komkov (1986)

$$\overline{u}(x, t_1) = 0, \qquad \dot{\overline{u}}(x, t_1) = 0 \tag{34}$$

Using (23) and (24) we transform (32) to the following explicit expression for the variation of G

$$\begin{aligned}\delta G \;=\; & \int_{t_0}^{t_1} \Big\{ \int_L \Big[-\rho\ddot{u}\overline{u} - \frac{\partial D}{\partial A} k\overline{k} \Big]\, \delta A\, dx \\ & + (\overline{M}'_z\, \kappa_z + M'_z\, \overline{\kappa}_z)\, \delta x_z - \kappa_z\, \overline{\kappa}_z\, \delta K_z \Big\}\, dt\end{aligned} \tag{35}$$

where M'_z represents the shear force at $x = x_z$. Note that the sensitivity operator with respect to δA has been known in the literature, Haug, Choi, Komkov (1986).

Formula (35) can be applied to particular problems. The most common practical design problem consists in minimizing the amplitude of vibrations induced by harmonic load $p = \hat{p}(x)\, sin\, \omega t$. Consider the steady state harmonic vibrations

$$u(x, t) = \hat{u}(x)\, sin\, \omega t \tag{36}$$

where $\hat{}$ indicates the amplitude. The objective functional can be assumed in the following form

$$G = \int_L \hat{u}^2\, dx = \frac{2}{T} \int_0^T \int_L [u(x, t)]^2\, dx\, dt \tag{37}$$

with $T = 2\pi/\omega$. According to (24), the load of the adjoint structure is also harmonic, namely

$$\overline{p} = \frac{4}{T} u(x, t) = \frac{4}{T} \hat{u}(x)\, sin\, \omega t \tag{38}$$

and hence the steady-state adjoint vibrations are

$$\overline{u}(x,t) = \hat{\overline{u}}(x)\, sin\, \omega t \tag{39}$$

Introduction of (36) and (39) into (32) and integration with respect to time $t = (0, T)$ with boundary conditions

$$u(x,t) = \overline{u}(x,t) = 0 \tag{40}$$

for $t = 0$ and $t = T$ yields the sensitivity derivative of (37)

$$\begin{aligned} \delta G &= \frac{T}{2}\Big\{ \int_L \Big[\rho\omega^2 \hat{u}\hat{\overline{u}} - (\partial D/\partial A)\hat{k}\hat{\overline{k}}\Big]\; \delta A\; dx + \\ &\quad + (\hat{\overline{M}}'_z\, \hat{\kappa}_z + \hat{M}'_z\, \hat{\overline{\kappa}}_z)\, \delta x_z - \hat{\kappa}_z\, \hat{\overline{\kappa}}_z\, \delta K_z \Big\} \end{aligned} \tag{41}$$

Finally, consider the special case when the concentrated load $\hat{P}$ is applied at the point $x = x_0$, namely

$$p(x,t) = \hat{P}\,\delta(x - x_0)\, sin\, \omega t \tag{42}$$

Assume the objective functional as the square of the amplitude at $x = x_0$

$$G = (\hat{u}_0)^2 = \int_L [\hat{u}(x)]^2\, \delta(x - x_0)\, dx \tag{43}$$

then, the adjoint structure is loaded with a concentrated force

$$\overline{P}(t) = \frac{4}{T}\hat{u}_0\, sin\, \omega t \tag{44}$$

The adjoint displacement $\hat{\overline{u}}$ is proportional to $\hat{u}$, namely $\hat{\overline{u}} = (4\hat{u}_0/\hat{P}T)\hat{u}$. Using the chain rule of differentiation $\delta(\hat{u}_0^2) = 2\hat{u}_0\,\delta\hat{u}_0$ we obtain from (41) the variation of the amplitude of the displacement under the concentrated force

$$\begin{aligned} \delta\hat{u}_0 &= \frac{1}{\hat{P}}\Big\{ \int_L \Big[\rho\omega^2 \hat{u}\hat{u} - (\partial D/\partial A)\hat{k}\hat{k}\Big]\; \delta A\; dx + \\ &\quad + 2\hat{M}'_z\, \hat{\kappa}_z\, \delta x_z - \hat{\kappa}_z\, \hat{\kappa}_z\, \delta K_z \Big\} \end{aligned} \tag{45}$$

