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Continuity with respect to the domain for the Laplacian : a survey

by

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In this paper, we present several results of continuity with respect to the domain for the solution of problems with the Laplacian. We consider the usual homogeneous Dirichlet problem, then the Poisson problem and, at least, the Neumann problem. Some different kinds of convergence of the domains are taken in account and we always assume the minimum of regularity for the domains.

1. Introduction

The aim of this paper is to give a good overview of the results of stability for the solution of a boundary value problem with the laplacian operator, when we let the domain vary. Of course almost all the results presented here are already known — excepting, perhaps, theorems 2.15 and 3.3 which are, to my knowledge, originals. Nevertheless, it seems to me important to gather these results which are often scattered in many papers or books. Moreover, since I have in mind possible applications to shape optimization or free boundary problems, I will consider here domains without any regularity or with a minimal assumption of regularity.

It is in part 2, for the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \omega \\ u = 0 & \text{on } \partial\omega \end{cases} \tag{1.1}$$

that I present a more complete exposition, considering different kinds of convergence of the domains ω_n to ω.

In part 3, I recall the results essentially obtained by V. Keldyš for the Poisson problem

$$\begin{cases} \Delta u = 0 & \text{in } \omega \\ u = \varphi & \text{on } \partial\omega \end{cases} \tag{1.2}$$

while in part 4 some results for Neumann boundary conditions are presented.

In this paper, I do not talk about the dependance of the spectrum of the laplacian with respect to the domain : I refer to the classical books Courant, Hilbert (1962) or Kato (1976) for the regular case, some results are also given in Rauch, Taylor (1975) with less of regularity assumptions. In this last paper, one can find some generalizations to other operators and equations : heat equation , Schrödinger and wave equation for example. For more general elliptic operators, see the paper of Saak (1972).

2. The Dirichlet problem

2.1. Statement of the problem

Let D be a ball in $\mathbb{R}^N$ which will contain all the domains considered here. We always denote by ω_n a sequence of open subsets in D which is assumed to converge to an open subset ω of D, the kind of convergence being specified in each statement. Let f be an element of the Sobolev space $H^{-1}(D)$, i.e. a continuous linear form on the Sobolev space $H_0^1(D)$ ($H_0^1(D)$ being equipped with the norm :

$$||v|| := \left(\int_D |\nabla v|^2 \right)^{\frac{1}{2}} \text{ for } v \in H_0^1(D)$$

which is equivalent to the usual H^1-norm by Poincaré inequality).

We denote by u (resp. u_n) the variational solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \omega \\ u = 0 & \text{on } \partial\omega \end{cases} \tag{2.1}$$

that is to say, the unique function in $H_0^1(\omega)$ which satisfies

$$\forall\ v \in H_0^1(\omega) \quad \int_\omega \nabla u \cdot \nabla v = \langle f, v \rangle_{H^{-1}(\omega)\times H_0^1(\omega)} \tag{2.2}$$

(and similarly for u_n, replacing everywhere ω by ω_n).

We still denote by u (resp u_n) the function u extended by 0 in D :

$$u(x) = \begin{cases} u(x) & \text{if } x \in \omega \\ 0 & \text{if } x \in D|\omega \end{cases} \qquad \text{(and in the same way for } u_n) \tag{2.3}$$

so u and u_n could be considered as functions of $H_0^1(D)$.

In all the following, I am interested in the convergence of u_n to u in $H_0^1(D)$ when ω_n converge to ω in some sense. So when it is claimed that u_n converge to u, it will always mean :

$$u_n \to u \text{ in } H_0^1(D) \qquad (\text{i.e. } \|u_n - u\| \to 0 \text{ when } n \to +\infty). \tag{2.4}$$

In all the following, we consider the distribution f as fixed and if most of the convergence results claimed here would be true for every f, one of them (theorem 2.15) will depend implicitly on the distribution f. Anyway, I mention a recent result of V. Šverak which shows that it could be sufficient to consider the case $f \equiv 1$.

THEOREM 2.1 *If the convergence of u_n to u takes place for $f \equiv 1$, it will take place for every f in $H^{-1}(D)$.*

PROOF. See Šverak (1992).

In the convergence of the domains ω_n to ω, we will always prescribe :

$$\begin{array}{l}\text{For every compact subset } K \text{ of } \omega,\\ \text{there exists } n_0 \in \mathbb{N} \text{ such that } K \subset \omega_n, \text{ for } n \geq n_0.\end{array} \tag{2.5}$$

It is well known, see for instance Pironneau (1984), that

PROPOSITION 2.2 *If ω_n converge to ω for the Hausdorff metric, then (2.5) is satisfied.*

Using only the property (2.5), we are able to prove the basic following result which is the starting point of all our theory :

PROPOSITION 2.3 *Assume that (2.5) is satisfied, then there exists $u^\star$ in $H_0^1(D)$ and a subsequence u_{n_k} such that*

$$u_{n_k} \rightharpoonup u^\star \text{ in } H_0^1(D) \qquad (\text{weak convergence}) \tag{2.6}$$

$$\forall\, v \in H_0^1(\omega) \qquad \int_\omega \nabla u^\star \cdot \nabla v = \langle f, v\rangle_{H^{-1}(\omega)\times H_0^1(\omega)}. \tag{2.7}$$

PROOF. The functions u_n are solution of

$$u_n \in H_0^1(\omega_n) \text{ and } \forall\, \varphi \in H_0^1(\omega_n) \quad \int_{\omega_n} \nabla u_n \cdot \nabla \varphi = \langle f, \varphi\rangle_{H^{-1}(\omega_n)\times H_0^1(\omega_n)}$$

so replacing φ by u_n yields :

$$\int_{\omega_n} |\nabla u_n|^2 = \langle f, u_n\rangle_{H^{-1}(\omega_n)\times H_0^1(\omega_n)}.$$

