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Saddle point approach to stiffness optimization of discrete structures including unilateral contact[1]

by

Joakim Petersson and **Anders Klarbring**

Department of Mechanical Engineering
Division of Solid Mechanics
Linköping Institute of Technology
S-581 83 Linköping
Sweden

This paper treats the problem of maximum stiffness topology optimization for discrete structures and makes two extensions to include unilateral contact. It is demonstrated how a saddle point approach to these problems gives a clear and concise theory which results in simple proofs of existence of a solution and optimality conditions. The extensions to include unilateral contact are, firstly, the direct one where unilateral contact conditions are simply allowed alongside classical boundary conditions and, secondly, a problem where initial gaps between contact nodes and obstacles are treated as design variables in addition to the volume ones. Both these extensions, as well as the classical problem, result in uniformly stressed structures, but the second one also gives a uniform contact force distribution. A simple two-bar truss is optimized in order to exemplify different features of optimal structures in unilateral contact.

1. Introduction

Optimization problems where the energy function of the state problem is used as objective function (or, in some cases, equivalently as a single constraint function) have been and continue to be of central importance in the field of structural optimization. Typical problems of this kind are plastic limit design problems, maximum stiffness design problems, design for maximum buckling load and design for maximum first natural frequency. We refer to Save and Prager (1985) for a classical treatment and to Bendsøe and Mota Soares (1993) for a view of the modern use in topology optimization. In this paper we study a discrete version of the elastic maximum stiffness design problem, extend this problem to

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include unilateral contact in two different ways and demonstrate the usefulness of viewing this classical problem as a saddle point problem.

There are several resons for the importance of the above-mentioned problems. We like to explicitly mention the following two:

1. Structural optimization problems in terms of energy functions are convex, which has obvious implications for the possibility to make qualitative analysis and obtain numerical solutions.
2. These problems frequently result in optimum structures that are optimal in very fundamental ways, beyond the immediate result of, for instance, maximum stiffness. This is seen by deriving optimality conditions, which due to convexity are both necessary and sufficient for optimality. One finds that (i) optimal trusses, sheets and sandwich-type beams and plates are uniformly stressed, (ii) in shape optimization the optimized contour will have a uniform strain energy density and (iii) in contact problems the contact stress distribution is uniform (or "almost" uniform, see Klarbring and Haslinger 1993).

As mentioned, we study a discrete problem of maximum stiffness. It can be viewed as a model for truss structures and finite element discretized sheets, but other interpretations are also possible.

In section 3 the case of bilateral non-contacting structures is considered. The problem treated is in a sense identical to that considered by Ben–Tal and Bendsøe (1991) and Achtziger, Bendsøe, Ben–Tal and Zowe (1992). However, our approach through the saddle point theory is more direct and gives shorter proofs of existence of a solution and optimality conditions. Also, since with any saddle point problem a primal and a dual optimization problem are connected, we indicate how in the present case the primal one is the problem treated by Ben-Tal *et al.* (see above) and the dual optimization problem is related to the classical nested approach of structural optimization, where the state variables are eliminated. Note also that since design variables, that represent volumes, are allowed to reach zero during the optimization process, structural elements can be removed and, therefore, this problem is usually considered to be the one of topology optimization.

In section 4 we extend the problem to include the possibility of the structure coming into unilateral contact. This is a discrete version of the continuous problem formulated by Benedict (1982) and recently mathematically studied by Petersson (1994). The saddle point formulation gives existence of a solution and optimality conditions by almost trivial extensions of the same results for the non-contact case. Moreover, a simple two-bar truss example is investigated to give insight into the nature of optimal trusses in unilateral contact.

A second generalization to include unilateral contact is given in section 5. Here, in addition to the volume design variables, the initial gaps between contact nodes and obstacles are treated as design variables. The result is an optimal structure that is uniformly stressed and has a uniform contact force distribution. Again, existence and optimality conditions follow directly from the saddle point theory. Presented results are slight extensions of the ones given in Klarbring, Petersson and Rönnqvist (1993).

As a prerequisite to the mechanical problems we give in section 2 some general results from non-linear programming and saddle point theory. The classical Karush-Kuhn-Tucker (KKT)-conditions are derived through a saddle point approach that has much in common with the method of proof used for optimality conditions of the mechanical problems in the upcoming sections. This step is done mainly for didactic reasons but we also explicitly need the KKT-conditions in the following.

2. Saddle points and duality

We first give a saddle point treatment of a standard non-linear programming problem, including a derivation of the KKT-conditions. Then a general saddle point problem is defined and a theorem establishing existence of a solution is stated, and afterwards the non-linear programming problem is revisited in light of the general theory.

Let J_i be real-valued functions defined on $\mathbf{R}^n$, for $i = 0, 1, \ldots, k$. The zero-indexed function is the objective function, and the remaining k ones are used to define a permissible subset of $\mathbf{R}^n$, on which the optimization will be performed.

DEFINITION 2.1 The set $\mathcal{U} = \{u \in \mathbf{R}^n \mid J_i(u) \leq 0;\ i = 1, \ldots, k\}$ is said to satisfy the CQ[2]-condition if at least one of the following holds:

(*i*) $J_i(\cdot)$ is affine on $\mathbf{R}^n$ for $i = 1, \ldots, k$.

(*ii*) $\nexists\, \alpha_i \in \mathbf{R}^+;\ i = 1, \ldots, k,\ \sum_{i=1}^{k} \alpha_i > 0$
such that $\sum_{i=1}^{k} \alpha_i \nabla J_i(v) = 0 \qquad \forall v \in \mathcal{U}$. ■

The condition (*ii*) is naturally applicable only if the gradients exist, and it is due to Kuhn and Tucker.

The problem of minimizing J_0 over the set $\mathcal{U}$ defined above will be referred to as the *primal* problem:

$$(\mathcal{P}) \quad \left\{ \min_{u \in \mathcal{U}} \quad J_0(u) \right.$$

If we define Λ to be the following cone in $\mathbf{R}^k$:

$$\Lambda = \{v \in \mathbf{R}^k \mid v_i \geq 0\,; i = 1, \ldots, k\},$$

then we can also define the Lagrangian function $\mathcal{L}$ associated with $(\mathcal{P})$, as the following mapping from $\mathbf{R}^n \times \Lambda$ into $\mathbf{R}$:

$$\mathcal{L}(u, \lambda) = J_0(u) + \sum_{i=1}^{k} \lambda_i J_i(u) = J_0(u) + \lambda^T J(u)$$

[2]Constraint qualification

Here we have used the following notation:

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} ; \; J(u) = \begin{pmatrix} J_1(u) \\ \vdots \\ J_k(u) \end{pmatrix}$$

We will relate the primal problem to the problem of finding a saddle point to $\mathcal{L}$:

$$(\mathcal{SL}) \quad \begin{cases} \text{Find } (\tilde{u}, \tilde{\lambda}) \in \mathbf{R}^n \times \Lambda \; : \\ \mathcal{L}(\tilde{u}, \lambda) \leq \mathcal{L}(\tilde{u}, \tilde{\lambda}) \leq \mathcal{L}(u, \tilde{\lambda}) \quad \forall (u, \lambda) \in \mathbf{R}^n \times \Lambda \end{cases}$$

Note that here the variation of u is unconstrained, i.e. over the whole of $\mathbf{R}^n$, instead of over the set $\mathcal{U}$.

