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On optimal reinforcement of plates and choice of design parameters

by

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The first part of the paper is expository and outlines the theory and methodology for optimal design of heterogeneous continua as it applies to reinforced Kirchoff plates. A similar approach for Mindlin plates has been developed in Diaz, Lipton (1994).

The second part of this paper introduces a set of design parameters well suited for this problem. New results are presented that facilitate the identification of optimal stiffener geometries from these design parameters.

1. Introduction

We consider the problem of optimal design of a stiffener reinforced Kirchoff plate subject to an ensemble of random static transverse loads. For a prescribed area fraction of stiffeners, the objective is to reinforce the plate so as to minimize the average compliance with respect to the ensemble of loads.

We suppose that a plate of midplane thickness h_1 is reinforced using ribs or stiffeners of thickness $h_2 > h_1$. Let the plate domain be given by R, then for a random transverse load $f(x,\omega)$ (where the realization ω is taken from some probability space Ω) the midplane deflection w and the bending moment σ satisfy

$$\sigma_{ij}(x,\omega) = M_{ijkl}\partial^2_{x_k x_l} w(x,\omega) \tag{1.1}$$

$$\partial^2_{x_i x_j}\sigma_{ij}(x,\omega) = f(x,\omega) \text{ on } R.$$

We suppose the plate is clamped at the edges, so that

$$w(x,\omega) = \partial_n w(x,\omega) = 0 \text{ on } \partial R. \tag{1.2}$$

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Here ∂_n represents the outward normal derivative. The tensor M introduced in (1.1) is the bending rigidity of the plate and relates the bending moment σ_{ij} to the midplane curvature $\partial^2_{x_i x_j} w$. We shall assume that both plate and stiffeners are made from the same isotropic elastic material with Young's modulus and Poisson's ratio E and ν respectively. Letting $\mathbb{P}_h$ and $\mathbb{P}_s$ denote the projections onto the space of hydrostatic and shear strains, the rigidity tensors of the plate and stiffeners are given by

$$M_1 = \frac{2}{3} h_1^3 (2\mu \mathbb{P}_s + 2\kappa \mathbb{P}_h) \tag{1.3}$$

and

$$M_2 = \frac{2}{3} h_2^3 (2\mu \mathbb{P}_s + 2\kappa \mathbb{P}_h) \tag{1.4}$$

respectively. Here $\mu = E/2(1+\nu)$ is the shear modulus and $\kappa = E/2(1-\nu)$. Thus for a reinforced plate the layout of stiffeners is given by the piecewise constant rigidity:

$$M \equiv \chi_1 M_1 + \chi_2 M_2 \tag{1.5}$$

where χ_1 and χ_2 are the indicator functions of plate and stiffeners respectively and $\chi_1 = 1 - \chi_2$. The prescribed area "c_2" of stiffeners is given by

$$c_2 = \int_R \chi_2(x) dx. \tag{1.6}$$

The compliance or work done by the load $f(x,\omega)$ for a particular layout is defined by

$$H(M,\omega) \equiv \int_R w(x,\omega) f(x,\omega) dx. \tag{1.7}$$

Our goal is to minimize the average compliance $J(M)$ given by

$$J(M) = \langle H(M,\omega) \rangle \tag{1.8}$$

over all admissible layouts. Here $\langle\ \rangle$ denotes ensemble averaging.

Appealing to the definition of G-convergence (see Spagnolo 1976) it is readily seen that the average compliance is a continuous function of the rigidity with respect to the topology of G-convergence, (cf. Cabib and DalMaso, 1988). We may then appeal to the direct method of the calculus of variations to conclude the existence of a minimizing layout, provided that the admissible set of rigidities is compact with respect to the G-convergence topology. Unfortunately, the set of admissible rigidities given by (1.5) is not closed under the G-convergence topology. Indeed, there exists chattering layouts of isotropic-stiffeners of the type (1.5)-(1.6) whose G-limit is associated with an anisotropic plate, see Lurie,

