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Interface optimization problems for parabolic equations

by

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An optimization problem for the heat equation with respect to the geometrical domain is considered. The existence of an optimal solution is obtained and the first order necessary optimality conditions are derived.

1. Introduction

In this paper an optimization problem for the heat equation with respect to the geometrical domain is considered. Two C^1 curves are selected in an optimal way, the curves being the interfaces between the subsets of the lateral boundary of parabolic cylinder where the Neumann and Dirichlet boundary conditions are prescribed for the equation. The existence of an optimal solution is obtained by adding a regularizing term to the cost functional and the first order necessary optimality conditions are derived.

The proofs of the results presented in the paper as well as some applications and numerical results are given in the forthcoming paper by Hoffmann and Sokołowski (1991).

Let $Q = \Omega \times (0,T)$, $T > 0$, denote a cylinder in $\mathbb{R}^3$, where $\Omega \in \mathbb{R}^2$ is a bounded domain with the smooth boundary $\partial\Omega$. We assume that there are given compact, simply connected subsets $\mathcal{K}_i \subset \partial\Omega$, $i = 1,2$, such that $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$ and two sufficiently smooth curves $\mathcal{X}_i(\cdot) : [0,T] \to \mathcal{K}_i \subset \mathbb{R}^3$ on $\overline{\Sigma} = \partial\Omega \times [0,T]$. The set $\overline{\Sigma}$ consists of two subsets $\overline{\Sigma}_i$, $i = 1,2$, with

$$\overline{\Sigma}_1 \cap \overline{\Sigma}_2 = \bigcup_{t\in[0,T]} \{\mathcal{X}_1(t) \times t\} \cup \{\mathcal{X}_2(t) \times t\} .$$

We assume that $\mathcal{X}_i(\cdot) \in H^2(0,T;\mathbb{R}^2)$ for $i = 1,2$, and denote $\mathcal{X} = \{\mathcal{X}_1, \mathcal{X}_2\} \in \mathcal{U} = H^2(0,T;\mathbb{R}^4)$. The set of admissible curves $\mathcal{U}_{ad} = \{\mathcal{X} \in H^2(0,T;\mathbb{R}^4) | \mathcal{X}(t) \in \mathcal{K}_1 \times \mathcal{K}_2\}$ obviously is not convex.

The domain optimization problem can be formulated as follows

$$\inf_{\mathcal{X}\in\mathcal{U}_{ad}} J(\mathcal{X}) \tag{1}$$

$$\text{with } J(\mathcal{X}) = \frac{1}{2}\int_Q (y(\mathcal{X}) - y_d)^2 dQ + \frac{\alpha}{2}\|\mathcal{X}\|^2_{\mathcal{U}}$$

where $y_d \in L^2(Q)$ is given, $\alpha \geqq 0$ is a regularization parameter. The function $y(\mathcal{X})(x,t)$, $\mathcal{X} \in \mathcal{U}_{ad}$, $(x,t) \in Q$, satisfies for a given function $F \in L^2(Q)$ the heat equation

$$\frac{\partial y}{\partial t} - \Delta y = F, \qquad \text{in } Q = \Omega \times (0,T)$$

with given initial condition in Ω and mixed boundary conditions depending on $\mathcal{X} \in \mathcal{U}_{ad}$ prescribed on the lateral boundary of Q.

The boundary conditions are prescribed as follows: $y = y(\mathcal{X})$ satisfies the nonhomegenuous Dirichlet condition on Σ_2 (resp. nonhomogenuous Neumann condition on Σ_1). In order to obtain the existence of an optimal domain i.e. the existence of an element $\mathcal{X}^\star \in \mathcal{U}_{ad}$ such that $J(\mathcal{X}^\star) \leqq J(\mathcal{X})$ for all $\mathcal{X} \in \mathcal{U}_{ad}$, it is sufficient to select a family of admissible domains which is compact in an appropriate sense. For the problem under consideration it is sufficient to assume that for a minimizing sequence of domains $\{Q_m\}$ there exists a subsequence, still denoted $\{Q_m\}$, such that the sequence of characteristic functions $\chi_m =$ characteristic function of Σ_2^m, converges in $L^2(\Sigma)$ to a characteristic function $\overline{\chi}$.