The proof of the following theorem can be found in e.g. Ciarlet (1989). [3]

THEOREM 2.1 *Suppose that the functions J_i are everywhere differentiable and convex; $i = 0, 1, \ldots, k$. Then $\tilde{u} \in \mathcal{U}$ solves $(\mathcal{P})$ if and only if (for some existing $\tilde{\lambda} \in \mathbf{R}^k$) $(\tilde{u}, \tilde{\lambda}) \in \mathbf{R}^n \times \Lambda$ solves $(\mathcal{SL})$, provided the CQ-condition is satisfied.*

Note that if $(\tilde{u}, \tilde{\lambda}) \in \mathbf{R}^n \times \Lambda$ solves $(\mathcal{SL})$, then $\tilde{u}$ necessarily belongs to $\mathcal{U}$.

LEMMA 2.1 *Suppose that J_i is convex and differentiable for $i = 0, 1, \ldots, k$. Then $(\tilde{u}, \tilde{\lambda}) \in \mathcal{U} \times \Lambda$ solves $(\mathcal{SL})$ if and only if*

$$\left. \begin{array}{r} J_i(\tilde{u}) \leq 0 \\ \tilde{\lambda}_i \geq 0 \end{array} \right\} \; i = 1, \ldots, k \quad \textit{and} \begin{cases} \nabla J_0(\tilde{u}) + \sum_{i=1}^k \tilde{\lambda}_i \nabla J_i(\tilde{u}) = 0 \in \mathbf{R}^n \\ \tilde{\lambda}^T J(\tilde{u}) = 0 \end{cases}$$

PROOF. The pair $(\tilde{u}, \tilde{\lambda})$ solves $(\mathcal{SL})$ if and only if (i): $\tilde{u} \in \mathcal{U}$ minimizes $u \longmapsto \mathcal{L}(u, \tilde{\lambda})$ and (ii): $\tilde{\lambda}$ maximizes $\lambda \longmapsto \mathcal{L}(\tilde{u}, \lambda)$. In both directions of the proof, $J_i(\tilde{u}) \leq 0$, $\tilde{\lambda}_i \geq 0$ for $i = 1, \ldots, k$, is a presumption, and hence we only have to handle the equalities.

(i) This is a minimization of a convex and differentiable functional, since these features are possessed by all the J_i's, and the λ_i's are all non-negative. The admissible set is a Banach space (namely $\mathbf{R}^n$), and, (Theorem 2.2 in Céa (1978) page 28 yields that), the minimum is attained at $\tilde{u} \in \mathbf{R}^n$ if and only if the first Gateaux derivative has the property that

$$\frac{\partial}{\partial u} \mathcal{L}(u, \tilde{\lambda}; \varphi)|_{u=\tilde{u}} = 0 \quad \forall \varphi \in \mathbf{R}^n \iff$$

$$J_0'(\tilde{u}; \varphi) + \sum_{i=1}^{k} \tilde{\lambda}_i J_i'(\tilde{u}; \varphi) = 0 \quad \forall \varphi \in \mathbf{R}^n.$$

[3] In Céa (1978), a result corresponding to Theorem 2.1 is proved without the differentiability assumption, extensively using the Hahn-Banach theorem. However, the assumed CQ in Cea (1978) does not include (i) of Definition 2.1.

Writing this in terms of gradients of J_i one obtains

$$\nabla J_0(\tilde{u}) + \sum_{i=1}^{k} \tilde{\lambda}_i \nabla J_i(\tilde{u}) = 0 \in \mathbf{R}^n.$$

(ii) This maximization holds if and only if $\lambda^T J(\tilde{u}) \leq \tilde{\lambda}^T J(\tilde{u})$ for all $\lambda \in \Lambda$. Now $\tilde{\lambda}^T J(\tilde{u}) \leq 0$ since $\tilde{u} \in \mathcal{U}$ and $\tilde{\lambda} \in \Lambda$. Taking $\lambda = 0 \in \mathbf{R}^k$ one gets $0 \leq \tilde{\lambda}^T J(\tilde{u})$, and hence $\tilde{\lambda}^T J(\tilde{u}) = 0$ is equivalent to the maximization. ■

THEOREM 2.2 *Let J_i; $i = 0, 1, \ldots, k$, be differentiable and convex, and suppose that the CQ-condition holds. Then, $\tilde{u} \in \mathcal{U}$ solves $(\mathcal{P})$ if and only if (for some existing $\tilde{\lambda} \in \mathbf{R}^k$)*

$$\left.\begin{array}{r} J_i(\tilde{u}) \leq 0 \\ \tilde{\lambda}_i \geq 0 \\ \lambda_i J_i(\tilde{u}) = 0 \end{array}\right\} \; i = 1, \ldots, k \quad \textit{and } \nabla J_0(\tilde{u}) + \sum_{i=1}^{k} \tilde{\lambda}_i \nabla J_i(\tilde{u}) = 0 \in \mathbf{R}^n.$$

PROOF. Follows from Lemma 2.1 and Theorem 2.1. ■

The necessary and sufficient inequalities and equalities in the above theorem will be referred to as KKT-conditions.