Cherkaev and Fedorov (1982) and Olhoff, Lurie, Cherkaev, and Fedorov (1981). This phenomenon is well known and observed in many contexts Armand, Lurie, and Cherkaev (1984), Cheng (1981), Cheng and Olhoff (1981), Murat and Tartar (1985). To insure compactness, one relaxes the problem and extends the design space to include all G-limits of classical layouts. The resulting extension of the design space is compact with respect to G-convergence (cf. Murat and Tartar, 1985), and so one can claim for the existence of an optimal design within the extended set of controls. The extended set of controls is often called the G-closure of the original set of controls. This direct approach to the optimal design problem was developed in the work of Murat and Tartar (1985), Tartar (1985), and independently by Lurie and Cherkaev (1986), see also Armand, Lurie, and Cherkaev (1984), Lurie, Cherkaev and Fedorov (1982). Most interestingly the G-closure of the set of classical designs (1.5) can be easily described in terms of a local effective bending rigidity associated with a local density $\theta_2(x)$ of stiffeners. This local representation of the G-closure set is what facilitates numerical solution of the optimal design problem. The local character of the G-closure set is elucidated in the works Murat and Tartar (1985) and Armand, Lurie and Cherkaev (1984). Dal Maso and Kohn (Kohn and Dal Maso 1985) have produced an elegant proof of the local character of the G-closure set. They show that the set of effective tensors associated with periodic microstructure are dense in the set of effective tensors.

In view of the above discussion we denote the set of effective bending rigidities associated with a local density of stiffeners θ_2 by G_{θ_2}. The relaxed problem is given by

$$\min_{0\leq\theta_2\leq 1} \quad \min_{M\epsilon G_{\theta_2}} J(M) \tag{1.9}$$

$$\text{subject to} \int_\Omega \theta_2 dx = c_2. \tag{1.10}$$

As of this writing the set of effective bending rigidities G_{θ_2} is still unknown, however because of the special form of the functional $J(M)$ we do not need a full characterization of G_{θ_2}.

It is shown in Lipton (1993), that only a subset of effective tensors denoted by GL_{θ_2} participate in the relaxed problem (1.9)-(1.10). This set of effective rigidities correspond to plates reinforced with "j" families of stiffeners oriented along different directions on widely separated length scales. These effective tensors are the generalization of the relaxation introduced for the one dimensional problem by Cheng and Olhoff (Cheng 1981, Cheng and Olhoff 1981) when the plate thickness is allowed to take two values.

For fixed local volume fraction, the set GL_{θ_2} is described by all effective rigidities $\overline{M}$ given by the formula:

$$\overline{M} = \overline{M}(\theta_2, P^j) \equiv M_2 - (1-\theta_2)[(M_2 - M_1)^{-1} - \\ - \frac{\theta_2}{\frac{2}{3}h_2^3(\mu+\kappa)} \int_{S^1} \hat{\Gamma}_1(n) P^j(dn)^{-1}. \quad (1.11)$$

Here $P^j(dn)$ is the discrete probability measure on the unit sphere defined by

$$P^j(dn) = \sum_{i=1}^{j} \rho_i \delta(n - n^i) dn \quad (1.12)$$

with $\rho_i \geq 0, \quad i = 1, \ldots, j, \sum_{i=1}^{j} \rho_i = 1$, and $\hat{\Gamma}_1$ is the scale invariant symbol associated with the projection operator onto mean zero periodic matrix valued fields of the form

$$E_{ij}(x) = \partial^2_{i_j} \phi_{(x)} \quad (1.13)$$

given by

$$\hat{\Gamma}_1(n) = n \otimes n \otimes n \otimes n \quad (1.14)$$

for n on the unit circle.

The directions n^i introduced in (1.12) are the normals to the stiffeners and the parameters ρ_i are related to the relative thickness of the stiffeners.

In this way we see that the local effective property participating in the relaxed design is determined explicitly by the local density θ_2 and the measure $P^j(dn)$ from which the stiffener orientation and thickness are determined. The relaxed problem becomes

$$\min_{0 \leq \theta_2 \leq 1} \min_{P^j} J(\overline{M}(\theta_2, P^j)) \quad (1.15)$$

$$\text{subject to} \int_\Omega \theta_2 dx = c_2. \quad (1.16)$$

Here P^j is allowed to vary from point to point in the design domain.

In the next Section we use the results of Avellaneda and Milton (1989) to eliminate the measure P^j in favor of a vector $\underline{m}$ of four design parameters $\underline{m} = (m_1, m_2, m_3, m_4)$.

