

Existence of extreme unilateral cracks in a plate

by

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An elastic plate is considered with middle surface limited by external boundary $\partial\Omega$. The plate is assumed to have a vertical crack whose shape may change. It is required to find a crack shape to be called an extreme one such that the displacements of the plate have a maximal deviation from given functions. In other words we define a functional on the set of functions describing crack shapes, which characterizes a deviation of displacements from prescribed elements. The problem is to maximize this functional. The cracks of finite length are considered with tips belonging both to the interior of the domain and to the external boundary $\partial\Omega$. Existence of extreme crack shapes will be proved in both cases. Similar problems with simpler nonpenetration conditions were considered in Khludnev (1992). The paper Khludnev (1989) was devoted to the case of normal displacements of a plate contacting with a rigid punch. A different approach to finding crack shapes was used in Banichuk (1970). Mathematical foundations of crack theory can be found for instance in Morozov (1984).

1. Internal cracks

A bounded domain $\Omega \subset R^2$ is assumed to have a smooth boundary $\partial\Omega$. A trace of the crack shape on the plane x, y is described by the function $y = \delta\psi(x)$, where $x \in (0, 1)$, δ is a parameter, $(x, y) \in \Omega$. The middle surface of the plate occupies the domain $\Omega_\delta = \Omega \setminus \Gamma_\delta$; Γ_δ is the graph of the function $y = \delta\psi(x)$. Horizontal and normal displacements of the middle surface points will be denoted by $W^\delta = (w_1^\delta, w_2^\delta)$ and w^δ , respectively, $\chi^\delta = (W^\delta, w^\delta)$. We introduce the energy functional

$$\Pi_\delta(\chi) = \frac{1}{2}b_\delta(w, w) + \frac{1}{2}B_\delta(W, W) - (F, \chi)_\delta.$$

Here

$$b_\delta(w, u) = \int_{\Omega_\delta} \{w_{xx}u_{xx} + w_{yy}u_{yy} + \sigma(w_{xx}u_{yy} + w_{yy}u_{xx}) + 2(1 - \sigma)w_{xy}u_{xy}\} d\Omega_\delta,$$

$$B_\delta(W, U) = \int_{\Omega_\delta} \{\epsilon_{11}(W)\epsilon_{11}(U) + \epsilon_{22}(W)\epsilon_{22}(U) + \sigma\epsilon_{11}(W)\epsilon_{22}(U) + \sigma\epsilon_{22}(W)\epsilon_{11}(U) + 2(1 - \sigma)\epsilon_{12}(W)\epsilon_{12}(U)\} d\Omega_\delta,$$

$\epsilon_{ij}(W)$ is the strain tensor, $\epsilon_{ij}(W) = \frac{1}{2}(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i})$, $x_1 = x$, $x_2 = y$, $0 < \sigma < \frac{1}{2}$, $\sigma = \text{const}$, $F = (f_1, f_2, f_3)$ is the vector of external forces, the brackets $(\cdot, \cdot)_\delta$ mean integration over Ω_δ . The model of the plate considered above is characterized by the following dependence on z of the horizontal displacements along the axis z : $w_i^z = w_i - zw_{x_i}$, $i = 1, 2$. It is assumed that $z = 0$ corresponds to the middle surface. Let us denote by $[\psi] = \psi^+ - \psi^-$ the jump of ψ on Γ_δ , where ψ^+ corresponds to the positive direction of the normal $\nu^\delta = (\nu_1^\delta, \nu_2^\delta)$ to Γ_δ , and ψ^- corresponds to the opposite direction. The aforesaid means that the nonpenetration condition for the crack banks may be written as follows

$$[W^\delta - z\nabla w^\delta]\nu^\delta \geq 0 \quad \text{on } \Gamma_\delta, \quad |z| \leq h. \quad (1)$$

Here $2h = \text{const}$ is the thickness of the plate. The jam conditions will be imposed on the external boundary:

$$W^\delta = w^\delta = \frac{\partial w^\delta}{\partial n} = 0 \quad \text{on } \partial\Omega. \quad (2)$$