We now turn to the general duality theory. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are any two sets, and that $L(\cdot,\cdot)$ is a function defined on them: $L(\cdot,\cdot) : \mathcal{A} \times \mathcal{B} \to \mathbf{R}$. Given any saddle point problem

$$(SL) \quad \left\{ \begin{array}{l} \text{Find } (\tilde{u}, \tilde{v}) \in \mathcal{A} \times \mathcal{B} : \\ L(\tilde{u}, v) \leq L(\tilde{u}, \tilde{v}) \leq L(u, \tilde{v}) \quad \forall (u, v) \in \mathcal{A} \times \mathcal{B} \end{array} \right.$$

one always has two associated problems

$$(p) \quad \left\{ \min_{u \in \mathcal{A}} \quad \Psi(u) \right.$$

where $\Psi(u) = \sup_{v \in \mathcal{B}} L(u, v)$ and

$$(d) \quad \left\{ \max_{v \in \mathcal{B}} \quad \Phi(v) \right.$$

where $\Phi(v) = \inf_{u \in \mathcal{A}} L(u, v)$. It is a consequence of the definitions that if $(\tilde{u}, \tilde{v})$ solves (SL), then $\tilde{u}$ solves (p) and $\tilde{v}$ solves (d). Conversely, given the existence of a saddle point, if $\tilde{u}$ solves (p) and $\tilde{v}$ solves (d), then $(\tilde{u}, \tilde{v})$ solves (SL). (This is shown by Ekeland and Temam 1976). Hence it is justified to study the existence of saddle points:

THEOREM 2.3 *Suppose that L maps $\mathcal{A} \times \mathcal{B}$ into $\mathbf{R}$, and:*
V and Q are reflexive Banach spaces.
$\mathcal{A} \subset V$ is non-empty, closed and convex.
$\mathcal{B} \subset Q$ is non-empty, closed and convex.
$\forall v \in \mathcal{B} : \mathcal{A} \ni u \longmapsto L(u, v)$ is convex and lower semi-continuous.

$\forall u \in \mathcal{A} \; : \; \mathcal{B} \ni v \longmapsto L(u,v)$ is concave and upper semi-continuous.
Moreover, assume that

$$\exists v_0 \in \mathcal{B} \; : \; \lim_{\substack{u \in \mathcal{A} \\ \|u\|_{\mathcal{V}} \to +\infty}} L(u, v_0) = +\infty \quad (\textit{coercivity})$$

and that $\mathcal{B}$ is bounded. Then there exists at least one solution to (SL).

The above statement is essentially due to Ekeland and Temam (1976), but based on older work of Ky Fan (1964) and Sion (1958).

In the case of $L = \mathcal{L}$, $\mathcal{A} = \mathbf{R}^n$ and $\mathcal{B} = \Lambda$ we get the saddle point problem $(\mathcal{SL})$ treated earlier, and the associated problems (p) and (d) are called the *primal* and *dual* problems, and we denote them as $(\mathcal{P})$ and $(\mathcal{D})$:

$$(\mathcal{P}) \quad \min_{u \in \mathbf{R}^n} \sup_{\lambda \in \Lambda} \mathcal{L}(u, \lambda) = \min_{u \in \mathcal{U}} J_0(u)$$

$$(\mathcal{D}) \quad \max_{\lambda \in \Lambda} \inf_{u \in \mathbf{R}^n} \mathcal{L}(u, \lambda)$$

(Concerning the identity in $(\mathcal{P})$, see Céa (1978), Proposition 1.1 in page 149).

If $(\mathcal{SL})$ has a solution, then objective values of $(\mathcal{P})$ and $(\mathcal{D})$ exist and coincide. This is not necessarily the case if $(\mathcal{SL})$ does not have any solution. However, the inequality

$$\sup_{\lambda \in \Lambda} \inf_{u \in \mathbf{R}^n} \mathcal{L}(u, \lambda) \leq \inf_{u \in \mathbf{R}^n} \sup_{\lambda \in \Lambda} \mathcal{L}(u, \lambda)$$

always holds, and hence in cases when solutions to $(\mathcal{P})$ and $(\mathcal{D})$ exist, the objective value of $(\mathcal{D})$ is always less than or equal to the one of $(\mathcal{P})$.[4]

EXAMPLE 2.1 Suppose $J_0(u) = \frac{1}{2} u^T K u - f^T u$ and $\mathcal{U} = \{u \in \mathbf{R}^n \mid Cu \leq g\}$ where K is a $n \times n$ symmetric positive definite matrix and C is a $r \times n$ matrix $(r < n)$, and $f \in \mathbf{R}^n$, $g \in \mathbf{R}^r$ are given. Then the primal problem can be identified with the displacement formulation of the equilibrium problem of a discrete elastic contact problem. It is straightforward to show, (with Theorem 2.2 applied to the infimum in $(\mathcal{D})$ that is actually attained), that the dual problem can be written as

$$(\mathcal{D}) \quad \min_{p \leq 0} \frac{1}{2} p^T A p + p^T (w^* - g)$$

where $A = CK^{-1}C^T$, $\lambda = -p$ and $w^* = CK^{-1}f$. This is the so called reciprocal energy principle. For details, see Klarbring (1986). ■

[4] Sometimes referred to as "duality gap"

3. Maximizing stiffness for non-contacting structures

We will study a discrete structure of linearly elastic material. More specifically, a truss structure is first considered and then the more general case of a finite element (FE) discretized structure is commented on.

Suppose there are N nodal points and m number of bars; $m \leq \frac{1}{2}N(N-1)$. Let $u \in \mathbf{R}^n$ be the vector of nodal displacements where zero-prescribed components are excluded; $n = dim \cdot N - n_0$ where dim is 2 or 3 and n_0 is the number of zero-prescribed displacements. External forces (work-conjugate to u) that act on the structure are $f \in \mathbf{R}^n$. Now the condition of equilibrium can be written as

$$f = K(t)u; \quad K(t) = \sum_{i=1}^{m} t_i K_i \tag{1}$$

where $t \in \mathbf{R}^m$ is a vector of bar volumes, $K(t)$ is the structural stiffness matrix and K_i is the (assembled) element stiffness matrix for bar number i. K_i can be written as

$$K_i = \frac{E}{L_i^2}\gamma_i\gamma_i^T \tag{2}$$

where $E > 0$ is the Young's modulus and $L_i > 0$ is the length of element i. Furthermore $\gamma_i \in \mathbf{R}^n$ is a vector of direction cosines describing the orientation of the bar. With non-negative thicknesses (or volumes) it is easy to see from (1) and (2) that $K(t)$ is positive semi-definite and symmetric. In the case of a truss, we assume that there is a sufficient number of zero displacement directions. This means that for some design $t_0 \in \mathcal{T}$, e.g. a so called ground structure $t_0 := \frac{V}{m}\mathbf{1}_m$, $K(t_0)$ is positive definite.