Extending u_n to D, we obtain :

$$||u_n||^2 = \int_D |\nabla u_n|^2 = \langle f, u_n \rangle_{H^{-1}(D) \times H_0^1(D)} \leq ||f||_{H^{-1}(D)} \, ||u_n||$$

so u_n is bounded in the reflexive space $H_0^1(D)$: we can extract a subsequence u_{n_k} which converges weakly to $u^\star \in H_0^1(D)$.
Now, let φ be given in $\mathcal{D}(\omega)$ (the space of infinitely differentiable functions with compact support in ω). By assumption (2.5), we have $\varphi \in \mathcal{D}(\omega_n)$ for n great enough so

$$\int_{\omega_n} \nabla u_n \cdot \nabla \varphi = \langle f, \varphi \rangle_{H^{-1}(\omega_n) \times H_0^1(\omega_n)}.$$

Extending to D, we still have

$$\int_D \nabla u_n \cdot \nabla \varphi = \langle f, \varphi \rangle_{H^{-1}(D) \times H_0^1(D)}$$

and going to the weak limit (for the subsequence u_{n_k}) we obtain :

$$\int_\omega \nabla u^\star \cdot \nabla \varphi = \int_D \nabla u^\star \cdot \nabla \varphi = \langle f, \varphi \rangle_{H^{-1}(D) \times H_0^1(D)} = \langle f, \varphi \rangle_{H^{-1}(\omega) \times H_0^1(\omega)}$$

this equality being true for every $\varphi \in \mathcal{D}(\omega)$, we obtain (2.7) by density of $\mathcal{D}(\omega)$ in $H_0^1(\omega)$. ■

Now, it is clear that to achieve convergence of u_n to u, it remains to prove that $u^\star$ belongs to $H_0^1(\omega)$ what is not, in general, true. All the following, in this section, goes around this crucial question :

$$\text{When are we able to say that } u^\star \in H_0^1(\omega) \text{ ?} \tag{2.8}$$

If it is the case, we can claim :

Proposition 2.4 *If we can prove that the weak limit $u^\star$ necessarily belongs to $H_0^1(\omega)$ then :*

- $u^\star = u$
- *all the sequence u_n converges to u*
- *the convergence is strong in $H_0^1(D)$.*

Proof. The two first assertions come from uniqueness of the solution of (2.2). Now

$$||u_n - u||^2 = \int_D |\nabla u_n - \nabla u|^2 = \int_D |\nabla u_n|^2 - 2\int_D \nabla u_n \cdot \nabla u + \int_D |\nabla u|^2$$

and when $n \to +\infty$ $\int_D \nabla u_n \cdot \nabla u \to \int_D |\nabla u|^2$. But, using (2.2) for ω and ω_n, we also have $\int_D |\nabla u_n|^2 = \langle f, u_n \rangle_{H^{-1} \times H_0^1} \to \langle f, u \rangle$ and $\int_D |\nabla u|^2 = \langle f, u \rangle_{H^{-1} \times H_0^1}$. So $\lim_{n \to +\infty} ||u_n - u||^2 = \langle f, u \rangle - 2\langle f, u \rangle + \langle f, u \rangle = 0$. ■

Let us now consider some favourable cases, where we can answer positively the question (2.8).

2.2. Increasing sequence

THEOREM 2.5 *Let* (ω_n) *be an increasing sequence and* $\omega = \bigcup_{n\in\mathbb{N}} \omega_n$ *then* u_n *converges to* u *in* $H_0^1(D)$.

PROOF. Assumption (2.5) is an immediate consequence of the definition of a compact set : since $\omega = \bigcup_{n\in\mathbb{N}} \omega_n$ is a recovering of K by a family of open subsets, we can extract a finite subsequence such that $K \subset \bigcup_{i=1}^{p} \omega_{n_i} = \omega_{n_p}$ and then $K \subset \omega_n \quad \forall n \geq n_p$. Let $u^\star$ be the weak limit of a subsequence like in proposition 2.3. Since u_{n_k} converge weakly to $u^\star$, there exists a sequence of convex combinations of the u_{n_k} which converges strongly to $u^\star$ and up to a subsequence, which converges quasi everywhere to $u^\star$ in the sense of the classical capacity associated to the Sobolev space $H^1(D)$, see for example Carleson (1967), Hedberg (1981) or Maz'ja (1985).
Now each $u_{n_k} = 0$ quasi everywhere on ω^c and so every convex combination of the u_{n_k} vanishes quasi everywhere on ω^c, and then

$$u^\star = 0 \quad \text{q.e. on } \omega^c \tag{2.9}$$

what implies, by classical spectral synthesis, see Hedberg (1981), that $u^\star \in H_0^1(\omega)$. The result follows using proposition 2.4. ■

2.3. Local perturbation

The same idea (spectral synthesis) can be used to prove convergence of u_n to u when the difference between ω_n and ω becomes very small in the sense of capacity of course. We will denote by $\omega_n \Delta\omega$ the symmetric difference of ω_n and ω, that is to say :

$$\omega_n \Delta\omega = \{x \in D \text{ such that } (x \in \omega \text{ and } x \notin \omega_n) \text{ or } (x \in \omega_n \text{ and } x \notin \omega)\},$$

then we claim :

THEOREM 2.6 *Let* $S \subset \partial\omega$ *a polar subset of* $\partial\omega$ *(i.e. a set of zero-capacity). Assume that* $\omega_n \Delta\omega \subset \bigcup_{x\in S} B(x, \frac{1}{n})$ *(where* $B(x, \frac{1}{n})$ *is the ball of center* x *and radius* $\frac{1}{n}$*). Then* u_n *converge to* u *in* $H_0^1(D)$.