4. Eigenvibrations

In the present Section we will examine the sensitivity of eigenfrequencies with respect to variations δx_z , δK_z and $\delta A(x)$. In the following one we will allow for a step-wise change of cross section $A_{z+} - A_{z-}$ and stiffness $D_{z+} - D_{z-}$ and hence the step of curvatures $k_{z+} - k_{z-}$. Consider once more a structure shown in Fig.1a assuming the external load to be equal to zero, $p(x,t) = 0$. Let ξ_i denote the square of eigenfrequency i, and u_i the respective eigenfunction. For brevity the subscript i will be neglected henceforth. The eigenfunction u and its variation $\delta u = u^* - u$ is shown in Fig.1a. The state equation can be written in the following form

$$\int_L (Dkk - A\rho\xi uu)\, dx + K_z\, \kappa_z\, \kappa_z + \sum_b K_b\, u_b\, u_b = 0 \tag{46}$$

where $\int_L \ldots dx = \int_0^{z-} \ldots dx + \int_{z+}^{L} \ldots dx$. The last term in (46) represents the work of elastic supports. The variation of (46) yields

$$\begin{aligned} &\int_L \left[\frac{\partial D}{\partial A} kk\, \delta A + 2Dk\, \delta k - \rho\xi uu\, \delta A - 2A\rho\xi u\, \delta u - A\rho uu\, \delta\xi \right] dx + \\ &\quad + [-(A_{z-} - A_{z+})\rho\xi u_z u_z + D_{z-}\, k_{z-}\, k_{z-} - D_{z+}\, k_{z+}\, k_{z+}]\, \delta x_z + \\ &\quad + \delta K_z\, \kappa_z\, \kappa_z + 2K_z\, \kappa_z\, \delta\kappa_z + 2\sum_b K_b\, u_b\, \delta u_b = 0 \end{aligned} \tag{47}$$

The terms with multiplier 2 in (47) can be transformed with the use of virtual work equation. Accouning for the singularities of variations δu and δk at $x = x_z$, we can write

$$\begin{aligned} &\int_L (Dk\, \delta k - A\rho\xi u\, \delta u)\, dx + K_z\, \kappa_z\, (\delta u'_{z-} - \delta u'_{z+}) + \\ &\quad + T_z\, (\delta u'_{z+} - \delta u'_{z-}) + \sum_b K_b\, u_b\, \delta u_b = 0 \end{aligned} \tag{48}$$

where $\delta u'_{z+} - \delta u'_{z-} = \kappa_z\, \delta x_z$ and $T_z = M'_z$. Subtracting (48) multiplied by 2 from (47) and using (49)

$$\delta\kappa_z = \kappa^*_{z*} - \kappa_z = \delta u'_{z-} - \delta u'_{z+} + (\delta u'_{z-} - \delta u'_{z+})\, \delta x_z \tag{49}$$

we obtain the formula for the variation of the eigenvalue

$$\begin{aligned} \delta\xi \quad = \quad & \frac{1}{\int_L A\rho uu\, dx} \left\{ \int_L \left[\frac{\partial D}{\partial A} kk - \rho\xi uu \right] \delta A\, dx \right. \\ & + [(A_{z+} - A_{z-})\rho\xi u_z u_z + -M_z(\, k_{z-} - k_{z+}) - 2T_z\, \kappa_z]\, \delta x_z \\ & \left. + \kappa_z\, \kappa_z\, \delta K_z \right\} \end{aligned} \tag{50}$$

The term $M_z(\, k_{z-} - k_{z+})$ represents transport of mass due to stepwise change of A at $z = z*$. Formula (50) can also be used for maximization of the distance of adjacent frequencies, Olhoff, Parbery (1984).