Once done, the problem is amenable to numerical solution; the optimal layout being described from point to point by the density θ_2 and the vector $\underline{m}$, see Section 2. In Section 3 we solve the inverse problem and show how to

construct a measure P^j associated with a point $\underline{m}$ in the feasible set of design parameters. The solution of this problem allows us to identify the optimal layout from the design parameters.

It follows from our solution of the inverse problem given in Section 3 that at most 3 families of stiffeners participate in the optimal design. We remark that this observation, Avellaneda and Milton (1993), also follows from theorems on extreme points for sets of probability measures with prescribed moments, see Karlen and Studden (1966).

We conclude the paper in Section 4 with observations indicating that the choice of structural design variable given by the vector $\underline{m}$ is well suited for problems of this type.

2. Relaxed formulation and a choice of design variables

In this Section we show how to eliminate the measure P^j in favor of a vector $\underline{m}$ of four local design variables. We then provide an alternate variational form of the relaxed design problem amenable to numerical solution. To expedite the presentation we introduce the basis for 2×2 constant curvature matrices $\epsilon_{ij} = \partial^2_{ij} w$ given by

$$\epsilon^1 = \frac{1}{\sqrt{2}}(ii - jj); \quad \epsilon^2 = \frac{1}{\sqrt{2}}(ij + ji); \quad \epsilon^3 = \frac{1}{\sqrt{2}}(ii + jj) \tag{2.1}$$

relative to this basis the matrix of the effective bending rigidity $\overline{\mathcal{M}}(\theta_2, P^j)$ is given by

$$\begin{aligned} \overline{\mathcal{M}}(\theta_2, P^j) &= \mathcal{M}_2 - (1-\theta_2)[(\mathcal{M}_2 - \mathcal{M}_1)^{-1} - \\ &- \frac{\theta_2}{\frac{2}{3}h_2^3(\mu + k)} \int_{S^1} \wedge(n) P^j(dn)]^{-1}. \end{aligned} \tag{2.2}$$

Here $\mathcal{M}_1, \mathcal{M}_2$ are the matrices associated with the rigidities M_1, M_2, and $\wedge(n)$ is associated with the tensor $\hat{\Gamma}_1(n)$. Following Avellaneda and Milton (1989) we set $n = (\cos\phi, \sin\phi)$ and introduce $\underline{m} = (m_1, m_2, m_3, m_4)$ to write:

$$\int_{S^1} \wedge(n) dP^j(n) = \int_0^{2\pi} \wedge(\cos\phi, \sin\phi) d\mu(\phi) = \overset{*}{\wedge}_{ij}\ (\underline{m}) \tag{2.3}$$

where $\overset{*}{\wedge}_{ij} = \overset{*}{\wedge}_{ji}$ and

$$\overset{*}{\wedge}_{11} = \frac{1}{4}(1 + m_3) \tag{2.4}$$

$$\overset{*}{\wedge}_{12} = \frac{m_4}{4}, \quad \overset{*}{\wedge}_{13} = \frac{m_2}{2} \tag{2.5}$$

$$\overset{*}{\Lambda}_{22} = \frac{1}{4}(1 - m_3) \tag{2.6}$$

$$\overset{*}{\Lambda}_{23} = \frac{m_1}{2}, \quad \overset{*}{\Lambda}_{33} = \frac{1}{2}. \tag{2.7}$$

Here the vector of geometric parameters is given by

$$\underline{m} = \int_0^{2\pi} (\sin 2\phi, \cos 2\phi, \cos 4\phi, \sin 4\phi) d\mu(\phi), \tag{2.8}$$

where the measure P^j is represented by μ in polar coordinates.

The feasible set for $\underline{m}$ as one ranges over all probability measures μ was determined in Avellaneda and Milton (1989) and is given by:

$$\Delta = \begin{cases} \underline{m}: & 1 - 2(m_1^2 + m_2^2) - (m_3^2 + m_4^2) \\ & \quad +2m_3m_2^2 - 2m_1^2m_3 + 4m_1m_2m_4 \geq 0, \\ & 1 \geq m_1^2 + m_2^2. \end{cases} \tag{2.9}$$

It is evident that Δ is convex.