Given a certain amount (volume) of material we will seek a structure of maximum stiffness by distributing the material to optimal positions. Of course the amount of material in an element is not allowed to be negative; $t_i \geq 0$, $i = 1, \ldots, m$. Since the lower bound is zero, bars may be removed (set equal to zero) if so required by the quest of maximal stiffness. Summing up, design constraints are

$$t \in \mathcal{T} = \left\{ t \in \mathbf{R}^m \;\middle|\; \sum_{i=1}^{m} t_i \equiv \mathbf{1}_m^T t = V, \quad 0 \leq t \right\} \tag{3}$$

where $0 \leq t$ means $t_i \geq 0$ for all i, $\mathbf{1}_a = (1, \ldots, 1)^T$ is vector of length a and $V > 0$ is the given total available volume of the bars.

The equation (1) and the design constraints also correspond to a FE discretization of a plane elasticity problem. Then the region occupied by the body is divided into a mesh of m elements and the thickness distribution is approximated in such a way that it takes uniform values t_i; $i = 1, \ldots, m$, in each element. If the body has a part of its boundary with positive measure where displacements are zero, and if the discretization on this boundary part is sufficiently dense, then (if $t_i > 0\ \forall i$) $K(t)$ is positive definite and symmetric, and $u \in \mathbf{R}^n$ contains the unknown nodal displacements; cf Hughes (1987).

The next lemma contains the principle of minimum potential energy and Clapeyron's theorem.

LEMMA 3.1 *The potential energy function, is given by*

$$\mathbf{R}^n \times \mathcal{T} \ni (u,t) \longmapsto J(u,t) = \frac{1}{2} u^T K(t) u - f^T u.$$

For any $t \in \mathcal{T}$, (1) is a necessary and sufficient condition for u to minimize $J(\cdot,t)$ over $\mathbf{R}^n$, and for such a minimizing displacement u it also holds that

$$J(u,t) = -\frac{1}{2} f^T u.$$

PROOF. The potential energy function is twice differentiable in u, and the first Gateaux derivative at u in any direction φ in $\mathbf{R}^n$, is

$$\frac{\partial}{\partial u} J(u,t;\varphi) = \varphi^T (K(t)u - f)$$

and the second derivative at u in any directions φ, φ is

$$\frac{\partial^2}{\partial u^2} J(u,t;\varphi,\varphi) = \varphi^T K(t) \varphi \geq 0.$$

The inequality is due to the positive semi-definiteness of $K(t)$, and it implies convexity of J. The first statement follows from Theorem 2.2 and the second one in turn follows from the first. ∎

We now turn to our main concern, namely, maximum stiffness structures. We will conveniently represent such structures (i.e. optimal designs $\tilde{t}$ and corresponding states $\tilde{u}$), as saddle points to the potential energy. Consider the problem

$$(SJ) \quad \begin{cases} \text{Find } (\tilde{u},\tilde{t}) \in \mathbf{R}^n \times \mathcal{T} : \\ J(\tilde{u},t) \leq J(\tilde{u},\tilde{t}) \leq J(u,\tilde{t}) \quad \forall (u,t) \in \mathbf{R}^n \times \mathcal{T} \end{cases}$$

Suppose $(\tilde{u},\tilde{t})$ solves (SJ) and that $t \in \mathcal{T}$ is any design with equilibrium state u_t, i.e. $f = K(t)u_t$. (If such a displacement u_t does not exist, the design t is useless). Then by Lemma 3.1 and the definition of (SJ)

$$J(u_t,t) \leq J(\tilde{u},t) \leq J(\tilde{u},\tilde{t}). \tag{4}$$

From the right inequality in (SJ) it follows that $\tilde{u}$ is an equilibrium state for $\tilde{t}$, and hence (4) says that a solution to (SJ) maximizes the *equilibrium* potential energy. Applying Lemma 3.1 once more to (4) we get

$$f^T \tilde{u} \leq f^T u_t \tag{5}$$

which proves that a solution to (SJ) minimizes the displacements weighted by f. This is the traditional objective in stiffness optimization.

That (SJ) is well-posed in some sense is shown in the following

THEOREM 3.1 *There exists a solution to* (SJ).

PROOF. We will verify all prerequisites of Theorem 2.3. Let us take $\mathcal{A} = V = \mathbf{R}^n$ and $\mathcal{B} = \mathcal{T} \subset Q = \mathbf{R}^m$. $\mathbf{R}^n$ is closed and convex in itself and it is straightforward to check that $\mathcal{T}$ has the same properties in $\mathbf{R}^m$. It is clear from the definitions that $K(t)$ is positive semi-definite for all $t \in \mathcal{T}$ and $J(u,t)$ is therefore convex in u for all $t \in \mathcal{T}$. Moreover, J is continuous in its both arguments and linear in t. By assumption, there is some ground structure $t_0 \in \mathcal{T}$ such that $K(t_0)$ is positive definite (and symmetric). Let $\lambda > 0$ be the smallest eigenvalue of $K(t_0)$. Then,

$$J(u,t_0) \geq \frac{\lambda}{2}\|u\|^2_{\mathbf{R}^n} - \|f\|_{\mathbf{R}^n}\|u\|_{\mathbf{R}^n} = \|u\|_{\mathbf{R}^n}\left\{\frac{\lambda\|u\|_{\mathbf{R}^n}}{2} - \|f\|_{\mathbf{R}^n}\right\}$$

by Cauchy-Schwartz inequality in $\mathbf{R}^n$. It is now immediate that

$$\lim_{\substack{u\in\mathbf{R}^n \\ \|u\|_{\mathbf{R}^n}\to+\infty}} J(u,t_0) = +\infty$$

and the definition of $\mathcal{T}$ contains sufficient constraints for $\mathcal{T}$ to be bounded. ■

We will now establish the necessary and sufficient optimality criteria for a solution to (SJ).

THEOREM 3.2 *A pair* $(\tilde{u},\tilde{t}) \in \mathbf{R}^n \times \mathcal{T}$ *solves* (SJ) *if and only if*
(j) $K(\tilde{t})\tilde{u} = f$, *and*
(jj) $\tilde{t} \in \mathcal{T}$, $\frac{1}{2}\tilde{u}^T K_i \tilde{u} < \max_{i=1,\ldots,m}(\frac{1}{2}\tilde{u}^T K_i \tilde{u}) \Rightarrow \tilde{t}_i = 0$.