5. Buckling

Consider a beam or frame structure schematically illustrated in Fig.1a. Assume that the transverse load p vanishes and that the structure is subject to the axial loading, so that before bifurcation only the axial inner forces are induced. For brevity we will limit the considerations to one axial force P applied at the point b and we will assume rigid supports, hence $u_b = 0$. The state equation and its variation take the following forms

$$\int_L (Dkk - Pu'u')\,dx + K_z\,\kappa_z\,\kappa_z = 0 \tag{51}$$

$$\begin{aligned}&\int_L \left[\frac{\partial D}{\partial A} kk\,\delta A + 2Dk\,\delta k - u'u'\,\delta P - 2Pu'\,\delta u'\right] dx + \\ &\quad + \left[D_{z-}\,k_{z-}\,k_{z-} - D_{z+}\,k_{z+}\,k_{z+} - Pu'_{z-}\,u'_{z-} + Pu'_{z+}\,u'_{z+}\right]\delta x_z + \\ &\quad + \delta K_z\,\kappa_z\,\kappa_z + 2K_z\,\kappa_z\,\delta\kappa_z = 0\end{aligned} \tag{52}$$

The virtual work theorem yields

$$\begin{aligned}&\int_L (Dk\,\delta k - Pu'\,\delta u')\,dx + K_z\,\kappa_z\,(\delta u'_{z-} - \delta u'_{z+}) + \\ &\quad + T_z\,(\delta u'_{z+} - \delta u'_{z-}) = 0\end{aligned} \tag{53}$$

Subtracting (53) multiplied by 2 from (52) we arrive at the formula for the variation of critical force P

$$\begin{aligned}\delta P &= \frac{1}{\int_L u'u'\,dx}\left\{\int_L \frac{\partial D}{\partial A} kk\,\delta A\,dx + [M_z(k_{z+} - k_{z-}) - 2\,T_z\,\kappa_z + \right. \\ &\quad \left. - P(u'_{z-} + u'_{z+})\,\kappa_z\,]\;\delta x_z + \kappa_z\,\kappa_z\,\delta K_z\right\}\end{aligned} \tag{54}$$

Note that $M'_{z-} = T_z + Pu'_z$ and $M'_{z+} = T_z + Pu'_{z+}$, hence the square bracketed term in (54) can be written in the equivalent form :

$$M_z(k_{z+} - k_{z-}) - (M'_{z-} + M'_{z+})\kappa_z.$$

The formula (54) remains valid for elastically supported structures and can easily be generalized for the case of proportional loading, when the structure is subject to a set of forces.

6. Generalizations

The theory presented above can readily be applied for problems of numerous hinges. In this case the subscript z in the derived sensitivity gradients specifies the consecutive number of a hinge, $z = 1, 2, \ldots, N$, and the portions standing with δx_z and δK_z represent N-dimensional sensitivity vectors. Note that the sensitivity gradient for numerous hinges is evaluated in one computation.

Structural symmetry constraints can easily be introduced here, too. Assume that groups of hinges have identical parameters. Such constraint reduces the number of independent components from m to μ. Let us denote the new design variable by $\boldsymbol{\xi}$, where $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_\mu\}$. Then the variations δG can be presented in the following way

$$\delta G = [\partial G/\partial \boldsymbol{s}]^T\,\boldsymbol{A}\,\delta\boldsymbol{\xi} \tag{55}$$

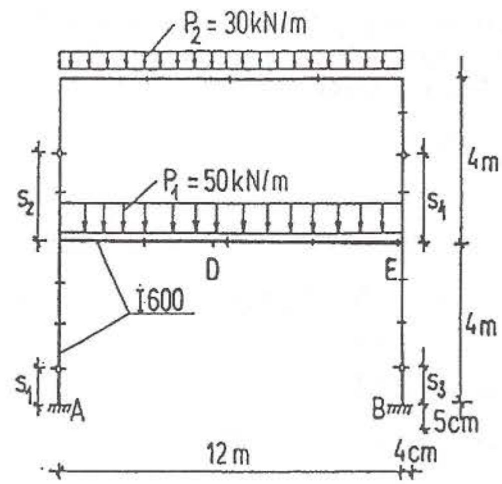

Figure 2. Optimal location of hinges in a steel frame subjected to load and support displacements.