In light of the above, we change variables and adopt the vector $\underline{m}$ as our local design variable and write

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}(\theta_2, \underline{m}) =$$

$$= \mathcal{M}_2 - (1 - \theta_2)[(\mathcal{M}_2 - \mathcal{M}_1)^{-1} - \frac{\theta_2}{\frac{2}{3}h_2^3(\mu + \kappa)} \overset{*}{\Lambda}(\underline{m})]^{-1}. \tag{2.10}$$

The optimal design problem becomes

$$\min_{0 \leq \theta_2 \leq 1} \min_{\underline{m} \in \Delta} J(\overline{\mathcal{M}}(\theta_2, \underline{m})) \tag{2.11}$$

$$\text{subject to} \int \theta_2 = c_2. \tag{2.12}$$

As in Lipton (1993) we introduce the appropriate class of moment tensors $\mathcal{C} = \{\tau_{ij} \mid \tau_{ij} = \tau_{ji}; \partial_{ij}^2 \tau_{ij} = f\}$ and write

$$J(\overline{\mathcal{M}}(\theta_2, \underline{m})) = \min_{\tau_{ij} \in \mathcal{C}} \langle \int_\Omega \overline{\mathcal{M}}^{-1}(\theta_2, \underline{m}) \tau \cdot \tau dx. \rangle. \tag{2.13}$$

$$= \min_{\tau \in \mathcal{C}} \int_\Omega \langle \overline{\mathcal{M}}^{-1}(\theta_2 \underline{m}) \tau \cdot \tau \rangle dx \tag{2.14}$$

$$= \min_{\tau \in \mathcal{C}} \int_\Omega \overline{\mathcal{M}}^{-1}(\theta_2, \underline{m}) \cdot \cdot \langle \tau \otimes \tau \rangle dx. \tag{2.15}$$

Here the last equality follows as $\overline{\mathcal{M}}^{-1}$ is deterministic and

$$\overline{\mathcal{M}}^{-1} \cdot \cdot \langle \tau \otimes \tau \rangle \equiv \overline{\mathcal{M}}^{-1}_{ijkl} \langle \tau_{ij} \tau_{kl} \rangle \tag{2.16}$$

The minimization (2.15) can be solved using the method of stochastic finite elements (cf. Gahenem, R.G. and Spanos, P.T.D. 1991).

Introducing a Lagrange multiplier λ for the constraint on the stiffener density the optimal design problem becomes

$$\min_{\theta_2} \min_{\underline{m}} \min_{\tau \in \mathcal{C}} [\int_\Omega [\overline{\mathcal{M}}^{-1}(\theta_2, \underline{m}) \cdot \cdot \langle \tau \otimes \tau \rangle + \lambda \theta_2] dx]. \tag{2.17}$$

Exchanging the order of minimizations one has

$$\min_{\tau \in \mathcal{C}} \int_\Omega \min_{\theta_2} \{ \min_{\underline{m} \in \Delta} (\overline{\mathcal{M}}^{-1}(\theta_2, \underline{m}) \cdot \cdot \langle \tau \otimes \tau \rangle) + \lambda \theta_2 \} dx. \tag{2.18}$$

It is evident from the formulation given above that the optimization procedure breaks into two parts; a local pointwise optimization over structural parameters $(\theta_2, \underline{m})$ and a global optimization over moment tensors τ. The formulation (2.18) is amenable to numerical analysis, see Diaz, Lipton, Soto (1994).

3. Obtaining layout from design variables

In this section we solve the following problem:

Given $\underline{m}$ in Δ find the measure $\mu(\phi)$ for which

$$\underline{m} = \int_{S^1} (\sin 2\phi, \cos 2\phi, \cos 4\phi, \sin 4\phi) d\mu(\phi). \tag{3.1}$$

We remind the reader that geometry of the local family of stiffeners is determined directly from the measure $\mu(\phi)$. Indeed, the measure associated with a local geometry consisting of three families of stiffeners with normals specified by the angles $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3$ and geometric parameters $0 \leq \rho_i \leq 1, \ \ i = 1, 2, 3, \sum_{i=1}^{3} \rho_i = 1$, (as in (1.12)), is given by

$$\mu(\phi) = \rho_1 \delta(\phi - \tilde{\phi}_1) + \rho_2 \delta(\phi - \chi - \tilde{\phi}_2) + \rho_3 \delta(\phi - \tilde{\phi}_3) \tag{3.2}$$

In what follows we provide explicit formulas for ρ_1, ρ_2, ρ_3 and $\tilde{\phi}_1, \tilde{\phi}_2$ and $\tilde{\phi}_3$ in terms of $\underline{m}$.