PROOF. If K is a compact subset of ω, we have $\delta = d(K, \partial\omega) > 0$ and then $K \subset \omega_n$ as soon as $n < \frac{1}{\delta}$ so assumption (2.5) is satisfied. Now following the proof of the previous proposition, we are able to prove that :

$$u^\star = 0 \text{ q.e. on every compact subset } L \text{ of } \omega^c \backslash S$$

(because $u_n = 0$ q.e. on L for n great enough) and then, since S has zero capacity, $u^\star = 0$ q.e. on ω^c so $u^\star \in H_0^1(\omega)$ what finishes the proof. ■

REMARK. *This proposition is very useful when you want to modify a domain ω only in a neighbourhood of one or some irregular points, see an application in Henrot (1994).*

In the same way, it is interesting to consider a situation, classical in homogeneization, where the domains ω_n are obtained from ω by removing a great number of little holes. Such a study is done in Rauch, Taylor (1975) and in Cioranescu, Murat (1982) for example. The general result depends essentially on the size of the holes, roughly speaking :

- if the holes are small enough, u_n will converge to u
- if the holes are too big, u_n will converge to 0
- there is a critical size for which $u^\star$ (the weak limit of u_n) is solution of an other boundary value problem on ω.

In order to give a more precise idea of this phenomenon, let me give a two–dimensional example taken from Cioranescu, Murat (1982).

For $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}^\star$ let $X_{ij} = \left(\frac{i}{n}, \frac{j}{n}\right)$ and r_n, $0 < r_n < \frac{1}{n}$. We consider a bounded open domain ω and the subdomains ω_n defined by

$$\omega_n = \omega \backslash \bigcup_{i,j} B(X_{i,j}, r_n)$$

(the union is taken over all balls which meet ω).

Like in proposition 2.3, it is easy to see that a subsequence of the u_n converges weakly in $H_0^1(D)$ to a function $u^\star \in H_0^1(D)$, and we can characterize $u^\star$ in accordance with the size of the holes.

PROPOSITION 2.7

- *If* $\dfrac{\log r_n}{n^2} \xrightarrow[n\to+\infty]{} -\infty$, *then* $u^\star = u$ *(and* $u_n \to u$ *in* $H_0^1(D)$*)*
- *if* $\dfrac{\log r_n}{n^2} \xrightarrow[n\to+\infty]{} 0$, *then* $u^\star = 0$ *(and* $u_n \to 0$ *in* $H_0^1(D)$*)*

the critical case :

- *if* $\dfrac{\log r_n}{n^2} \xrightarrow[n\to+\infty]{} -\ell < 0$, *then* $u^\star$ *is the solution of the problem :*

  $u^\star \in H_0^1(\omega)$ *and* $-\Delta u^\star + \frac{\pi}{2\ell} u^\star = f$ *in* ω *(and then* $u^\star \neq u$*).*

REMARK. *This result can be extended, of course, to dimension $N = 3$. In Rauch, Taylor (1975) many other interesting examples are presented.*

In two dimensions, V. Šverak proves a remarkable result which shows that when the number of holes is restricted, convergence takes place :
For $\ell \in \mathbb{N}^\star$, let us denote by :

$\mathcal{O}_\ell = \{\omega$ open subset of D such that the number of connected components of $D\backslash\omega$ is $\leq \ell\}$, then :

THEOREM 2.8 (see V. Šverak (1992)) *If $\omega_n \in \mathcal{O}_\ell$ and ω_n converge to ω for the Haussdorff metric, then $\omega \in \mathcal{O}_\ell$ and $u_n \to u$ in $H_0^1(D)$.*

2.4. The regular case

Up to now, I did not make any assumption on the regularity of the limit domain ω. If we want more general results, and particularly when the domains ω_n converge to ω from the exterior (decreasing sequence, for example), we are led to assume some regularity for ω. The minimal assumptions were essentially found by Keldyš and are the following :

DEFINITION (AND PROPOSITION) 2.9 (see Hedberg, 1980) *For an open subset ω of D the two following properties are equivalent*

(i) For all open Ω, $\operatorname{cap}(\Omega\backslash\overline{\omega}) = \operatorname{cap}(\Omega\backslash\omega)$

(ii) $\liminf_{r\to 0} \dfrac{\operatorname{cap}(B(x,r)\backslash\overline{\omega})}{\operatorname{cap}(B(x,r)\backslash\omega)} > 0$ *q.e. for $x \in \partial\omega$.*

If these properties are satisfied we will say that ω is stable.

REMARK. *Many sufficient conditions for (i) or (ii) to hold are known. For example, if ω^c has the restricted cone property (see Agmon, 1965 p.11) then it is satisfied.*

An other interesting example of sufficient condition is given in Frehse (1982) (see also Grigorieff, 1972 and Stummel, 1974):

PROPOSITION 2.10 *Assume that ω is a Caratheodory domain (i.e. $\partial\omega = \partial\overline{\omega}$) which is regular in the Wiener sense (see Keldyš, 1966; Landkof, 1972), then ω is stable.*

For other characterizations, I refer to the paper of Hedberg (1980) in which one can find the following proposition :

PROPOSITION 2.11 *ω is stable if and only if each function $v \in H^1(D)$ which vanishes q.e. on $\overline{\omega}^c$ belongs to $H_0^1(\omega)$.*

Using this last result, we are now able to state a general result of convergence when a natural convergence of ω_n to ω is assumed (see also Rauch, Taylor, 1975 for an other proof of the same result) :

THEOREM 2.12 *Let ω be a stable open subset of D and ω_n a sequence of open subsets converging to ω in the following sense : (2.5) is satisfied and*

$$\begin{aligned}&\text{For each } L \text{ compact subset of } \overline{\omega}^c,\\ &\text{there exist } n_0 \in \mathbb{N} \text{ such that } L \subset \overline{\omega}_n^c \text{ for } n \geq n_0\end{aligned} \tag{2.10}$$

then u_n converge to u in $H_0^1(D)$.