PROOF. A pair $(\tilde{u},\tilde{t})$ solves (SJ) if and only if (i): $\tilde{u}$ minimizes $u \longmapsto J(u,\tilde{t})$ over $\mathbf{R}^k$ and (ii): $\tilde{t}$ maximizes $t \longmapsto J(\tilde{u},t)$ over $\mathcal{T}$. (i) holds if and only if $K(\tilde{t})\tilde{u} = f$ (cf. Lemma 3.1), which is (j). Applying Theorem 2.2 to the maximization problem (ii) we get the following necessary and sufficient conditions:

$$\exists\tilde{\lambda}_i \in \mathbf{R}^+,\ \kappa \in \mathbf{R},\ \frac{1}{2}\tilde{u}^T K_i \tilde{u} = \kappa - \tilde{\lambda}_i,\ (i = 1,\ldots,m)$$

$$\tilde{\lambda}_i\tilde{t}_i = 0,\ (i = 1,\ldots,m),\ \tilde{t} \in \mathcal{T}$$

which (with $V > 0$) are seen to be equivalent to (jj) with

$$\kappa = \max_{i=1,\ldots,m}(\frac{1}{2}\tilde{u}^T K_i \tilde{u}).$$ ■

In the above proof we have actually implicitly rewritten $\mathbf{1}_m^T t = V$ as two inequalities and taken $J_0(t) = -J(\tilde{u},t)$ in order to match Theorem 2.2. The interpretation of the above result is that an optimal structure is an equilibrium structure where all non-removed elements have a constant (or the maximal) strain energy density (κ). (If it is less than κ, then the volume of the element is set equal to zero).

The specific strain energy in any non-removed element is constant and equals to $\kappa = \max_{i=1,\ldots,m}(\frac{1}{2}\tilde{u}^T K_i \tilde{u})$, but in the case of a truss this can also be written as $\sigma_i^2/2E$ (where σ_i is the stress in bar i) and hence the stress magnitude

$$|\sigma_i| = \sqrt{E\tilde{u}^T K_i \tilde{u}} = \sqrt{2E\kappa} \tag{6}$$

is the same for any bar with non-zero volume. Thus, (SJ) generates designs that distribute stresses in a very "democratic" way, and this is so as long as the optimality condition (jj) is valid.

As mentioned in Section 1, there are always two associated problems, (p) and (d), to any saddle formulation. It can be seen, cf. Bendsøe and Ben–Tal (1991), that the primal problem corresponding to (SJ), can be formulated as minimizing

$$\psi(u) = V \cdot \max_{i=1,\ldots,m}\{\frac{1}{2}u^T K_i u\} - f^T u$$

over the set of kinematically admissible displacements.[5] Demyanov and Malozemov (1990) give necessary and sufficient Karush-Kuhn-Tucker-like conditions for solutions to convex and non-differentiable problems, that in this case, not surprisingly turn out to be exactly (j) and (jj) in Theorem 3.2.

In the same manner as in (p) where $\min_u \psi(u)$ is obtained, one can formulate the dual problem (d), which results in a formulation essentially equivalent to minimizing $\frac{1}{2}f^T u$, subject to equilibrium and design constraints. In some sense, however, this is already clear from the discussion ending up with (5).

4. Maximizing stiffness for contacting structures

Now we add unilateral frictionless contact with rigid obstacles to the treatment of the last section. The essential change is that the set of kinematically admissible displacements must be such that the structure can not penetrate the obstacles.

Let $r(< n)$ be the number of directions of unilateral contact and $\nu_i \in \mathbf{R}^n$ a vector of direction cosines of the (inward) normal to the i^{th} obstacle; $i = 1, \ldots, r$. The non-penetration conditions can then be written as

$$\nu_i^T u \leq g_i, \; i = 1, \ldots, r \tag{7}$$

[5]In Ben–tal and Bendsøe (1991) numerical algorithms were developed to solve this primal problem.

where g_i is the initial distance between contact node and obstacle number i. If C is a $r \times n$-matrix whose rows are the ν_i^T's and g is a vector in $\mathbf{R}^n$ whose elements are g_i, then (7) can be written as

$$Cu \leq g. \tag{8}$$

We assume that C is such that there exists a displacement that satisfies (8). The total force F acting on the structure is now a sum of external forces f and contact force contributions $F_c = C^T p$, where $p \in \mathbf{R}^r$ is a vector of contact forces. The following lemma is the counterpart of Lemma 3.1 when contact conditions are present:

LEMMA 4.1 *For any $t \in \mathcal{T}$, the following conditions are necessary and sufficient for u to minimize $J(\cdot, t)$ over $\mathcal{U} = \{u \in \mathbf{R}^n \mid Cu \leq g\}$:*

$$K(t)u = F = C^T p + f, \tag{9}$$

$$Cu \leq g, \quad p \leq 0, \tag{10}$$

$$p^T (Cu - g) = 0. \tag{11}$$

In addition,

$$J(u,t) = \frac{1}{2} p^T g - \frac{1}{2} f^T u. \tag{12}$$

PROOF. Conditions (9) through (11) are obtained by applying Theorem 2.2, taking $p_i = -\lambda_i$, and (12) follows from (9) and (11). ∎

In the above lemma, (9) can be interpreted as force equilibrium, (10) as conditions for non-penetration and no adhesion, and (11) as the necessity of the contact force being zero when there is no contact.

In the non-contact case it was shown that the saddle point formulation (SJ) means minimizing the displacements weighted by the external forces, among all equilibrium structures with permissible designs. Lemma 4.1 shows that in the contact formulation something similar, but not identical, holds.

First, we have to reformulate (SJ) in terms of a new set of kinematically admissible displacements:

$$(SJ) \quad \begin{cases} \text{Find } (\tilde{u}, \tilde{t}) \in \mathcal{U} \times \mathcal{T} : \\ J(\tilde{u}, t) \leq J(\tilde{u}, \tilde{t}) \leq J(u, \tilde{t}) \quad \forall (u,t) \in \mathcal{U} \times \mathcal{T} \end{cases}$$

If $(\tilde{u}, \tilde{t})$ solves (SJ) and $t \in \mathcal{T}$ is an arbitrary design with equilibrium state u_t (that minimizes $J(\cdot, t)$ over $\mathcal{U}$), then (4) still holds, and as a consequence (SJ) means a maximization of the equilibrium potential energy. This quantity has been taken to be a measure of structural stiffness by Benedict (1982). From (12) we can see that our formulation (or the one used in Benedict 1982) means minimization of $f^T u + g^T(-p)$ meaning that the sum of the displacement u

weighted by f and the contact force magnitude $-p$ weighted by g, is minimized. Hence (SJ) is not purely a minimization of the compliance in the sense that $f^T u$ is minimized, unless the contact force or the initial gap is zero.[6] In a positive spirit, one can undeniably say that the saddle point formulation minimizes a weighted measure of the diplacements without obtaining "too large" contact forces.