To get started we introduce the orthogonal matrix associated with rotation of the plate through χ radians. The matrix is denoted by $Q(\chi)$ and is defined through:

$$Q_{11} = Q_{22} = \cos 2\chi \tag{3.3}$$

$$Q_{12} = -Q_{21} = -\sin 2\chi \tag{3.4}$$

$$Q_{33} = 1 \tag{3.5}$$

$$Q_{13} = Q_{31} = Q_{23} = Q_{32} = 0 \tag{3.6}$$

A short computation using equation (2.3) yields the identity:

$$Q^T(\chi) \overset{*}{\wedge} (\underline{m}) Q(\chi) = \int_0^{2\pi} \wedge(\cos\phi, \sin\phi) d\mu(\phi + \chi) \tag{3.7}$$

We set $\overset{*}{B} = Q^T(\chi) \overset{*}{\wedge} (\underline{m}) Q(\chi)$ to observe that one can always choose χ such that

$$\overset{*}{B}_{12} = \overset{*}{B}_{21} = 0. \tag{3.8}$$

We introduce the set $\tilde{\Delta}$ defined by

$$\tilde{\Delta} = \{\underline{m} : \ \underline{m} \text{ in } \Delta \text{ and } m_4 = 0\}. \tag{3.9}$$

From (2.9) it follows that

$$\tilde{\Delta} = \{\underline{m} : m_4 = 0, 1 \geq 2m_2^2/(1+m_3) + 2m_1^2/(1-m_3); m_3^2 \leq 1\}. \tag{3.10}$$

Motivated by (2.9), (3.7), (3.8), and (3.9) we state the following representation lemma.

LEMMA 3.1 *For $\underline{m}$ in Δ there exists an angle χ and a vector $\underline{\tilde{m}}$ in $\tilde{\Delta}$ such that $\overset{*}{\wedge} (\underline{m}) = Q^T(\chi) \wedge (\underline{\tilde{m}}) Q(\chi)$, conversely for every $\underline{\tilde{m}}$ in $\tilde{\Delta}$ and χ there exists $\underline{m}$ in Δ such that $Q^T(\chi) \overset{*}{\wedge} (\underline{\tilde{m}}) Q(\chi) = \overset{*}{\wedge} (\underline{m})$.*

The proof is straightforward and follows from (2.9), (3.7), and (3.9).

Next, we give explicit formulas for the angle χ and the moments $\underline{\tilde{m}}$ in terms of the vector $\underline{m}$. For $\underline{m}$ in Δ we write $\overset{*}{\wedge} (\underline{m}) = Q^T(\chi) \overset{*}{\wedge} (\underline{\tilde{m}}) Q(\chi)$, the angle χ and moments are given by

$$\chi = \frac{1}{2} \operatorname{tg}^{-1} \left(\frac{\sqrt{m_3^2 + m_4^2} - m_3}{m_4} \right) \tag{3.11}$$

and $\underline{\tilde{m}} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, 0)$ is specified by

$$\tilde{m}_1 = m_1 \cos 2\chi + m_2 \sin 2\chi \tag{3.12}$$

$$\tilde{m}_2 = m_2 \cos 2\chi - m_1 \sin 2\chi \tag{3.13}$$

$$\tilde{m}_3 = m_3 \cos 4\chi - m_4 \sin 4\chi. \tag{3.14}$$

We now observe from Lemma 3.1 and (3.7) that if the measure $\mu(\phi)$ is associated with the moments $(\tilde{m}_1, \tilde{m}_2.\tilde{m}_3, 0)$ then the measure $\mu(\phi-\chi)$ corresponds to $\underline{m} = (m_1, m_2, m_3, m_4)$. In what follows, we show how to determine the measure $\mu(\phi)$ associated with $\underline{\tilde{m}}$ such that

$$\underline{\tilde{m}} = (\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, 0) = \int_0^{2\pi} (\sin 2\phi, \cos 2\phi, \cos 4\phi, \sin 4\phi) d\mu(\phi) \tag{3.15}$$