In order to have the existence of an optimal domain the standard approach of control theory is used, namely a regularizing term is introduced. We refer the reader to Sokołowski and Zolesio (1992) for a description of the material derivative method in the shape sensitivity analysis. The related results on the shape sensitivity analysis of optimal control problems can be found in Sokołowski (1987, 1988).

2. Parabolic equation in variable domain

Let us consider the following parabolic equation

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = F, & \text{in } Q = \Omega \times (0,T) \\ \frac{\partial y}{\partial n} = f, & \text{on } \Sigma_1 \\ y = g, & \text{on } \Sigma_2 \\ y(x,0) = y_0(x), & \text{in } \Omega \end{cases}$$

where $f, g \in L^2(\Sigma)$ are given,

$$\Sigma_i = \{\ (x,t) \in \Gamma_i(t) \times \{t\},\ t \in (0,T)\ \} \quad i = 1,2$$

and

$$\overline{\Gamma_1(t)} \cap \overline{\Gamma_2(t)} = \mathcal{X}_1(t) \cup \mathcal{X}_1(t) \text{ for all } t \in [0,T]\ .$$

Here Ω is a given domain, $\overline{\partial\Omega} = \Gamma_1(0) \cup \Gamma_2(0) \cup \{\mathcal{X}_1(0)\} \cup \{\mathcal{X}_2(0)\}$. We denote $\Gamma_0 \equiv \Gamma_0(0)$, $\Gamma_1 \equiv \Gamma_1(0)$.

In order to derive the first order necessary optimality conditions for the optimization problem defined in the variable domain, we assume

$$\mathcal{X}_1(t) = T_t(V)(\mathcal{X}_1(0)),\ \mathcal{X}_2(t) = T_t(V)(\mathcal{X}_2(0)),\ \forall t \in [0,T]$$

for a given vector field $V(.,.) \in C(0, T+\delta_1; C^1(\mathbb{R}^2, \mathbb{R}^2))$, $\delta_1 > 0$ with the support in a compact of $\partial\Omega$.

First, we investigate the differential stability of the solution to the parabolic equation with respect to the perturbations of the curves $\mathcal{X}_1(t), \mathcal{X}_2(t) \in \partial\Omega \subset \mathbb{R}^2, t \in (0,T]$. Let $\mathcal{X}_1^\varepsilon(.), \mathcal{X}_2^\varepsilon(.)$ be perturbed curves defined below, where $\varepsilon \in [0,\delta)$ is a parameter. Denote by y_ε a solution to the parabolic equation in the perturbed domain,

$$\begin{cases} \frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon = F, & \text{in } Q = \Omega \times (0,T) \\ \frac{\partial y_\varepsilon}{\partial n} = f, & \text{on } \Sigma_1^\varepsilon \\ y_\varepsilon = g, & \text{on } \Sigma_0^\varepsilon \\ y_\varepsilon(x,0) = y_0(x), & \text{in } \Omega \end{cases}$$

here $f = f_\varepsilon$, $g = g_\varepsilon$ denote the restriction to Σ_2^ε (resp. Σ_1^ε) of functions f (resp. g) defined on Σ. We assume that for sufficiently small t the curves $\mathcal{X}_1(.), \mathcal{X}_2(.)$ are not perturbed

$$\mathcal{X}_1^\varepsilon(t) = \mathcal{X}_1(t)\ , \mathcal{X}_2^\varepsilon(t) = \mathcal{X}_2(t),\ t \in [0,\delta_1),\ \delta_1 > 0.$$

Furthermore, we assume that $\mathcal{X}_1^\varepsilon(t)$, $\mathcal{X}_2^\varepsilon(t) \in \partial\Omega$, $\mathcal{X}_1^\varepsilon(t) \neq \mathcal{X}_2^\varepsilon(t)$, $\forall t \in [0,T]$, $\forall\varepsilon \in [0,\delta)$ and there exist for $i = 1,2$ the limits in $H^2(0,T;\mathbb{R}^2)$

$$h_i = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}(\mathcal{X}_i^\varepsilon - \mathcal{X}_i^0),$$

where $h_1(t)$, $h_2(t)$, $t \in (0,T)$, are the tangent vectors on $\Sigma = \partial\Omega \times (0,T)$.