The extension of the saddle point formulation to the contact case is also well-posed; analogously to Theorem 3.1 we have

THEOREM 4.1 *There exists a solution to (SJ) in the contact case.*

PROOF. The only new thing in addition to what is included in the proof of Theorem 3.1, is the set $\mathcal{A} = \mathcal{U} = \{u \in \mathbf{R}^n \mid Cu \leq g\}$ instead of $\mathcal{A} = \mathbf{R}^n$. The requirements on $\mathcal{A}$, namely non-emptiness, convexity and closedness, are satisfied. ■

Again we will generalize to the contact case; Theorem 3.2 gives necessary and sufficient conditions for a solution to the saddle point problem, and this takes the following new form:

THEOREM 4.2 *A pair $(\tilde{u}, \tilde{t}) \in \mathcal{U} \times \mathcal{T}$ solves (SJ) if and only if*

(j) $K(\tilde{t})\tilde{u} = f + C^T\tilde{p},\ C\tilde{u} \leq g,\ \tilde{p} \leq 0,\ \tilde{p}^T(C\tilde{u} - g) = 0,$ *and*

(jj) $\tilde{t} \in \mathcal{T},\ \frac{1}{2}\tilde{u}^T K_i \tilde{u} < \max_{i=1,\ldots,m}(\frac{1}{2}\tilde{u}^T K_i \tilde{u}) \Rightarrow \tilde{t}_i = 0.$

PROOF. A pair $(\tilde{u}, \tilde{t})$ solves (SJ) (in the contact case) if and only if (i): $\tilde{u}$ minimizes $u \longmapsto J(u, \tilde{t})$ over $\mathcal{U}$ and (ii): $\tilde{t}$ maximizes $t \longmapsto J(\tilde{u}, t)$ over $\mathcal{T}$. By Lemma 4.1, (i) is equivalent to (j). Applying Theorem 2.2 to the maximization problem (ii) we get (jj) as necessary and sufficient conditions, exactly as in the proof of Theorem 3.2. ■

An example is given in order to see how optimal solutions in a contact (and non-contact) case may appear, and to compare and make clear different features.

EXAMPLE 4.1 Let us consider a two-bar truss subjected to a horizontal force $F > 0$ and a vertical one $2F$.

In Fig. 1 the element numbers are denoted inside circles, and the employed two–dimensional cartesian coordinate system is also shown. We have $t_1 + t_2 = V$, $K(t) = t_1 K_1 + t_2 K_2$, where $t = \{t_1\ t_2\}^T$, and $L_1 = \sqrt{2}L_2 = \sqrt{2}L$ and $g = \{g_1\}$. Obviously, the constants involving dimensions of vectors and other variables are $n = 2$, $m = 2$, $N = 3$, $n_0 = 4$, $dim = 2$ and $r = 1$. By standard truss-structure analysis methods one obtains

$$\frac{E}{4L^2}\begin{bmatrix} t_1 & -t_1 \\ -t_1 & t_1 + 4t_2 \end{bmatrix}\begin{Bmatrix} u_{2x} \\ u_{2y} \end{Bmatrix} = \begin{Bmatrix} F \\ -2F \end{Bmatrix}$$

which is $K(t)u = f$. We also have

$$K_1 = \frac{E}{4L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{E}{L^2} \end{bmatrix}. \qquad (13)$$

[6]This is sometimes categorized as a receding contact.

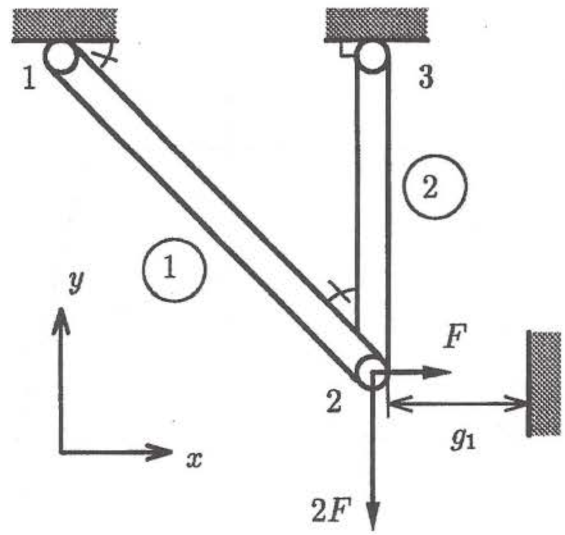

Figure 1. A two-bar truss.

The contact condition is $Cu \le g$ where $C = [\,1\;0\,]$.

We will study the appearance and behaviour of optimal solutions when g_1 is varied. We restrict ourselves to "not too negative" g_1's.

Optima, i.e. solutions to (SJ), can be obtained by straightforward use of the optimality criteria given in Theorem 4.2. When $g_1 > 3FL^2/EV$ solutions are:

$$\tilde{u} = \frac{3FL^2}{EV}\left\{\begin{array}{c} 1 \\ -1 \end{array}\right\},\ \tilde{t} = V\left\{\begin{array}{c} \frac{2}{3} \\ \frac{1}{3} \end{array}\right\},\ \tilde{p} = 0, \tag{14}$$

when $2FL^2/EV \le g_1 \le 3FL^2/EV$:

$$\tilde{u} = g_1\left\{\begin{array}{c} 1 \\ -1 \end{array}\right\},\ \tilde{t} = \left\{\begin{array}{c} 2V - \frac{4FL^2}{Eg_1} \\ \frac{4FL^2}{Eg_1} - V \end{array}\right\},\ \tilde{p} = \frac{EVg_1}{L^2} - 3F, \tag{15}$$

and finally when $-6FL^2/EV < g_1 < 2FL^2/EV$:

$$\tilde{u} = \left\{\begin{array}{c} g_1 \\ -\frac{2FL^2}{EV} \end{array}\right\},\ \tilde{t} = \left\{\begin{array}{c} 0 \\ V \end{array}\right\},\ \tilde{p} = -F. \tag{16}$$

Note that (14) is a solution to (SJ) in the non-contact case also since (j) and (jj) in Theorem 3.2 are satisfied. From (14) through (16), one can now construct diagrams for how the optimal design (represented by $\tilde{t}_1$) and contact force (represented by $-\tilde{p}$) depend on g_1. This is done in Fig. 2 where contact force characteristics for a "nominal" design $t = \frac{V}{2}\{\,1 \quad 1\,\}^T$ is also shown.