To do this it suffices to show how to construct all measures $\mu(\phi)$ corresponding to moments $\underline{\tilde{m}}$ on the boundary of the set $\tilde{\Delta}$. To see this, we observe that the point $(0, 1, 1, 0)$ lies on the boundary of $\tilde{\Delta}$ and is given by:

$$(0, 1, 1, 0) = \int_0^{2\pi} (\sin 2\phi, \cos 2\phi, \cos 4\phi, \sin 4\phi) d\mu(\phi) \tag{3.16}$$

for

$$\mu(\phi) = \delta(\phi). \tag{3.17}$$

It then follows from the convexity of $\tilde{\Delta}$ that for any point $\underline{\tilde{m}}$ in $\tilde{\Delta}$ there exists $0 \leq \rho \leq 1$ and a point $\underline{\tilde{m}}'$ on the boundary of $\tilde{\Delta}$ such that

$$\underline{\tilde{m}} = \rho(0, 1, 1, 0) + (1 - \rho)\underline{\tilde{m}}' \tag{3.18}$$

The point $\underline{\tilde{m}}'$ is associated with a measure $\nu(\phi)$ and so the measure $\mu(\phi)$ associated with $\underline{\tilde{m}}$ in $\tilde{\Delta}$ is given by

$$\mu(\phi) = \rho\delta(\phi) + (1 - \rho)\nu(\phi). \tag{3.19}$$

We now show how to construct all measures associated with points $\underline{\tilde{m}}$ on the boundary of $\tilde{\Delta}$. It is easily seen from (3.10) that points on the boundary of $\tilde{\Delta}$ admit the representation

$$(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, 0) = (\sin\beta \sin t, \cos\beta \cos t, \cos 2\beta, 0) \tag{3.20}$$

for $0 \leq t \leq 2\pi, 0 \leq \beta \leq \pi/2$.

We introduce the characteristic combination

$$s = \cos^2 2\beta - 2\cos 2\beta \cos 2t + 1 \tag{3.21}$$

and state the following Lemma;

LEMMA 3.2 *The boundary of the set $\tilde{\Delta}$ corresponds to measures of the form*

$$\mu = w_1\delta_{\phi_1} + w_2\delta_{\phi_1+\delta} \tag{3.22}$$

where $0 \leq w_1, w_1 + w_2 = 1$, *and*

$$\delta = \arctan\left[\pm\left(\frac{s}{1-\cos^2 2\beta}\right)^{\frac{1}{2}}\right] \tag{3.23}$$

$$w_1 = \frac{1}{2}\left[1 + \frac{\cos 2\beta \sin 2t}{\sqrt{s}}\right] \tag{3.24}$$

$$\phi_1 = \frac{1}{2}\arctan\left[\frac{-w_2 \sin 4\delta}{w_1 + w_2 \cos 4\delta + \cos 2\beta}\right] \tag{3.25}$$

Moreover the measure μ *given by (3.22)-(3.25) satisfies*

$$\int_{S^1} \sin 4\phi d\mu = 0. \tag{3.26}$$

PROOF.

The proof is constructive and follows from solution of the system,

$$w_1 + w_2 = 1 \tag{3.27}$$

$$w_1 \cos 4\phi_1 + w_2 \cos 4(\phi_1 + \delta) = \cos 2\beta \tag{3.28}$$

$$w_1 \sin 4\phi_1 + w_2 \sin 4(\phi_1 + \delta) = 0 \tag{3.29}$$

$$w_1 \sin 2\phi_1 + w_2 \sin 2(\phi_1 + \delta) = \sin\beta \sin t \tag{3.30}$$

$$w_1 \cos 2\phi_1 + w_2 \cos 2(\phi_1 + \delta) = \cos\beta \cos t. \tag{3.31}$$

Squaring and adding equations (3.28) to (3.29) and squaring and adding (3.30) to (3.31) yields the two equations

$$w_1^2 + w_2^2 + 2w_1 w_2 \cos 4\delta = \cos^2 2\beta \tag{3.32}$$

$$w_1^2 + w_2^2 + 2w_1 w_2 \cos 2\delta = \frac{1}{2}[1 + \cos 2\beta \cos 2t]. \tag{3.33}$$