where $\boldsymbol{A}$ is a coincidence matrix with dimension $(m \times \mu)$. For example for the frame shown in Fig.2 we have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{56}$$

7. Illustrative examples

EXAMPLE 1. *Steel frame designed for minimum of maximum stress.*

Consider a frame structure shown in Fig.2, subject to load p_1 and p_2 and to settlement of the support B. Assume symmetrical location of hinges $z_1 = s_1 = s_3$ and $z_2 = s_2 = s_4$. Our task is to find the coordinates of hinges z_1 and z_2 , which minimize the maximum of the absolute value of the bending moment $min\, max\, |M|$. To make the problem differentiable we reformulate it to the following form:

$$objective\quad function: \qquad min\ \ c \tag{57}$$

$$constraints: \qquad M(x) \leq c, \qquad -M(x) \leq c \tag{58}$$

The optimization problem was solved using the feasible direction method. The FE Method was employed for solution of primary and adjoint structures. Gradients of constraints (58) were computed using formulae (3),(5),(19), (55) and (56), at these nodal points x_0, where constraints were violated or were active. In the present case the function $F(u, M)$ in (3) has the form $F = M(x)\,\delta(x - x_0)$, therefore according to (5) concentrated unit initial distortions $\overline{k}^i = \delta(x - x_0)$ are induced in the adjoint structure. The starting point was $s_1 = s_2 = 0$. The optimal coordinates of hinges are $s_1 = 0.91\,m$ and $s_2 = 2.09\,m$. The optimal value of the bending moment $G = min\,max\,|M| = 486.6\,kNm$, whereas for the same structure without hinges we have $G = 1004\,kNm$.

EXAMPLE 2. *Eigenvibrations of a beam.*

To illustrate the sensitivity of eigenfrequencies to variation of hinge position consider a beam of constant cross section, shown in Fig.3a.

Using FE Method we compute eigenfrequencies ω and eigenmodes u for various hinge positions. After normalization $\int A\rho uu\,dx = 1$ we obtain from (50)

$$\frac{d(\omega^2)}{dx_z} = -2T_z\,\kappa_z, \qquad \frac{d\omega}{dx_z} = -\frac{1}{\omega}T_z\,\kappa_z \tag{59}$$

where ω, T_z and κ_z denote the frequency, shear force and hinge rotation, respectively. Fig.3b shows the first three nondimensional eigenfrequencies $\omega_i^* = \omega_i(\rho AL^4/D)^{\frac{1}{2}}$ as functions of hinge position $\eta = x_z/L$. The sensitivity derivatives $(59)_2$ are depicted in Fig.3c. All extrema of ω_i^* are regular and occur at points where the derivatives change signs. At points of maximum the hinge rotations κ_z vanish and the maximal eigenfrequencies and associated eigenmodes are equal to the ones of a beam without a hinge. In fact, insertion of a hinge at an inflexion point of an eigenmode does not change the mode. Local minima of ω_2 and ω_3 occur at points where the shear force T_z vanish, hence, the substructures vibrate with the respective frequency without interaction. The global infima of ω_1^* , ω_2^* , and ω_3^* occur for $\eta \Rightarrow 1$, and in the limit they are equal to eigenfrequencies of a cantilever beam.

The eigenmodes for characteristic hinge positions are plotted in Fig.4. The solid lines in Figs.4a,b,c represent the eigenmodes for hinge positions associated with maximal values of eigenfrequencies. The eigenmodes associated with minimal values of ω_2^* at $\eta = 0.33$ and ω_3^* at $\eta = 0.21$ and $\eta = 0.55$ are depicted by dashed lines in Figs.4b,c, respectively. For comparison the first eigenmodes for $\eta = 0.15$ and $\eta = 0.75$ are shown in Fig.4a and the second eigenmode for $\eta = 0.75$ is shown in Fig.4b.