Let also

$$H(\Omega_\delta) = H^{1,0}(\Omega_\delta) \times H^{1,0}(\Omega_\delta) \times H^{2,0}(\Omega_\delta),$$

where $H^{s,0}(\Omega_\delta)$ is the Sobolev space obtained by the closure in $H^s(\Omega_\delta)$ of smooth functions equal to zero near $\partial\Omega$. The norm in $H^s(\Omega_\delta)$ will be denoted by $\|\cdot\|_{s,\Omega_\delta}$. By introduction of the closed and convex set

$$K_\delta(\Omega_\delta) = \{(W, w) \in H(\Omega_\delta) \mid (W, w) \text{ satisfy (1)}\}$$

the equilibrium state of the plate may be described by the following variational problem

$$\inf \Pi_\delta(\chi) \quad \text{on } K_\delta(\Omega_\delta).$$

The latter is equivalent to the inequality

$$\chi^\delta \in K_\delta(\Omega_\delta) : \quad \langle \Pi'_\delta(\chi^\delta), \bar{\chi} - \chi^\delta \rangle \geq 0 \quad \text{for all } \bar{\chi} \in K_\delta(\Omega_\delta), \quad (3)$$

where $\Pi'_\delta(\chi^\delta)$ is the derivative of the functional Π_δ at the point χ^δ . It is easily seen that a solution $\chi^\delta = (W^\delta, w^\delta)$ of the problem (3) exists for every fixed δ and that the inequality (3) may be written in the form

$$b_\delta(w^\delta, \bar{w} - w^\delta) + B_\delta(W^\delta, \bar{W} - W^\delta) \geq (F, \bar{\chi} - \chi^\delta)_\delta \quad \text{for all } \bar{\chi} \in K_\delta(\Omega_\delta). \quad (4)$$

At the first step we shall study the behaviour of the solution when $\delta \rightarrow 0$. Convergence of the solution will be proved in an appropriate sense. To this end a one-to-one correspondence between Ω_δ and Ω_0 will be established. The function ψ is assumed to belong to the space $H_0^2(0, 1)$ and, moreover, ψ will be extended beyond the interval $(0, 1)$ by zero. Let us choose domains Ω_1, Ω_2 such that $\bar{\Omega}_1 \subset \Omega_2, \bar{\Omega}_2 \subset \Omega, \Gamma_\delta \subset \Omega_1$ for all small δ and take a smooth nonnegative function ξ possessing the properties: $\xi = 1$ on $\Omega_1, \xi = 0$ in $\Omega \setminus \Omega_2$. The transformation of the independent variables

$$\tilde{x} = x, \quad \tilde{y} = y - \delta\psi\xi \quad (5)$$

is mapping Ω_δ onto Ω_0 . The Jacobian $q_\delta = 1 - \delta\psi\xi_y$ of this transformation is positive for all small δ . We denote $u^\delta(\tilde{x}, \tilde{y}) = w^\delta(x, y), U^\delta(\tilde{x}, \tilde{y}) = W^\delta(x, y), \omega^\delta = (U^\delta, u^\delta)$. Then the restriction (1) may be rewritten as

$$[U^\delta - z(u_x^\delta - \delta\psi_x u_y^\delta, u_y^\delta)](-\delta\psi_x, 1) \geq 0 \quad \text{on } \Gamma_0, \quad |z| \leq h. \quad (6)$$

A set of functions from $H(\Omega_0)$ satisfying (6) will be denoted by $K_\delta(\Omega_0)$. As a result the inequality (4) is reduced to the following

$$\omega^\delta \in K_\delta(\Omega_0) : \quad b_0^\delta(u^\delta, \bar{u} - u^\delta) + B_0^\delta(U^\delta, \bar{U} - U^\delta) \geq (F^\delta, \bar{\omega} - \omega^\delta)_0 \quad (7)$$

for all $\bar{\omega} \in K_\delta(\Omega_0)$.