Solution of (3.32) and (3.34) for w_1 and δ yields the expression (3.23) and (3.24) of Lemma 3.2. It is evident from (3.24) that $0 \leq w_1 \leq 1$. Expanding (3.28)-(3.31) in $\operatorname{tg}\phi_1$ and $\operatorname{tg} 2\phi_1$ provides the system of quadratic equations in $\operatorname{tg}\phi_1$ and $\operatorname{tg} 2\phi_1$ given by:

$$\begin{aligned}(w_1 + w_2 \cos 4\delta + \cos 2\beta)\operatorname{tg}^2 2\phi_1 + 2w_2 \sin 4\delta \operatorname{tg} 2\phi_1 - \\ -(w_1 + w_2 \cos 4\delta - \cos 2\beta) = 0\end{aligned} \tag{3.34}$$

$$-(w_2 \sin 4\delta)\operatorname{tg}^2 \phi_1 + 2(w_1 + w_2 \cos 4\delta)\operatorname{tg} 2\phi_1 + w_2 \sin 4\delta = 0 \tag{3.35}$$

$$-(w_2 \sin 2\delta + \sin\beta \sin t)\,\mathrm{tg}^2\,\phi_1 + 2(w_1 + w_2 \cos 2\delta)\,\mathrm{tg}\,\phi_1 + \\ +(w_2 \sin 2\delta - \sin\beta \sin t) = 0 \tag{3.36}$$

$$(w_1 + w_2 \cos 2\delta + \cos\beta \cos t)\,\mathrm{tg}^2\,\phi_1 + 2(w_2 \sin 2\delta)\,\mathrm{tg}\,\phi_1 - \\ -(w_1 + w_2 \cos 2\delta - \cos\beta \cos t) = 0 \tag{3.37}$$

Applying (3.32) to (3.36) and (3.37) it is easy to show that their roots agree and are given by a single root of multiplicity two written as

$$\mathrm{tg}\,2\phi_1 = \frac{-w_2 \sin 4\delta}{w_1 + w_2 \cos 4\delta + \cos 2\beta}. \tag{3.38}$$

Similarly applying (3.34) to (3.36) and (3.37) shows that their roots agree and are given by

$$\mathrm{tg}\,\phi_1 = \frac{w_2 \sin 2\delta \pm \sin\beta \sin t}{-(w_1 + w_2 \cos 2\delta + \cos\beta \cos t)}. \tag{3.39}$$

We observe that the choice of ϕ_1, δ, w_1 given by (3.23), (3.24) and (3.38) is a solution of the system (3.27)-(3.31) provided that the angles ϕ_1 delivered by (3.38) and (3.39) are identical. Indeed we choose the root in (3.39) given by

$$\mathrm{tg}\,\phi_1 = \frac{w_2 \sin 2\delta - \sin\beta \sin t}{-(w_1 + w_2 \cos 2\delta + \cos\beta \cos t)}. \tag{3.40}$$

Denoting the righthand sides of (3.38) and (3.40) by x and y respectively, we establish the identity

$$x = \frac{2y}{1 - y^2} \tag{3.41}$$

from which it follows that the angles ϕ_1 delivered by (3.38) and (3.40) are identical. We remark that (3.41) was established using Maple. ■

Collecting our results we construct an algorithm for identifying the measure associated with the vector $\underline{m}$ in Δ: The first step is the computation of χ using (3.11); the second is the computation of $\underline{\tilde{m}}$ using (3.12)-(3.14); the third step amounts to solving for ρ and the angles β and t using (3.18); once these parameters have been calculated one applies Lemma 3.2 to obtain:

THEOREM 3.1 *Given $\underline{m}$ in Δ, the associated measure $\mu(\phi)$ is of the form (3.2) with angles given by $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3$ where*

$$\tilde{\phi}_1 = -\chi, \quad \tilde{\phi}_2 = \phi_1 - \chi \tag{3.42}$$

$$\tilde{\phi}_3 = \phi_1 + \delta - \chi \tag{3.43}$$

here χ, δ and ϕ_1 are given by 3.11, (3.23) and (3.25). The geometric parameters ρ_1, ρ_2, ρ_3 are of the form

$$\rho_1 = \rho, \quad \rho_2 = (1-\rho)w_1, \quad \rho_3 = (1-\rho)(1-w_1) \tag{3.44}$$

where w_1 is given by (3.24)