EXAMPLE 3. *Eigenvibrations of a frame.*

In order to study the sensitivity of a real framework structure to variations of hinge position consider the steel frame shown in Fig.5. Assume the following cross sectional areas A, moments of inertia I and masses m: column 1 - $A = 84.5\,cm^2$, $I = 23130\,cm^4$, $m = 300\,kg/m$, column 2 - $A = 53.8\,cm^2$, $I = 8360\,cm^4$, $m = 300\,kg/m$, beam 3 - $A = 84.5\,cm^2$, $I = 23130\,cm^4$,

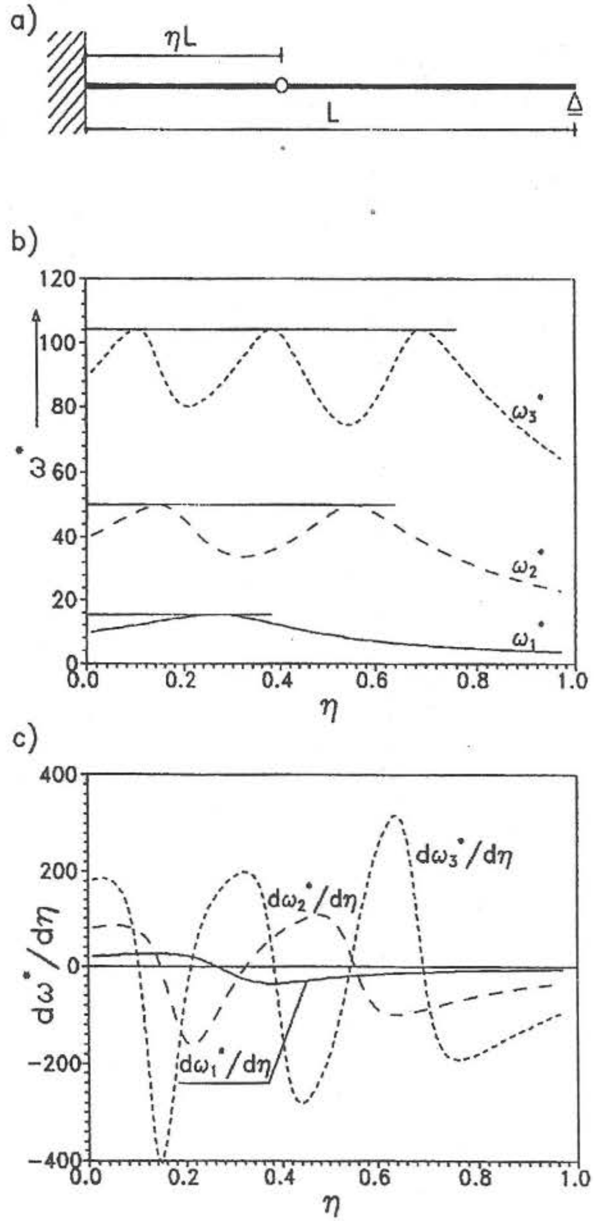

Figure 3. Eigenvibrations of a beam. a) The beam. b) Eigenfrequencies vs. position of hinge. c) Sensitivity of eigenfrequencies for variable hinge position.