PROOF. Let $u^\star$ be the weak limit of a subsequence u_{n_k} like in proposition 2.3. According to assumption (2.10), and using the same trick as in the proof of proposition 2.5 we are able to prove that $u^\star = 0$ q.e. on $\overline{\omega}^c$. Using stability of ω and proposition 2.11 we can claim that $u^\star \in H_0^1(\omega)$ what finishes the proof with the help of proposition 2.4. ■

In Frehse (1982), one can find an other stability result assuming different kind of convergence of ω_n to ω :

PROPOSITION 2.13 *Let ω be a Caratheodory domain which is regular in the Wiener sense. Let ω_n be a sequence of open bounded domains converging to ω in the following sense: (2.5) is satisfied and*

$$\text{the Lebesgue measure of } \omega_n \setminus (\omega_n \cap \omega) \text{ tends to zero} \tag{2.11}$$

then u_n converge to u in $H_0^1(D)$.

2.5. Case of bounded gradient

Up to now, we have used spectral synthesis to characterize the functions in $H_0^1(\omega)$. But there exists other criteria for a function in $H^1(D)$ to belong to $H_0^1(\omega)$. For example, the following one we can find in Mikhailov (1980) for the regular case and in Osipov, Suetov (1990) for the general case.

We denote by $I_\epsilon = \{x \in \omega\,;\, d(x, \partial\omega) < \epsilon\}$ where $d(x, \partial\omega)$ is the distance of x to the boundary of ω, and then :

PROPOSITION 2.14 *Let V be a function in $H^1(D)$. Assume there exists $C > 0$ and $\epsilon_0 > 0$ such that for every ϵ, $0 < \epsilon < \epsilon_0$ we have :*

$$\int_{I_\epsilon} V^2(x)\, dx \le C\,\epsilon^2 \tag{2.12}$$

then $V \in H_0^1(\omega)$.

REMARK. *This sufficient condition expresses that V vanishes quickly enough when we approach the boundary of ω.*

PROOF. Let $(P_\epsilon)_{\epsilon>0}$ be a sequence of mollifiers, i.e. a sequence of functions in $\mathcal{D}(\mathbb{R}^N)$ which satisfies :

$$\operatorname{supp}\, \rho_\epsilon \subset B(0,\epsilon)\ ;\ \rho_\epsilon \ge 0\ ;\ \int_{\mathbb{R}^N} \rho_\epsilon = 1\ ;\ |\nabla\rho_\epsilon| \le \frac{C}{\epsilon^{N+1}}$$

with C independent of ϵ.

Let us define the functions

$$\eta_\epsilon(x) = \int_{\omega\setminus I_{2\epsilon}} \rho_\epsilon(x-y)dy = \int_{(\omega\setminus I_{2\epsilon})\cap B(x,\epsilon)} \rho_\epsilon(x-y)dy.$$

It is easy to verify that : $\eta_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^N)$; $\eta_\epsilon \equiv 1$ on $\omega \backslash I_{3\epsilon}$; $\eta_\epsilon \equiv 0$ on $I_\epsilon \cup \omega^c$ (and then $\eta_\epsilon \in \mathcal{D}(\omega)$).

Furthermore $\forall\, x \in \mathbb{R}^N \quad 0 \leq \eta_\epsilon(x) \leq 1$ and $|\nabla \eta_\epsilon(x)| \leq C_2/\epsilon$ with a constant C_2 independent of ϵ.

Now, by Meyers–Serrin theorem (see Adams, 1975), there exists a sequence $\{V_n\}$ of functions in $\mathcal{C}^\infty(\omega) \cap H^1(\omega)$ which converges to V in $H^1(\omega)$: so for any $\epsilon > 0$ there exists $n = n(\epsilon)$ such that $\|V - V_n\|_{H^1(\omega)} < \epsilon$. In particular, using assumption (2.12) and the above inequality, we obtain that for ϵ such that $3\epsilon < \epsilon_0$:

$$\|V_n\|_{L^2(I_{3\epsilon})} \leq \|V\|_{L^2(I_{3\epsilon})} + \epsilon \leq (3\sqrt{c} + 1)\epsilon. \tag{2.13}$$

Now, let us consider the sequence $V_{n(\epsilon)}\,\eta_\epsilon$ which belongs to $\mathcal{D}(\omega)$. Thanks to properties of η_ϵ and assumption (2.12), it is obvious that $V_{n(\epsilon)}\,\eta_\epsilon$ converges to V in $L^2(\omega)$ when $\epsilon \to 0$. Let us prove that the sequence $V_{n(\epsilon)}\,\eta_\epsilon$ is bounded in $H^1(\omega)$. Since $V_{n(\epsilon)}\,\eta_\epsilon = V_{n(\epsilon)}$ on $\omega \backslash I_{3\epsilon}$, which is bounded in $H^1(\omega \backslash I_\epsilon)$ because it is convergent, it remains to consider :

$$\int_{I_{3\epsilon}} \eta_\epsilon^2\, |\nabla V_n|^2 \text{ and } \int_{I_{3\epsilon}} V_n^2\, |\nabla \eta_\epsilon|^2.$$