4. Conclusion

From Theorem 3.3 it follows that the local optimal layout consists of at most 3 families of stiffeners. In view of this one could use the 3 rib directions n^1, n^2, n^3 and the two independent parameters ρ_1, ρ_2 as local design variables instead of the vector $\underline{m}$. However, working in this way one has five design variables instead of four. A second and more important observation is that the local energy $\overline{\mathcal{M}}^1, \cdots \langle \tau \otimes \tau \rangle$ is convex in the design variable $\underline{m}$. Since the feasible set is convex we see that the local energy has no local minima. This feature rules out the possibly of local minima when using the design vector $\underline{m}$.

References

ARMAND, J.L., LURIE, K.A., AND CHERKAEV, A.V. (1984) Optimal control theory and structural design in new directions in optimal structural design, E. Atrek, K.M. Ragsdell, R.H. Gallagher, and O.C. Zienkiewicz, eds. Wiley, pp. 211–231.

AVELLANEDA, M., AND MILTON, G.W. (1989) Bounds on the effective elasticity tensor of composites based on two point correlations, *Proceedings, ASME the 5th Energy–Technology Conference and Exhibition*, Houston, Texas.

AVELLANEDA, M., AND MILTON, G.W. (1993) Private communication.

CABIB, E., AND DALMASO, G. (1988) A class of optimum problems in structural design, *JOTA*, **56**, pp. 39–65.

CHENG, K.T. (1981) Non-smoothness in optimal design of solid, elastic plates, *Int. J. Solids Structures*, **17**, 795–810.

CHENG, K.T. AND OLHOFF, N. (1981) An investigation concerning optimal design of solid elastic plates, *Int. J. Solids Structures*, **17**, 305–323.

DIAZ, A., LIPTON, R. (1994) Relaxation of optimal design problems for Mindlen plate. Technical Report CDL 94–001, Computational Design Laboratory, Mechanical Engineering, Michigan State University, East Lansing, Michigan.

DIAZ, A., LIPTON, R., SOTO, C. (1994) *In preparation.*

GIBIANSKI, L. AND CHERKAEV, A. (1984) Design of composite plates of extremal rigidity; Ioffe Physicotechnical Institute, Preprint, (in Russian).

GAHENEM, R.G. AND SPANOS, P.T.D. (1991) A spectral formulation of stochastic finite elements, *Proccedings, 10th International Invitational UFEM Symposium*, Worcester, MA, 60–82.

KARLEN, S., AND STUDDEN, W.J. (1966) *Tchebycheff Systems: with applications in analysis and statistics*, New York, John Wiley and Sons.

KOHN, R.V. AND DAL MASO, G. (1985) *In preparation.*

LIPTON, R. (1993) Optimal design and relaxation for reinforced plates subject to random loads. *Probabilistic Engineering Mechanics.*

LURIE, K.A., CHERKAEV, A., AND FEDOROV, A. (1982) Regularization of optimal design problems for bars and plates I, II, *J. Optim. Th. Appl.*, **37**, 499–521 and 523–543.

LURIE, K.A. AND CHERKAEV, A. (1986) Exact estimates of the conductivity of a binary mixture of isotropic materials, *Proc. Roy Soc, Edinburgh*, **104A**, 21–38.

MURAT, F. AND TARTAR, L., (1985) Calcul des variations et homogenization, In: Les Methodes de l'Homogenisation: Theorie et Applications en Physique, Coll. de la Dir. des Etudes et Recherche d'Electricité de France, Eyrolle, 319–369.

OLHOFF, N., LURIE, K., CHERKAEV, A. AND FEDOROV, A., (1981) Sliding regimes and anisotropy in optimal design of vibrating axisymmetric plates, *Int. J. Solids Structures*, **17**, 931–948.

SPAGNOLO, S. (1976) Convergence in energy for elliptic operators, In: *Numerical Solutions of Partial Differential Equations II — Synspade 1975*, B. Hubbard, ed., Academic Press.

TARTAR, L., (1985) Estimations fines des coefficients homogénéisés, in Ennio de Georgi Collquium, P. Kreé (ed.), *Pitman Research Notes in Math*, **125**, 168–187.