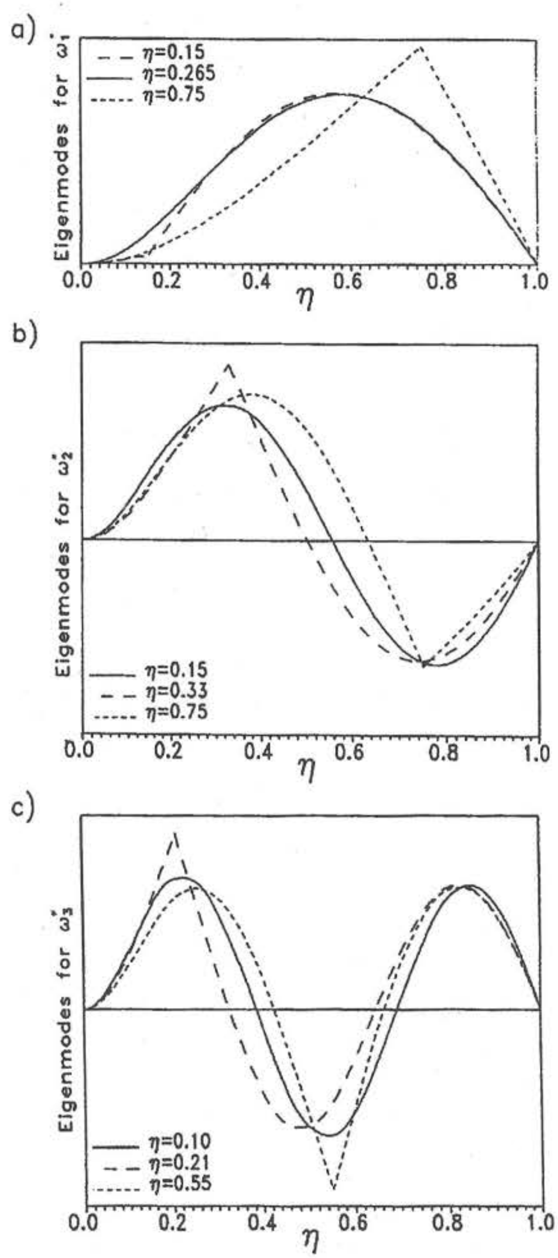

Figure 4. Eigenmodes for various hinge positions. a) First eigenmodes. b) Second eigenmodes. c) Third eigenmodes.

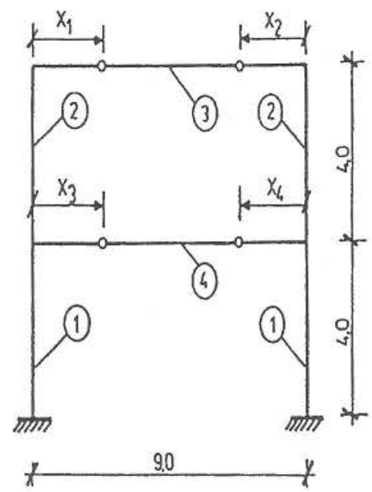

Figure 5. Steel frame structure.

$m = 1000\,kg/m$, beam 4 - $A = 116\,cm^2$, $I = 48200\,cm^4$, $m = 1500\,kg/m$. Let the structure be symmetric and let $x_1 = x_2 = x_3 = x_4 = x$.

We employ FE Method with the normalization of eigenmodes. Using (50) and the above symmetry constraint we obtain

$$\frac{d\omega_i}{dx} = \sum_{z=1}^{4} \frac{(-1)^z}{\omega_i} T_z^i \kappa_z^i \tag{60}$$

were i denotes the number of eigenvalue. The coefficient $(-1)^z$ follows from the negative direction of x_2 and x_4. The first four eigenfrequencies ω_i, $i = 1, \ldots, 4$, are depicted in Fig.6a as functions of x. The intersection point A demonstrates the bimodal eigenvalues $\omega_2 = \omega_3 = 29.71\,rad/s$ at $x = 0.287\,m$. For $x < 0.287\,m$ we have ω_1^a, ω_2^s , ω_3^a , ω_4^s , where superscripts s and a indicate symmetric and antisymmetric eigenmodes, respectively. For $x > 0.287\,m$ we observe ω_1^a, ω_2^a , ω_3^s , ω_4^s. The sensitivity derivatives $d\omega_i/dx$ are shown in Fig.6b. The derivatives of ω_2 and ω_3 are discontinues at $x = 0.287\,m$. The preferable region of hinge position is $0.8\,m < x < 1.7\,m$, where the eigenvalues are well separated. Fig.6 demonstrates that ω_1 remains nearly constant. The plots of ω_3 and ω_4 have regular maxima. Note that the optimality condition $\kappa_z = 0$, observed in example 2, is not valid now, because Equ.(60) involves summation.