Here $F^\delta = q_\delta^{-1}F_\delta, F_\delta(\tilde{x}, \tilde{y}) = F(x, y), b_0^\delta(u^\delta, \bar{u}) = b_\delta(w^\delta, \bar{w}), B_0^\delta(U^\delta, \bar{U}) = B_\delta(W^\delta, \bar{W})$. It is clear that $0 \in K_\delta(\Omega_0)$ and $\frac{3}{4} < q_\delta < \frac{5}{4}$ for all δ small enough. We can substitute $\bar{\omega} = 0$ in (7) as the test function. Taking into account the estimate

$$b_0(u, u) \geq c \|u\|_{2, \Omega_0}^2$$

and the first Korn's inequality which are valid in Ω_0 one arrives therefore at the estimate

$$\|U^\delta\|_{1, \Omega_0} + \|u^\delta\|_{2, \Omega_0} \leq c \quad (8)$$

uniform in $\delta \leq \delta_0$. It turns out that every element from $K_0(\Omega_0)$ may be approximated by elements from $K_\delta(\Omega_0)$. This allows us to establish the following statement.

LEMMA. For every element $(u_1, u_2, u) \in K_0(\Omega_0)$ there exists a sequence $(u_1^\delta, u_2^\delta, u^\delta) \in K_\delta(\Omega_0)$ such that as $\delta \rightarrow 0$

$$(u_1^\delta, u_2^\delta, u^\delta) \rightarrow (u_1, u_2, u) \quad \text{strongly in } H(\Omega_0).$$

PROOF. The inclusion $(u_1, u_2, u) \in K_0(\Omega_0)$ means the validity of the following inequality

$$[u_2] - z[u_y] \geq 0 \quad \text{on } \Gamma_0, \quad |z| \leq h.$$

The function $u_{\bar{x}} - \delta\psi_x u_{\bar{y}}$ belongs to the space $H^{1,0}(\Omega_0)$. Hence, its traces on the lines $\bar{y} = 0+$, $\bar{y} = 0-$ are the elements of $H^{\frac{1}{2}}(\bar{y} = 0\pm)$. The difference of these traces belongs to $H^{\frac{1}{2}}(\bar{y} = 0)$ and coincides with $[u_{\bar{x}} - \delta\psi_x u_{\bar{y}}]$ on Γ_0 . Let us choose the extension of this difference from the space $H^1(R^2)$ denoting it by Q . Consequently, the restriction of the function $\xi h |\delta\psi_x Q|$ to Ω is the element of $H_0^1(\Omega)$. Now we may define in Ω_0

$$(u_1^\delta, u_2^\delta, u^\delta) = (u_1, u_2, u) + (0, \delta\psi_x u_1 + \xi h |\delta\psi_x Q|, 0).$$

First of all the inclusion $(u_1^\delta, u_2^\delta, u^\delta) \in K_\delta(\Omega_0)$ will be proved. In so doing we have to notice that the needed boundary conditions on $\partial\Omega$ are fulfilled. Hence, it suffices to prove (6). It follows from the above considerations that the inequality

$$h |\delta\psi_x Q| \geq -\delta z \psi_x [u_{\bar{x}} - \delta\psi_x u_{\bar{y}}] \quad \text{for all } z, |z| \leq h$$

holds on Γ_0 . Whence, one has on Γ_0

$$\begin{aligned} & [u_1](-\delta\psi_x) + [u_2] + \delta\psi_x [u_1] + h |\delta\psi_x Q| + \\ & \delta z \psi_x [u_{\bar{x}} - \delta\psi_x u_{\bar{y}}] - z [u_{\bar{y}}] \geq [u_2] - z [u_{\bar{y}}] \geq 0, \quad |z| \leq h, \end{aligned}$$

that is to say the inequality (6) takes place. The strong convergence of the sequence $(u_1^\delta, u_2^\delta, u^\delta)$ to (u_1, u_2, u) in $H(\Omega_0)$ is obvious. Lemma has been therefore proved. ■