In this section we derive the form of the derivative y' of the solution $y_\varepsilon = y(\mathcal{X}_\varepsilon) \in L^2(Q)$ to the parabolic equation, with respect to ε, at $\varepsilon = 0$, assuming that the derivative exists in the space $L^2(Q)$. For the proof of the existence we refer the reader to Hoffmann and Sokołowski (1991). Using the transposition method it follows that the parabolic equation is equivalent to the following integral identity

$$y_\varepsilon \in L^2(Q): \qquad \int_0^T \int_\Omega y_\varepsilon\Big(-\frac{\partial\varphi}{\partial t} - \Delta\varphi\Big)dxdt = \int_0^T \int_\Omega F\varphi dxdt$$

$$+ \int_{\Sigma_2^\varepsilon} g\frac{\partial\varphi}{\partial n}d\Sigma + \int_{\Sigma_1^\varepsilon} f\varphi d\Sigma + \int_\Omega y_0(x)\varphi(0,x)dx,$$

$$\forall\varphi \in H^{2,1}(Q),\ \varphi(T) = 0 \text{ in } \Omega,\ \varphi = 0 \ \text{ on } \Sigma_2^\varepsilon$$

Let us select a sufficiently smooth test function φ and let us assume that the test function is independent of the parameter ε i.e., $\varphi = 0$ in a small neighbourhood of $\Sigma_2^\varepsilon \subset \mathbb{R}^3$. If y_ε is differentiable with respect to ε, at $\varepsilon = 0$, the derivative y' satisfies the following integral identity

$$y' \in L^2(Q): \qquad \int_0^T \int_\Omega y'(-\frac{\partial\varphi}{\partial t} - \Delta\varphi)dxdt$$

$$= \int_{\partial\Sigma_2} g\frac{\partial\varphi}{\partial n}\mathcal{L}(h)d\ell + \int_{\partial\Sigma_1} f\varphi\mathcal{L}(h)d\ell$$

$$\forall\varphi \in H^{2,1}(Q),\ \varphi(T) = 0 \text{ in } \Omega,\ \varphi = 0 \ \text{ on } \Sigma_2^\varepsilon\ ,$$

where we denote

$$h = \begin{cases} h_1 = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}(\mathcal{X}_1^\varepsilon - \mathcal{X}_0), & \text{on } \mathcal{X}_1^0 \equiv \mathcal{X}_1 \\ h_2 = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}(\mathcal{X}_2^\varepsilon - \mathcal{X}_2^0), & \text{on } X_2^0 \equiv \mathcal{X}_2 \end{cases}$$

$$dl = (1 + \Big|\frac{dX_i}{dt}(t)\Big|^2)^{1/2}dt, \qquad \text{on } \mathcal{X}_i\ ,\quad i = 1,2$$

$$\mathcal{L}(h)(t) = \frac{1}{\alpha(t)}[h_i(t) - (h_i(t), V(t,\mathcal{X}_i(t))_{\mathbb{R}^2}/\|V(t,\mathcal{X}_i(t))\|_{\mathbb{R}^2}], \text{ on } \mathcal{X}_i,$$

$$i = 1,2$$

REMARK 1
If the test function φ is not sufficiently smooth, the right hand side of the integral identity for y' is not well defined. In such a case, assuming the differentiability of $\varepsilon \to y_\varepsilon$, the form of the derivative y' is defined by an auxiliary parabolic problem.

PROPOSITION 1
There exist distributions $\mathcal{G}_0$, $\mathcal{G}_1$ supported on $\partial\Sigma_2$ (resp. on $\partial\Sigma_1$) such that

$$y' \in L^2(Q): \qquad \int_0^T \int_\Omega y'(-\frac{\partial \varphi}{\partial t} - \Delta\varphi)dxdt$$

$$=< \mathcal{G}_0(\varphi), h_1 >_{\partial\Sigma_2} + < \mathcal{G}_1(\varphi), h_2 >_{\partial\Sigma_1}$$

$$\forall \varphi \in H^{2,1}(Q),\ \varphi(T) = 0 \ in\ \Omega,\ \varphi = 0 \quad on\ \Sigma_2 = \Sigma_2^0$$

3. Optimization problem

Finally, the necessary optimality conditions are derived for the following optimization problem.