EXAMPLE 4. *Buckling of a column.*

Examine the sensitivity of the lowest critical load P with respect to variations of hinge position x_z in the axially loaded structure shown in Fig.7a. Assume the

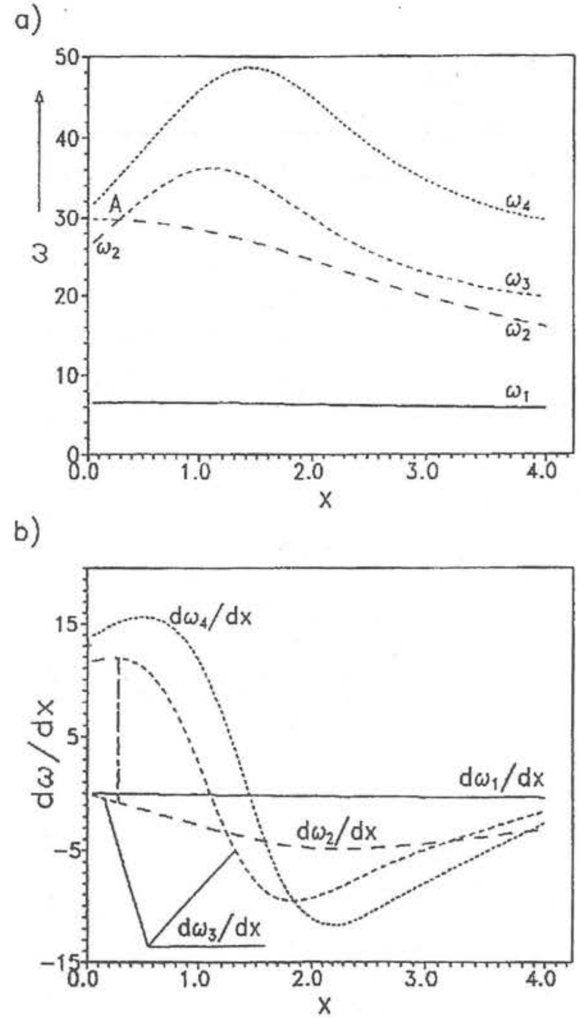

Figure 6. Eigenvibrations of steel frame with symmetrically located hinges. a) Eigenfrequencies vs. hinge position. b) Sensitivity of eigenfrequencies for variable hinge position.