Since $0 \leq \eta_\epsilon \leq 1$, the first integral can be estimated from above by $\int_{I_{3\epsilon}} |\nabla V_n|^2$ which is bounded since V_n converge in $H^1(\omega)$. The second integral is estimated using $|\nabla \eta_\epsilon| \leq \frac{C_2}{\epsilon}$ and assumption (2.12) as soon as $3\epsilon < \epsilon_0$ by :

$$\int_{I_{3\epsilon}} V_n^2\, |\nabla \eta_\epsilon|^2 \leq \frac{C_2^2}{\epsilon^2}\, \|V_n\|^2_{L^2(I_{3\epsilon})} \leq C_2^2 (3\sqrt{c} + 1)^2 \text{ by (2.13)}.$$

Finally since $V_{n(\epsilon)}\,\eta_\epsilon$ converges to V in $L^2(\omega)$ and are bounded in $H^1(\omega)$, we have $V_{n(\epsilon)}\,\eta_\epsilon$ which converges weakly to V in $H^1(\omega)$ and since $V_{n(\epsilon)}\,\eta_\epsilon$ belongs to $H_0^1(\omega)$, $V \in H_0^1(\omega)$. ■

This criterion may be used, in particular, when the gradients of the sequence u_n are bounded in a neighbourhood of $\partial\omega$. In the following, we will assume that f is regular enough so that $u_n \in W^{1,\infty}(I_\epsilon)$ for ϵ little enough. We state a result, an application of which is given for the study of a free boundary problem in Henrot (1994).

THEOREM 2.15 *Let ω_n be a sequence of domains converging to ω in the following sense :*

- *(2.5) is satisfied*
- *For each $x \in \partial\omega$, $\lim_{n \to +\infty} d(x, \partial\omega_n) = 0$.* (2.14)

Assume moreover that ∇u_n is bounded uniformly in a neighbourhood of $\partial\omega$:

$$\exists\, M > 0,\ \exists\, \alpha > 0 \text{ such that } \forall\, x \in \omega_n,\ d(x, \partial\omega) < \alpha \Rightarrow |\nabla u_n(x)| \leq M. \tag{2.15}$$

Then u_n converges to u in $H_0^1(D)$.

REMARKS.

- *It is easy to prove that Hausdorff convergence of ω_n to ω in the usual sense (that is to say, Hausdorff convergence of $\overline{D}\backslash\omega_n$ to $\overline{D}\backslash\omega$) implies properties (2.5) and (2.14). The converse is obviously wrong : in $\mathbb{R}^2$ take*
  $\omega =]0,1[\times]0,1[$ *and*
  $\omega_n = (]-1/n, 1+1/n[\times]-1/n, 1+1/n[) \cup (]1,2[\times]1,2[)$.
- *If we denote by $d_n(x)$ the functions : $d_n(x) = d(x, \partial\omega_n)$ defined on $\partial\omega$, the assumption (2.14) expresses that the sequence d_n converges simply to 0 on $\partial\omega$. But since this sequence is equicontinuous (every function d_n is 1-lipschitzian) and $\partial\omega$ is compact, the convergence of the sequence d_n to 0 is therefore uniform.*

PROOF OF THEOREM 2.15. Let $\epsilon > 0$ be fixed, with $\epsilon < \alpha$ and n_0 such that the compact set : $\{x \in \omega / d(x, \partial\omega) \geq \alpha\} \subset \omega_n$ for $n \geq n_0$ (thanks to (2.5)). Let us fix now $n \geq n_0$. For every $x \in I_\epsilon$, there exists $y = y(x) \in \partial\omega$ such that $|y - x| < \epsilon$ and for this y, there exists, thanks to (2.14), $z = z(x,y) \in \partial\omega_n$ such that $|z - y| < \epsilon$.
Now, since $u_n \in H_0^1(\omega_n)$, there exists a sequence, say φ_q, of functions in $\mathcal{D}(\omega_n)$ which converges to u_n in $H_0^1(\omega_n)$ and we can assume moreover that $\nabla\varphi_q$ converges almost everywhere to ∇u_n, hence $|\nabla\varphi_q| \leq M+1$ on the neighbourhood of $\partial\omega$ described above. Now for every $x \in I_\epsilon$:

$$|\varphi_q(x)| = |\varphi_q(x) - \varphi_q(z)| = |\int_0^1 \nabla\varphi_q(z + t(x-z)).(x-z)dt|$$

hence :

$$|\varphi_q(x)| \leq (M+1)\,|x - z| \leq (M+1)2\epsilon$$

which yields, integrating on I_ϵ :

$$\int_{I_\epsilon} |\varphi_q(x)|^2 dx \leq (M+1)^2\, 4\,\epsilon^2 \text{ mes } (I_\epsilon) \leq (M+1)^2\, 4 \text{ mes } (\omega)\, \epsilon^2 = C\epsilon^2.$$

But since $\varphi_q \rightarrow u_n$ strongly in L^2 : $\int_{I_\epsilon} |u_n(x)|^2\, dx \leq C\epsilon^2$.
Now let $u^\star$ be the weak limit in $H_0^1(D)$ of a sequence u_{n_k} like in proposition 1.3. By Rellich theorem, we can assume that, up to a subsequence, u_{n_k} converges strongly to $u^\star$ in $L^2(D)$ so going to the limit in the above inequality yields

$$\int_{I_\epsilon} |u^\star(x)|^2\, dx \leq C\,\epsilon^2$$

this being true for each $\epsilon < \alpha$. So, by applying proposition 2.14 we have proved that $u^\star \in H_0^1(\omega)$ and the result follows thanks to proposition 2.4. ■