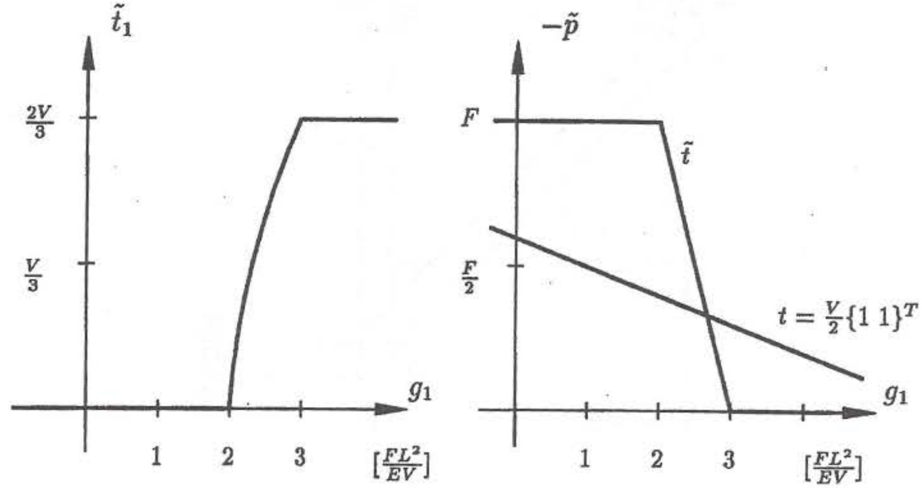

Figure 2. Optimal volume distribution for the bar number one and contact force, as functions of the initial gap.

We will also investigate how displacements and stresses change during the advancing of the support, and compare the optimal design with the "nominal" one. In the region $2 \leq g_1 EV/FL^2 \leq 3$ we have from (15)

$$\tilde{u} = \left\{ \begin{array}{c} g_1 \\ -g_1 \end{array} \right\} \text{ and } \tilde{u}^T K_1 \tilde{u} = \tilde{u}^T K_2 \tilde{u} = 2\kappa = \frac{E g_1^2}{L^2}$$

and hence by (6),

$$|\tilde{\sigma}_i| = \frac{E g_1}{L} \tag{17}$$

and

$$|\tilde{u}| = \sqrt{2} g_1. \tag{18}$$

When $g_1 < 2FL^2/EV$ Fig. 2 shows that $\tilde{t}_1 = 0$, and one can deduce that

$$\tilde{\sigma}_2 = \frac{2FL}{V} \tag{19}$$

and

$$|\tilde{u}| = \sqrt{g_1^2 + 4\frac{F^2 L^4}{V^2 E^2}}. \tag{20}$$

With the aid of (17)-(20) the diagrams in Fig. 3 are constructed. Note that there is a small region where the "nominal" design is better, that is, it has lower

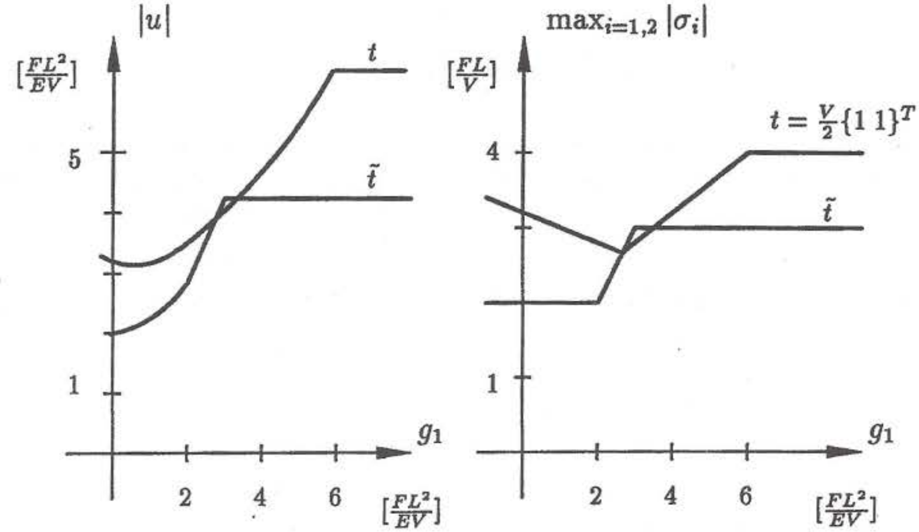

Figure 3. Stresses and displacements for the optimal and "nominal" designs as functions of the initial gap.

magnitudes of displacement and stress, but from Fig. 2 it can be seen that in this entire region the contact force is worse for the "nominal" design.

Two conclusions can be made:

- When the support is far away, the optimal design coincides with traditional maximum stiffness designs, and when the support is moved sufficiently close to the structure, the optimal design alters rather drastically but in a continuous fashion, whilst both displacements and stresses decrease and contact force increases.
- When the initial gap is between 2 and 3 units of FL^2/EV, there exist admissible (non-optimal) designs with slightly smaller displacements than for the solution to (SJ), but they generate larger contact forces. ■

5. Maximizing stiffness for structures with uniform contact force distribution

In this section we will replace the set $\mathcal{U}$ with a new, closed and convex set $\mathcal{U}_0$. This will imply that the saddle point formulation will in effect mean a problem of finding $t \in \mathcal{T}$ and $g \in \mathbf{R}^r$ such that the structure represented by (9)-(11), or by the equivalent minimization problem, is as stiff as possible among all structures that have a constant contact force distribution. The design variables are the volume of bars $t = \{t_i\} \in \mathbf{R}^m$ as before, and also the contact distances

$g = \{g_i\} \in \mathbf{R}^r$. Design constraints on initial distances are

$$g \in \mathcal{G} = \left\{ g \in \mathbf{R}^r \;\middle|\; \sum_{i=1}^{r} t_i \equiv \mathbf{1}_r^T g = V_g \right\}. \tag{21}$$

where V_g is a constant representing a given "volume" of the total contact gap. The set $\mathcal{U}_0$ will play the role of $\mathcal{U}$ in (SJ), and it is a set of kinematically admissible displacements in a contact case for *some* permissible g, as shown below:

LEMMA 5.1 *The following sets contain exactly the same elements:*

$$A = \{u \in \mathbf{R}^n \mid \exists g \in \mathbf{R}^r \text{ such that } Cu \le g,\ \mathbf{1}_r^T g = V_g\}$$

$$\mathcal{U}_0 = \{u \in \mathbf{R}^n \mid \mathbf{1}_r^T Cu \le V_g\}$$

PROOF. $(A \subset \mathcal{U}_0)$ Follows directly from $\mathbf{1}_r^T Cu \le \mathbf{1}_r^T g = V_g$.