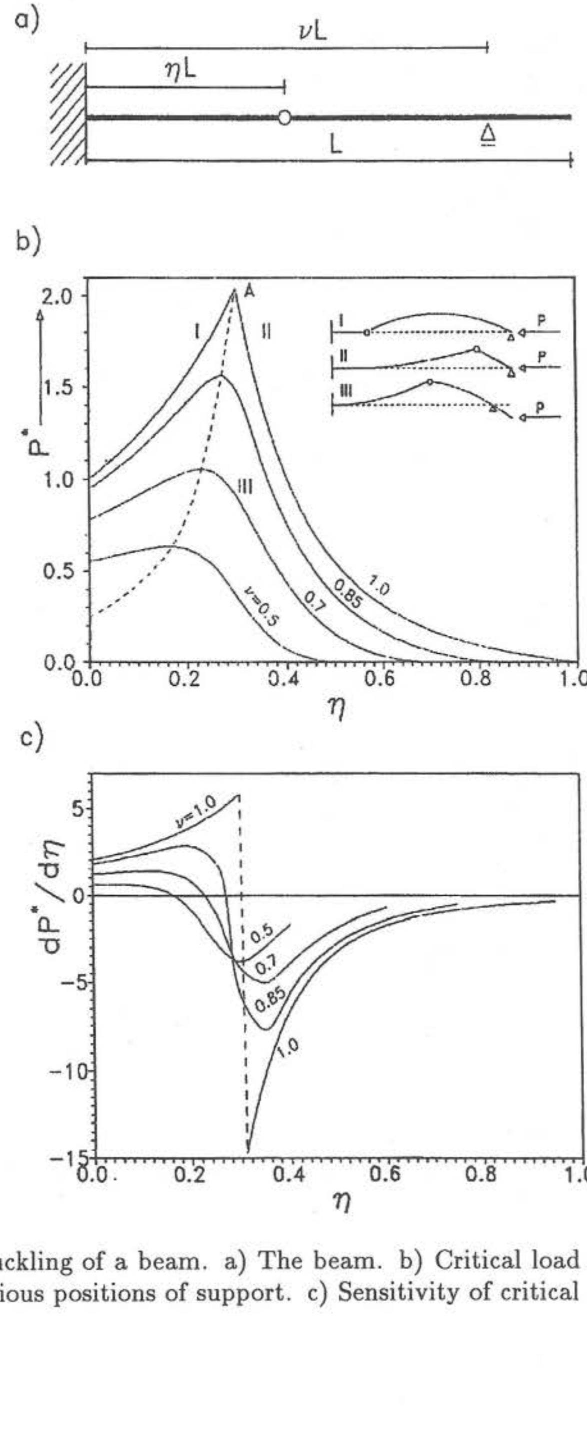

Figure 7. Buckling of a beam. a) The beam. b) Critical load vs. position of hinge for various positions of support. c) Sensitivity of critical force vs. hinge position.

cross section to be constant, $A = const$. Solve the problem for various positions of the support B.

Using the normalization $\int u'u'\,dx = 1$ we obtain from (54)

$$\frac{dP}{dx} = -2\,T_z\,\kappa_z + P(u'_{z-} + u'_{z+})\,\kappa_z \tag{61}$$

The dependence of the nondimensional critical load $P^* = PL^2/\pi D$, on hinge position $\eta = x_z/L$ for various support positions $\nu = x_B/L$ is plotted in Fig.7b, whereas the sensitivity derivative $dP^*/d\eta$ vs. η is plotted in Fig.7c.

For $\nu = 1$ and $\eta < 0.301$ the buckling mode N_I^0 occurs with the left-hand bar remaining straight, whereas for $\eta > 0.301$ the mode N_{II}^0 is observed with the right-hand bar rotating as a rigid body. For $\nu = 1$ and $\eta = 0.301$ the bimodal buckling and a jump of the sensitivity derivative is observed. For $\nu < 1$ the buckling mode N_{III}^0 occurs for all hinge positions, and the curves $P^*(\eta)$ have regular maxima, where $dP^*/dx = 0$, and $\kappa_z = 0$. The optimal hinge locations, denoted in Fig.7b by the dotted line, provide the buckling loads equal to the critical loads of beams without a hinge.

8. Concluding remarks

This paper presents the extension of the theory of optimal synthesis of structures by consideration the sensitivity of static, dynamic and buckling response of structures with respect to variations of the position and stiffness of a set of elastic hinges. Insertion of hinges to a structure results in increasing its compliance. Therefore it can be beneficial in the case of interaction of external loads and distortions or support settlement, since more flexible structures can deform without high stress levels. Designing structures with proper compliance can play essential role in the case of dynamic loading, when the danger of resonance occurs. As is illustrated by the examples, varying the hinge position, the distance between two adjacent eigenfrequencies can be maximized. The numerical examples demonstrate substantial structural sensitivity to variation of hinge location.

The sensitivity gradients derived in the paper can be used in optimal design and active control of structures. The subprogram of sensitivity analysis with respect to hinge parameters can easily be linked to standard programs of structural analysis. The access to the source code is not necessary since the sensitivity derivatives are expressed in terms of total displacements and stresses in hinges.

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