Thanks to (8) one may choose a subsequence from the sequence ω^δ with the previous notation such that as $\delta \rightarrow 0$

$$\omega^\delta \rightarrow \omega \quad \text{weakly in } H(\Omega_0).$$

Let us take any fixed element $\bar{\omega} \in K_0(\Omega_0)$ and construct a sequence $\bar{\omega}^\delta \in K_\delta(\Omega_0)$ strongly converging in $H(\Omega_0)$ to $\bar{\omega}$. Taking into consideration the strong convergence $F_\delta \rightarrow F$, $q_\delta \rightarrow 1$ in $L^2(\Omega)$ we may therefore justify the passage to the limit $\delta \rightarrow 0$ in (7) and get

$$b_0(u, \bar{u} - u) + \bar{B}_0(U, \bar{U} - U) \geq (F, \bar{\omega} - \omega)_0 \quad \text{for all } \bar{\omega} \in K_0(\Omega_0). \quad (9)$$

This inequality means that the limiting function $\omega = (U, u)$ corresponds to the crack shape $\bar{y} = 0$. Thus the following statement has been proved :

THEOREM 1 *From the sequence $\chi^\delta = \omega^\delta$ of solutions of the problem (3) one can choose a subsequence weakly converging in $H(\Omega_0)$ to the solution ω of the problem (9).*

Now we are in a position to prove the existence of the extreme crack shape. The formulation of the appropriate problem will be as follows. Let Ψ be a closed convex and bounded set in $H_0^3(0, 1)$. Every element $\psi \in \Psi$ is assumed to

describe a crack shape. The space of functions analogous to $H(\Omega_\delta)$ is denoted by $H(\Omega_\psi)$. The nonpenetration condition in this case has the form

$$[W_\psi - z \nabla w_\psi] \nu_\psi \geq 0 \quad \text{on} \quad \Gamma_\psi, \quad |z| \leq h. \quad (10)$$

Here (W_ψ, w_ψ) is the displacement, Γ_ψ is the graph of the function $y = \psi(x)$, $\nu_\psi = (-\psi_x, 1)(1 + \psi_x^2)^{-\frac{1}{2}}$ is the normal to Γ_ψ . We denote by $K_\psi(\Omega_\psi)$ the set of functions from $H(\Omega_\psi)$ satisfying (10). The solution $\chi_\psi = (W_\psi, w_\psi)$ can be found from the variational inequality

$$\chi_\psi \in K_\psi(\Omega_\psi) : \quad \langle \Pi'_\psi(\chi_\psi), \bar{\chi} - \chi_\psi \rangle \geq 0 \quad \text{for all} \quad \bar{\chi} \in K_\psi(\Omega_\psi).$$

Let the functions W_0, w_0 be given belonging to the space $L^2(\Omega)$. We introduce the cost functional

$$J(\psi) = \|W_\psi - W_0\|_{0, \Omega_\psi} + \|w_\psi - w_0\|_{0, \Omega_\psi}$$

and consider the optimal control problem

$$\sup J(\psi) \quad \text{on} \quad \Psi. \quad (11)$$

THEOREM 2 *There exists a solution of the problem (11).*

PROOF. Let ψ^n be a maximizing sequence. It is bounded in the space $H_0^3(0, 1)$. Choosing a subsequence, if necessary, we may assume that as $n \rightarrow \infty$

$$\psi^n \rightarrow \psi \quad \text{weakly in} \quad H_0^3(0, 1).$$

In view of the imbedding theorems the additional convergence

$$|\psi_{xx}^n - \psi_{xx}| < \frac{1}{n} \quad \text{on} \quad (0, 1)$$

takes place. The unique solution $\chi^n = (W^n, w^n)$ satisfying the following relations can be found for every n

$$\chi^n \in K_{\psi^n}(\Omega_{\psi^n}) : \quad \langle \Pi'_{\psi^n}(\chi^n), \bar{\chi} - \chi^n \rangle \geq 0 \quad \text{for all} \quad \bar{\chi} \in K_{\psi^n}(\Omega_{\psi^n}). \quad (12)$$