$(\mathcal{U}_0 \subset A)$ For arbitrary $u \in \mathcal{U}_0$ we will find some g such that $Cu \le g$ and $\mathbf{1}_r^T g = V_g$. If $\mathbf{1}_r^T Cu = V_g$, one can simply take $g := Cu$. Suppose $\mathbf{1}_r^T Cu = \sum_{i=1}^{r} (Cu)_i < V_g$. Define the two disjoint index sets

$$J = \{i \in \{1, \ldots, r\} \mid (Cu)_i < V_g/r\}, \tag{22}$$

$$K = \{i \in \{1, \ldots, r\} \mid (Cu)_i \ge V_g/r\}. \tag{23}$$

Clearly, $J \neq \emptyset$. Denote the number of elements of J by M, $M > 0$, and define

$$\kappa := -M^{-1}(\mathbf{1}_r^T Cu - V_g) > 0. \tag{24}$$

Now, $g \in \mathbf{R}^r$ is constructed as

$$g_i := \begin{cases} (Cu)_i + \kappa & \text{if } i \in J \\ (Cu)_i & \text{if } i \in K. \end{cases}$$

It is easy to see that $g \in \mathcal{G}$ and $Cu \le g$. ∎

An element of $\mathcal{G}$ corresponding to $u \in \mathcal{U}_0$ as in the lemma will be denoted by g_u. Lemma 5.1 enables one to decouple g from the formulation, and *a posteriori* it will be possible to pick an admissible gap (from u as described in the above proof) in congruence with optimal thicknesses and displacements.

We give a theorem that characterizes all instances of (9)-(11) that correspond to a constant contact force distribution.

THEOREM 5.1 *For fixed $t \in T$ let $(u, p, g, \Lambda) \in \mathbf{R}^n \times \mathbf{R}^r \times \mathcal{G} \times \mathbf{R}$ satisfy*

$$K(t)u = f + C^T p \tag{25}$$

$$Cu \le g,\ p \le 0,\ p^T(Cu - g) = 0 \tag{26}$$

$$p = -\Lambda \mathbf{1}_r,\ \Lambda \geq 0 \tag{27}$$

i.e. the structure is in an equilibrium state with constant contact force distribution. Then $(u, \Lambda) \in \mathbf{R}^n \times \mathbf{R}$ *satisfy*

$$K(t)u = f - W\Lambda,\ W := C^T \mathbf{1}_r \tag{28}$$

$$W^T u \leq V_g,\ \Lambda \geq 0,\ \Lambda(W^T u - V_g) = 0. \tag{29}$$

Conversely, let $(u, \Lambda) \in \mathbf{R}^n \times \mathbf{R}$ *satisfy (28) and (29). Then there exist* $p \in \mathbf{R}^r$ *and* $g \in \mathcal{G}$ *such that* $(u, p, g, \Lambda) \in \mathbf{R}^n \times \mathbf{R}^r \times \mathcal{G} \times \mathbf{R}$ *satisfy (25), (26) and (27).*

PROOF. The first claim follows since $g \in \mathcal{G}$ satisfies $\mathbf{1}_r^T g = V_g$. The converse follows from the definition $p := -\Lambda \mathbf{1}_r$ and Lemma 5.1. ■

We note that (28) and (29) are sufficient and necessary (KKT-)conditions for u to minimize the potential energy function over the set $\mathcal{U}_0$; cf. Theorem 2.2.

In this section the problem (SJ) takes the following new form:

$$(SJ) \quad \left\{ \begin{array}{l} \text{Find } (\tilde{u}, \tilde{t}) \in \mathcal{U}_0 \times \mathcal{T} \ : \\ J(\tilde{u}, t) \leq J(\tilde{u}, \tilde{t}) \leq J(u, \tilde{t}) \quad \forall (u, t) \in \mathcal{U}_0 \times \mathcal{T} \end{array} \right.$$

The interpretation of (SJ) is that among the whole class of admissible designs with constant contact forces we pick the one that optimizes the stiffness and the contact forces in a sense similar to that of the previous section. This is a result of the following lemma:

LEMMA 5.2 *For any* u, Λ *as in (28) and (29) and* $t \in \mathcal{T}$ *it holds that*

$$J(u, t) = \inf_{v \in \mathcal{U}_0} J(v, t) = -\frac{1}{2} f^T u - \frac{1}{2} \Lambda \cdot V_g.$$

PROOF. (28) and (29) and $t \in \mathcal{T}$ mean that the sufficient Kuhn-Tucker conditions for

$$J(u, t) = \min_{v \in \mathcal{U}_0} J(v, t)$$

to hold are satisfied. Moreover,

$$J(u, t) = \frac{1}{2} u^T K(t) u - f^T u = \frac{1}{2} u^T (f - \Lambda W) - f^T u$$

$$= -\frac{1}{2} f^T u - \frac{1}{2} \Lambda \cdot W^T u = -\frac{1}{2} f^T u - \frac{1}{2} \Lambda \cdot V_g$$

from (28) and (29). ■

Note that if $V_g = 0$, as was the case in Klarbring, Petersson and Rönnqvist (1993), we can say that Clapeyron's theorem holds also in the contact case, and (SJ) has the interpretation that we find the stiffest structure in the same sense as for non-contacting structures.

The statements for existence and optimality criteria for (SJ) are here repeated in a new case. The proofs are very similar to those of Theorems 3.1, 3.2, 4.1 and 4.2, and therefore omitted.

THEOREM 5.2 *There exists a solution to (SJ) in the case of constant contact forces.*

THEOREM 5.3 *A pair $(\tilde{u}, \tilde{t}) \in \mathcal{U}_0 \times \mathcal{T}$ solves (SJ) if and only if*
(j) $K(\tilde{t})\tilde{u} = f - \tilde{\Lambda}W$, $W^T\tilde{u} \leq V_g$, $\tilde{\Lambda} \geq 0$, $\tilde{\Lambda}(W^T\tilde{u} - V_g) = 0$, *and*
(jj) $\tilde{t} \in \mathcal{T}$, $\frac{1}{2}\tilde{u}^T K_i \tilde{u} < \max_{i=1,\ldots,m}(\frac{1}{2}\tilde{u}^T K_i \tilde{u}) \Rightarrow \tilde{t}_i = 0$.

Here, (jj) is the usual criterion of optimal design, and (j) represents, according to Theorem 5.1, nothing but the equilibrium state conditions for a case of uniform contact forces for some permissible g.

In Klarbring, Petersson and Rönnqvist (1993) it is shown that in case of a truss and $V_g = 0$ solutions to (j) and (jj) can be obtained by solving a dual pair of linear programming problems. The structure of these is identical to the one of plastic limit design problems.

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