3. The Poisson problem

3.1. Introduction

We consider here the classical Poisson problem :

$$\begin{cases} \Delta v = 0 & \text{in } \omega \\ v = \varphi & \text{on } \partial\omega \end{cases} \tag{3.1}$$

where φ is a continuous function defined on $\partial\omega$. It is well known that if ω has the usual Wiener regularity (see e.g. Dautray, Lions, 1984; Landkof, 1972) the solution of (3.1) is the classical one, while if ω fails to have this regularity, the solution is to be taken in a generalized sense. This generalized solution is unique and can be defined in several ways (for example : Perron's method in Dautray, Lions, 1984, with the balayage operator in Landkof, 1972, by increasing sequences of regular domains in Keldyš, 1966).

Most of the work concerning the stability of the Poisson problem were done by V. Keldyš in the 40's and is exposed in his fundamental work Keldyš (1966). This is the reason why we are going to give most of the results without any proof : we refer to Hedberg (1980), Keldyš (1966) and Landkof (1972) for more details (see also Saak, 1970, for some complementary results).

If the continuous function φ is given on $\partial\omega$, we are able to extend it by Tietze theorem to all $\mathbb{R}^N$, we will still denote by φ this extension. The solution of (3.1) in an open domain ω_n and with the boundary data φ will be denoted by v_n and we are concerned, here, with the convergence of v_n to v on any compact subset of ω.

The simpler case, which is also the most favourable one, is still the case of an increasing sequence of domains :

PROPOSITION 3.1 *Assume that ω_n is an increasing sequence of open regions and let $\omega = \bigcup_{n=1}^{\infty} \omega_n$ (all the boundaries $\partial\omega_n$ and $\partial\omega$ are assumed to be compact). Then v_n converge to v uniformly on any compact subset of ω.*

3.2. The regular case

Let us consider now a decreasing sequence of open regions ω_n such that

$$\begin{cases} \omega \subset \overline{\omega} \subset \omega_n \text{ for } n = 1, 2 \cdots \\ \text{and } \partial\omega_n \text{ converge to } \partial\omega \text{ in the following sense :} \\ \forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N} \text{ such that for } n \geq n_0 \text{ each of the sets} \\ \partial\omega_n \text{ and } \partial\omega \text{ lies in a } \epsilon-\text{neighbourhood of the other one.} \end{cases} \tag{3.2}$$

It is easy to see that the functions v_n converge uniformly on every compact subset of ω to a function $v^\star$ which is harmonic on ω (use, for example, Harnack inequality) and, like in the previous section, the main difficulty is to prove that $v^\star = v$. The criterion of regularity which is necessary to state this result is the same as in section 2.

THEOREM 3.2 (see Hedberg, 1980 or Keldyš, 1966) *Assume that ω_n is a decreasing sequence of open regions satisfying (3.2) and that ω is stable in the sense of definition (2.9), then v_n converge to v uniformly on every compact subset of ω.*

3.3. Case of bounded gradients

Like in the previous section, we will develop more the case where $|\nabla v_n|$ is bounded near $\partial\omega_n$. This kind of results seems to be new and an interesting application to a classical free boundary problem is given in Henrot, Seck (1994).

Here, we do not assume any regularity (i.e. stability) property for ω, nevertheless we will assume that the domains ω_n have the Wiener regularity property. It means that the solution of (3.1) for ω_n is classical (see Dautray, Lions, 1984 or Landkof, 1972 for geometric characterization of this property).

Since the case of "internal convergence" of ω_n to ω is quite clear, we will restrict ourselves here to external convergence, that is to say the case where $\omega \subset \omega_n$.

THEOREM 3.3 *Let ω_n be a sequence of "Wiener regular" open domains containing ω, we assume that :*

(i) $\forall\ x \in \partial\omega,\ d(x, \partial\omega_n) \to 0$ when $n \to +\infty$

(ii) $\exists\ \alpha > 0,\ \exists\ M > 0$ such that $\forall\ y \in \omega_n,\ d(y, \partial\omega) \leq \alpha \Rightarrow |\nabla v_n(y)| \leq M$ (where v_n is the solution of problem (3.1) in ω_n)

then v_n converge to v uniformly on every compact subset of ω.

PROOF. By Harnack inequality, see also Dautray, Lions (1984), there exists a subsequence v_{n_k} and an harmonic function $v^\star$ in ω such that v_{n_k} converge to $v^\star$ in the usual harmonic sense, that is to say $v_{n_k} \to v^\star$ uniformly on every compact subset of ω, like all their derivatives. So we obtain immediately, passing to the limit in (ii) that

$$\forall\ y \in \omega,\ d(y, \partial\omega) < \alpha \Rightarrow |\nabla v^\star(y)| \leq M. \tag{3.3}$$

Now let x be fixed on $\partial\omega$, and y, y' two points in $B(x,\alpha) \cap \omega$. By the mean value property, we have :

$$|v^\star(y) - v^\star(y')| \leq M|y - y'|$$

and then $v^\star$ satisfies the Cauchy condition at the point x, it means that $\lim_{y\to x} v^\star(y)$ exists, we denote it by $\varphi^\star(x)$.

We are going to prove that $\varphi^\star = \varphi$ on $\partial\omega$. If it is the case, then $v^\star$ is a (classical) solution of (2.1) and then, by uniqueness, $v^\star = v$, what proves the theorem.