We choose domains Ω_1, Ω_2 and a function ξ as above assuming $\Gamma_{\psi^n} \subset \Omega_1$ for all n and consider the transformation of the independent variables

$$\tilde{x} = x, \quad \tilde{y} = y + (\psi - \psi^n)\xi.$$

As above the functions ψ, ψ^n are assumed to be extended by zero beyond the interval $(0, 1)$. We obtain therefore a one-to-one mapping between Ω_{ψ^n} and Ω_ψ with the positive Jacobian $q_n = 1 + (\psi - \psi^n)\xi_y$ for all sufficiently large n . The further reasoning remind this used to prove convergence of W^δ, w^δ . Let $\chi^n(x, y) = \omega^n(\tilde{x}, \tilde{y})$. The inequality (12) may be rewritten in the variables \tilde{x}, \tilde{y}

$$\omega^n \in K_{\psi^n}(\Omega_\psi) : \quad b_n^\psi(u^n, \bar{u} - u^n) + B_n^\psi(U^n, \bar{U} - U^n) \geq (F^n, \bar{\omega} - \omega^n)_\psi \quad (13)$$

for all $\bar{\omega} \in K_{\psi^n}(\Omega_\psi)$,

where

$$\begin{aligned} F^n &= q_n^{-1} F_n, F_n(\tilde{x}, \tilde{y}) = F(x, y), \\ K_{\psi^n}(\Omega_\psi) &= \{(U, u) \in H(\Omega_\psi) \mid [U - z(u_{\tilde{x}} + (\psi_x - \psi_x^n)u_{\tilde{y}}, u_{\tilde{y}})](-\psi_x^n, 1) \geq 0 \\ &\quad \text{on } \Gamma_\psi, |z| \leq h\}. \end{aligned}$$

It is of importance that thanks to the above convergence of ψ^n all derivatives of $n(\psi - \psi^n)\xi$ up to the second order being included in (13) are bounded. A priori estimates of solutions will be as follows

$$\|U^n\|_{1, \Omega_\psi} + \|u^n\|_{2, \Omega_\psi} \leq c$$

with a constant c independent of n . Choosing a subsequence with the same notation one may suppose that as $n \rightarrow \infty$

$$\omega^n \rightarrow \omega \quad \text{weakly in } H(\Omega_\psi), \quad \text{strongly in } L^2(\Omega_\psi).$$

To justify the passage to limit in (13) we have to take into account that for every fixed $\bar{\omega} \in K_\psi(\Omega_\psi)$ there exists a sequence $\bar{\omega}^n \in K_{\psi^n}(\Omega_\psi)$ such that

$$\bar{\omega}^n \rightarrow \bar{\omega} \quad \text{strongly in } H(\Omega_\psi).$$

This convergence can be proved as that of Lemma. Hence, it follows from (13) that

$$\omega \in K_\psi(\Omega_\psi) : \quad \langle \Pi'_\psi(\omega), \bar{\omega} - \omega \rangle \geq 0 \quad \text{for all } \bar{\omega} \in K_\psi(\Omega_\psi),$$

i.e. $\omega = \omega(\psi)$. Moreover

$$\begin{aligned} \sup J(\bar{\psi}) &= \limsup \{ \|W^n - W_0\|_{0, \Omega_{\psi^n}} + \|w^n - w_0\|_{0, \Omega_{\psi^n}} \} \\ &= \lim \{ \|q_n^{-\frac{1}{2}}(U^n - W_{0n})\|_{0, \Omega_\psi} + \|q_n^{-\frac{1}{2}}(u^n - w_{0n})\|_{0, \Omega_\psi} \} \\ &= \|U_\psi - W_0\|_{0, \Omega_\psi} + \|u_\psi - w_0\|_{0, \Omega_\psi} = J(\psi). \end{aligned}$$

The latter means that the function ψ solves the problem (11). Theorem 2 has been proved. \blacksquare