Minimize the cost functional

$$J(\mathcal{X}) = \frac{1}{2}\int_0^T \int_\Omega (y_\varepsilon - y_d)^2 dxdt$$

$$+\frac{\alpha}{2}\|\mathcal{X}\|_{\mathcal{U}} + \frac{\beta_1}{2}\int_{\Sigma_2} g^2 d\Sigma + \frac{\beta_2}{2}\int_{\Sigma_1} f^2 d\Sigma$$

subject to the following nonconvex constraints $\mathcal{X}(t) = (\mathcal{X}_1(t), \mathcal{X}_2(t)) \in \mathcal{U}_{ad}$, for all $t \in [0,T]$, where $\alpha > 0$, $\beta_1, \beta_2 \geq 0$ are given, $\mathcal{U} = H^2(0,T;\mathbb{R}^4)$.

PROPOSITION 2
There exists an optimal solution $\overline{\mathcal{X}} \in \mathcal{U}_{ad}$ for the optimization problem.

In order to derive the first order necessary optimality conditions we assume that there exists a vector field $\overline{V}$ such that

$$\overline{\mathcal{X}}_i(t) \equiv T_t(\overline{V})(\mathcal{X}_i(0)) \qquad \forall t \in [0,T], \quad i = 1,2$$

A vector field $H(.,.)$ defines an admissible perturbation of the optimal solution $\overline{\mathcal{X}}$ provided that for $\varepsilon > 0$, ε small enough, $T_t(V_\varepsilon)(\mathcal{X}_1(0)) \in \mathcal{K}_1$, $T_t(V_\varepsilon)(\mathcal{X}_2(0)) \in \mathcal{K}_2$, $\forall t \in [0,T]$ where $V_\varepsilon = \overline{V} + \varepsilon H$.

For simplicity, we denote an optimal solution of the optimization problem by $\mathcal{X}$.

THEOREM 1
An optimal solution $\mathcal{X} \in \mathcal{U}_{ad}$ satisfies the following optimality system
State equation:

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = F, & in\ Q = \Omega \times (0,T) \\ \frac{\partial y}{\partial n} = f, & on\ \Sigma_1 \\ y = g, & on\ \Sigma_2 \\ y(x,0) = y_0(x), & in\ \Omega \end{cases}$$

Adjoint state equation:

$$\begin{cases} -\frac{\partial p}{\partial t} - \Delta p = y - y_d, & in\ Q = \Omega \times (0,T) \\ \frac{\partial p}{\partial n} = 0, & on\ \Sigma_1 \\ y = 0, & on\ \Sigma_2 \\ p(x,T) = 0, & in\ \Omega \end{cases}$$

Optimality conditions:

$$\int_{\partial\Sigma_2} g\frac{\partial p}{\partial n}\mathcal{L}(h)d\Sigma + \int_{\partial\Sigma_1} fp\mathcal{L}(h)d\Sigma + \alpha\Sigma_{i=1}^{2}(\mathcal{X}_i, h_i)_{H^2(0,T;\mathbb{R}^2)}$$

$$+\beta_1\int_{\partial\Sigma_2} g^2\mathcal{L}(h_1)dl + \beta_2\int_{\partial\Sigma_1} f^2\mathcal{L}(h_2)dl \geq 0, \qquad \forall h = (h_1, h_2)$$

for any admissible vector field $H(.,.)$.

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References

HOFFMANN, K.-H. AND SOKOŁOWSKI, J. (1991) Domain optimization problem for parabolic equation, submitted for publication, and DFG Report No. 342, Augsburg.

LIONS, J.L. AND MAGENES, E. (1968) *Problèmes aux Limites Non Homogènes et Applications.* Vol.1, Dunod, Paris.

SOKOŁOWSKI, J. (1987) Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. *SIAM J. Control and Optimization* **25**, *1542–1556.*

SOKOŁOWSKI, J. (1988) Shape sensitivity analysis of boundary optimal control problems for parabolic systems. *SIAM Journal on Control and Optimization* **26**, *763–787*.

SOKOŁOWSKI, J. AND ZOLESIO, J.-P. (1992) *Introduction to Shape Optimization. Shape sensitivity analysis.* Springer Verlag.