So let x_0 be a boundary point of ω and $\epsilon > 0$.
By continuity of φ, there exists $\eta_1 > 0$ such that

$$\forall\ y \in B(x_0, \eta_1)\ |\varphi(y) - \varphi(x_0)| < \frac{\epsilon}{2}.$$

Let us set $\eta = \inf\ (\eta_1, \frac{\epsilon}{4M}, \alpha)$, by assumption (i) there exists for n great enough a point y_n which belongs to $\partial\omega_n \cap B(x_0, \eta)$. Let us fix also $x \in B(x_0, \eta) \cap \omega$. We have, for n great enough, and by assumption (ii) :

$$|v_n(x) - v_n(y_n)| = |v_n(x) - \varphi(y_n)| \leq M|x - y_n| \leq 2M\eta \leq \frac{\epsilon}{2}$$

and then

$$|v_n(x) - \varphi(x_0)| \leq |v_n(x) - \varphi(y_n)| + |\varphi(y_n) - \varphi(x_0)| \leq \epsilon.$$

When n goes to $+\infty$, we obtain

$$|v^\star(x) - \varphi(x_0)| \leq \epsilon \quad \text{for every } x \text{ in } \omega \cap B(x_0, \eta).$$

Then, we let x tending to x_0, which yields :

$$\forall\ \epsilon > 0 \quad |\varphi^\star(x_0) - \varphi(x_0)| \leq \epsilon \quad \text{which achieves the proof.}$$ ■

REMARK. *If ω is an open domain such that, for a given φ, the solution v of (3.1) has a bounded gradient in a neighbourhood of $\partial\omega$, the previous proof shows that there exists a function $\varphi^\star$ such that $\lim\limits_{\substack{y \to x \\ y \in \omega}} v(y) = \varphi^\star(x)$. It is easy to see that $\varphi^\star$ is continuous. Now, if ω is a Caratheodory domain (i.e. all the boundary points of ω are closed to the exterior of ω), Bouligand's lemma (see Keldyš, 1966) proves that $\varphi^\star = \varphi$, and then v is a classical solution of (3.1). This remark shows that if we assume $|\nabla v_n|$ bounded on ω_n instead of (ii), we do not need that ω_n be regular in the sense of Wiener for the theorem 3.3 to be true.*

4. The Neumann problem

4.1. Introduction

We consider here the homogeneous Neumann problem whose variational formulation is (for a general open domain ω) :

$$\begin{cases} \text{find } u \text{ in the Sobolev space } H^1(\omega) \text{ such that for every } v \text{ in } H^1(\omega) \\ \text{we have } \int_\omega \nabla u . \nabla v + \int_\omega uv = \int_\omega fv \end{cases} \tag{4.1}$$

where f is a given function in $L^2(\omega)$. According to Lax-Milgram's lemma, this problem has a unique solution and when ω is regular, the function u solves :

$$\begin{cases} -\Delta u + u = f & \text{in } \omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\omega \end{cases} \tag{4.2}$$

where $\frac{\partial u}{\partial n}$ is the normal derivative of u on the boundary.

We are going to consider a family of domains ω_n always included in a ball D converging in some sense to ω, and our purpose is to study the convergence of the functions u_n solutions of (4.1) (with the domain ω_n) to the function u.

In section 2, for the homogeneous Dirichlet condition, we worked on the Sobolev space $H_0^1(\omega)$ and whatever the regularity of ω was, we had a natural extension for functions in $H_0^1(\omega)$:

if $y \in H_0^1(\omega)$, $\widehat{y}$ defined by $\widehat{y}(x) = \begin{cases} y(x) & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases}$ belongs to $H_0^1(D)$ (and $\|\widehat{y}\|_{H_0^1(D)} = \|y\|_{H_0^1(\omega)}$). Unfortunately, in the case of Neumann boundary condition, it is not so simple since if ω is not regular, there may not exist such an extension for every function of $H^1(\omega)$ to $H^1(D)$ (see Agmon, 1965; Maz'ja, 1985). So we are led to consider here domains with some regularity. A sufficient condition for such an extension to exist is (see Agmon, 1965) the restricted cone property.

Moreover we will need in the proof below that this extension be, in some sense, uniform for all the domains we will consider. So, we restrict ourselves in this section to work with domains belonging to a class $\mathcal{S}$ such that :

$$\begin{cases} \bullet \text{ For each domain } \omega \text{ in } \mathcal{S}, \text{ there exists a linear continuous} \\ \quad \text{extension operator } P_\omega \text{ from } H^1(\omega) \text{ into } H^1(D) \\ \bullet \text{ Moreover, there exists a constant } K \text{ such that for every } \omega \text{ in } \mathcal{S}, \\ \quad \|P_\omega\| \leq K \text{ (where } \|.\| \text{ denotes the classical norm of operators).} \end{cases} \tag{4.3}$$

In Chenais (1975), D. Chenais considered the class $\mathcal{S}_\epsilon$ constituted of the set of all open domains satisfying the restricted cone property with a given height and angle of the cone, say ϵ (each open domain is said to satisfy the $\epsilon-$cone property). She proves, then, that $\mathcal{S}_\epsilon$ verifies assumption (4.3).

To conclude, let me mention that the results exposed in this section are also taken from D. Chenais's paper, Chenais (1975). In Rauch, Taylor (1975) more general boundary conditions (including mixed conditions) with a quite similar approach are considered (they need also a uniform extension property as (4.3), but they restrict themselves to the case where $\omega_n \subset \omega$). For a different approach, see Saak (1970).