The same reasoning allows us to prove an existence of solutions of the problem

$$\inf J(\psi) \quad \text{on } \Psi.$$

2. Boundary cracks

Let us consider the case when tips of a crack may belong to the external boundary $\partial\Omega$. We assume that the points $(0, 0) \in \partial\Omega$ and $(0, 1) \in \Omega$ correspond to the tips of the crack. As above $\Omega_\delta = \Omega \setminus \Gamma_\delta$, Γ_δ is the graph of the function $y = \delta\psi(x)$. In this case the function ψ is supposed to belong to the space

$H_0^4(0, 1)$. Moreover, the tangent to the boundary $\partial\Omega$ at the point $(0, 0)$ and the tangent to the graph $y = \delta\psi(x)$ at the same point are assumed to be different. The displacements of the middle surface of the plate can be found from the inequality

$$\chi^\delta \in K_\delta(\Omega_\delta) : \quad \langle \Pi'_\delta(\chi^\delta), \bar{\chi} - \chi^\delta \rangle \geq 0 \quad \text{for all } \bar{\chi} \in K_\delta(\Omega_\delta). \quad (14)$$

At the first step we intend to investigate the behaviour of the solution χ^δ when $\delta \rightarrow 0$. To do this we choose domains Ω_1, Ω_2 with boundaries $\partial\Omega_1, \partial\Omega_2$ and a smooth function $\xi \geq 0$ such that $\xi = 1$ in Ω_1 , $\xi \in C^\infty(\Omega)$, $\Omega_1 \subset \Omega_2$, $\Omega_2 \subset \Omega$, $\xi = 0$ beyond Ω_2 , $\{\Gamma_\delta \setminus (0, 0)\} \subset \Omega_1$. In particular, it can be done if the equations of the boundaries near the point $(0, 0)$ are chosen in the form $y = \alpha_2 x$, $y = \alpha_1 x$, $y = \beta_2 x$, $y = \beta_1 x$, $\beta_2 > 0$, $\beta_1 < 0$, $\alpha_i, \beta_i = \text{const}$, $\alpha_2 > \beta_2 > \beta_1 > \alpha_1$ (see Khludnev, 1992). As earlier, the transformation of the variables x, y has the form $\tilde{x} = x$, $\tilde{y} = y - \delta\psi\xi$. In view of the inclusion $\psi \in H_0^4(0, 1)$ we obtain $\psi(x) = o(x^2)$, $\psi_x(x) = o(x)$, $\psi_{xx}(x) = o(1)$ near $x = 0$, so that the function $\psi\xi$ and its derivatives up to the second order are bounded when $x \rightarrow 0$, $(x, y) \in \Omega$. The scheme of further reasoning is analogous to that of section 1. Hence, we arrive at the following statement :

THEOREM 3 *From the sequence ω^δ one can choose a subsequence with the same notation such that as $\delta \rightarrow 0$*

$$\omega^\delta \rightarrow \omega \quad \text{weakly in } H(\Omega_0),$$

where ω is the solution of the problem

$$\omega \in K_0(\Omega_0) : \quad \langle \Pi'_0(\omega), \bar{\omega} - \omega \rangle \geq 0 \quad \text{for all } \bar{\omega} \in K_0(\Omega_0).$$

Using the above arguments we may study the existence of extreme crack shapes. Let $y = \psi(x)$ be the equation of the crack shape with tips at the points $(0, 0)$ and $(0, 1)$; $(0, 0) \in \partial\Omega$. The convex closed and bounded set $\Psi \subset H_0^4(0, 1)$ is assumed to be chosen such that the above mentioned hypothesis concerning the discrepancy of the tangents at the point $(0, 0)$ is fulfilled. The formulation of the problem will be as follows. It is required to maximize the cost functional

$$\sup J(\psi) \quad \text{on } \Psi, \quad (15)$$

where as before

$$J(\psi) = \|W_\psi - W_0\|_{0, \Omega_\psi} + \|w_\psi - w_0\|_{0, \Omega_\psi}.$$

The solution of the problem (15) also exists. We will omit the arguments.

In conclusion we notice that the same reasoning allows us to prove the existence of extreme crack shapes in the case when the both tips of the crack belong to the external boundary: $(0, 0) \in \partial\Omega$, $(0, 1) \in \partial\Omega$.

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