4.2. The convergence result

We are going to consider here a different kind of convergence of domains of those used in the previous sections :

DEFINITION. *We will say that a sequence of domain ω_n converges "in L^1" to ω if χ_n, the characteristic function of ω_n, converges in $L^1(D)$ to χ, the characteristic function of ω. This is equivalent to the statement that*

$$mes\ (\omega_n \backslash \omega) + \ mes\ (\omega \backslash \omega_n) \to 0. \tag{4.4}$$

REMARK. *This notion of convergence is neither weaker nor stronger than compact convergence (2.5), (2.10) considered in section 2 as shown by the following examples :*

EXAMPLE 1 *Let* $\omega_n = [0,1]\backslash \bigcup_{k=0}^{2^n} \{\frac{k}{2^n}\}$ *in* $\mathbb{R}$, *then* ω_n *converge in* L^1 *to* $]0,1[$, *but* ω_n *do not converge for the compact convergence (2.5), (2.10).*

EXAMPLE 2 *Let* $(x_k)_{k\in\mathbb{N}}$ *be a dense subsequence of* $]0,1[$ *and* r_k *positive numbers chosen so that* $\sum_{k=0}^{+\infty} r_k = \frac{\epsilon}{2}$ *a given positive number.*

Let us consider the open domains :

$$\omega = \bigcup_{k\in\mathbb{N}}]x_k - r_k, x_k + r_k[\text{ and } \omega_n =]-\frac{1}{n}, 1+\frac{1}{n}[$$

then ω_n *converge to* ω *for the compact convergence (observe that the exterior of* ω ; $\overline{\omega}^c = \mathbb{R}\backslash[0,1]$), *but* ω_n *do not converge to* ω *for the* L^1 *convergence since* $mes\ (\omega_n\backslash\omega) \geq 1-\epsilon$.

Let us now state the convergence result for the Neumann problem :

THEOREM 4.1 *Let* $(\omega_n)_{n\in\mathbb{N}}$ *and* ω *be open domains in a class* $\mathcal{S}$ *satisfying (4.3) and assume that* ω_n *converge in* L^1 *to* ω. *Let us denote, respectively, by* u_n *and* u *the solutions of the homogeneous Neumann problem (4.1) in* ω_n *and* ω. *Let us also denote by* $\widehat{u}_n = P_{\omega_n}(u_n)$ *the extension of* u_n *to* D.

Then $\widehat{u}_n/\omega$ *converge to* u *in* $L^2(\omega)$.

PROOF. By (4.1), we have $||u_n||^2_{H^1(\omega_n)} = \int_{\omega_n} |\nabla u_n|^2 + \int_{\omega_n} u_n^2 = \int_{\omega_n} f\, u_n$ and then, by Cauchy–Schwartz inequality, $||u_n||_{H^1(\omega_n)} \leq ||f||_{L^2(\omega_n)}$.

For the sequence of extensions $\widehat{u}_n$, we have :

$$||\widehat{u}_n||_{H^1(D)} \leq ||P_{\omega_n}||\ ||u_n||_{H^1(\omega_n)} \leq K||f||_{L^2(D)}$$

what proves that the sequence $\widehat{u}_n$ is bounded in $H^1(D)$. So, we can extract a subsequence $\widehat{u}_{n_k}$ which converges weakly in $H^1(D)$, and strongly in $L^2(D)$ by Rellich theorem, to a function u^* of $H^1(D)$. It remains to prove that u^* satisfies variational formulation (3.1) on ω.
By definition of u_n, we have for each $v \in H^1(D)$:

$$\int_{\omega_n} \nabla u_n \nabla v + \int_{\omega_n} u_n\, v = \int_{\omega_n} fv,$$

which yields, introducing the characteristic functions :

$$\int_D \chi_n \nabla\widehat{u}_n \nabla v + \int_D \chi_n\, \widehat{u}_n\, v = \int_D \chi_n\, fv. \tag{4.5}$$

Now, since χ_n converge to χ in $L^1(D)$, we can extract a subsequence, still denoted by χ_{n_k} which converges almost everywhere to χ on D.
Then :

- $\chi_{n_k} fv \to \chi fv$ in $L^1(D)$ by Lebesgue's dominated convergence theorem
- $(\chi_{n_k} v \to \chi v$ strongly in $L^2(D)$ and $\widehat{u}_n \rightharpoonup u^\star$ weakly in $L^2(D))$

$\Rightarrow (\chi_{n_k} v, \widehat{u}_n) \to (\chi v, u^\star)$
where $(.,.)$ denotes the L^2 scalar product on D.
Similarly :

- $(\chi_{n_k} \frac{\partial v}{\partial x_i} \to \chi \frac{\partial v}{\partial x_i}$ strongly in $L^2(D)$ and $\frac{\partial \widehat{u}_n}{\partial x_i} \rightharpoonup \frac{\partial u^\star}{\partial x_i}$ weakly in $L^2(D))$

$\Rightarrow (\chi_{n_k} \frac{\partial v}{\partial x_i}, \frac{\partial \widehat{u}_n}{\partial x_i}) \to (\chi \frac{\partial v}{\partial x_i}, \frac{\partial u^\star}{\partial x_i})$.

Therefore, passing to the limit in (3.5) yields :

$$\int_D \chi \nabla u^\star \nabla v + \int_D \chi u^\star v = \int_D \chi fv$$

or :

$$\int_\omega \nabla u^\star . \nabla v + \int_\omega u^\star v = \int_\omega fv \quad \text{for every } v \text{ in } H^1(D)$$

what proves that $u^\star/\omega = u$.

Now, this proof being valid for every subsequence of $(\widehat{u}_n)$, we have proved that $\widehat{u}_n/\omega \to u$ in $L^2(\omega)$. ■

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