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Optimization problems on manifolds and the shape optimization of elastic solids

by

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We consider the problem of optimal shape of an elastic solid occupying a multiply-connected bounded domain on the plane. The problem consists in finding a shape that minimizes the area (weight) of the elastic solid under the restrictions on displacements, stresses, geometry and so on. By using the fundamental solution we reduce the state equation (boundary value problem) to the singular integral equations on the boundary, and so we reduce the above problem to the optimization on manifolds. By applying smooth maps we obtain the optimization problem on the unit circle, prove the existence of an optimal solution, and establish the Fréchet differentiability of the mapping "control – function of state".

1. Introduction

General approaches to the domain shape optimization for elliptic equations and their applications to the optimal shape design for various problems of mechanics are given in Litvinov (1987, 1989, 1990, 1994), Pironneau (1984). Usually the domain shape optimization problems are formulated as follows. Let M be a set of controls. To each $q \in M$, a domain Ω_q in R^n is assigned, and one considers the problem of finding a function u_q defined on $\bar{\Omega}_q$ that satisfies $A_q u_q = f_q$. Here A_q is some elliptic operator acting from space V_q to space H_q, V_q and H_q consisting of functions defined in Ω_q and on its boundary S_q. The optimization problem is to minimize or maximize a goal functional under some restrictions. But in the general case, the goal and restriction functionals cannot be defined on various spaces V_q, $q \in M$. Besides that, for the existence of the solution, it is necessary to have some continuity of the goal and restriction functionals with respect to the control q, but it is inconvienient to establish continuity when we work with various V_q's. So the following approach is used, Litvinov (1987, 1989, 1990, 1994). A diffeomorphism P_q of the set $\bar{\Omega}_q$ onto a fixed set $\bar{\Omega}$ is applied and after the replacement of variables corresponding to the diffeomorphism P_q one obtains the problems $A(q)u(q) = f(q)$ in the fixed domain Ω and on its boundary S for all $q \in M$. In this case, $u(q) \in V$, where V is a space of functions defined on $\bar{\Omega}$, and the goal and restriction functionals may be defined on V. Thereby the general shape optimization problems and various optimization problems of mechanics were formulated, and the existence theorems, the differentiability of the function $q \to u(q)$, the differentiability of the goal and restriction functionals with respect to the control q, the necessary optimality conditions etc. were established, Litvinov (1987, 1989, 1990, 1994). But in some cases construction of the diffeomorphisms P_q for all $q \in M$, transition to the problems $A(q)u(q) = f(q)$ in the fixed domain $\bar{\Omega}$ and solving of these problems may be difficult. We introduce and study another approach to the shape optimization, which is based on the transition to equations on the boundary, and so on solving optimization problem on manifolds. In this case, instead of the diffeomorphisms P_q we should define maps I_q of the boundaries S_q, and the domains of the maps I_q should be same for all $q \in M$. Denoting it by T, we obtain state equations on the fixed set $T \subset R^{n-1}$, while $\Omega_q \in R^n$. Of course, such an approach may be used when fundamental solutions of state equations are calculated.

The outline of the paper is as follows. In Section 2 we formulate the optimal shape problem for two-dimensional elastic solid, and reduce state equations (boundary value problems) in domains to singular integral equations on the unit circle. Further we prove some auxiliary results of singular operators on the unit circle (Section 3). In Section 4 we prove the existence theorem for the optimization problem, and establish the Fréchet–differentiabilility of the mapping "control – function of state on the unit circle".

2. Optimization problem for an elastic solid

2.1. Formulation of the problem

We consider a shape optimization problem for a two-dimensional elastic solid. Let M be a set of controls and to each $q \in M$ a bounded domain $\Omega_q \subset R^2$ with a smooth boundary S_q be assigned. We suppose Ω_q to be multiply-connected, and denote by S_{q0} the external boundary of Ω_q, and by S_{qi}, $1 \le i \le p$, the other components of S_q, where $S_{qi} \cap S_{qj} = \emptyset$ for $i \neq j$, and S_{q0} envelopes the other S_{qk}, and S_{qk} $k = 1, \ldots, p$ do not envelop each other. S_{qi} is defined by a periodic function $q_i : (-\pi, \pi) \to R^2$, $i = 0, 1, \ldots, p$, see Fig 1. So we define a set of controls M by

$$\begin{aligned} M \;=\; \Big\{ & \; q = (q_0, q_1, \ldots, q_p),\; q_i = (q_{i1}, q_{i2}) \in \tilde{C}^{m+1}(-\pi,\pi)^2, \\ & m \text{ is an integer, } m \ge 3,\; \left(\frac{dq_{i1}}{dt}(t)\right)^2 + \left(\frac{dq_{i2}}{dt}(t)\right)^2 > c_0 > 0, \\ & q_i(t) \in Q_i \;\; \forall t \in (-\pi, \pi],\; i = 0, 1, \ldots, p \;\Big\}. \end{aligned} \tag{2.1}$$

Here Q_i are some open sets in R^2 such that $\forall q \in M$ the above conditions on Ω_q

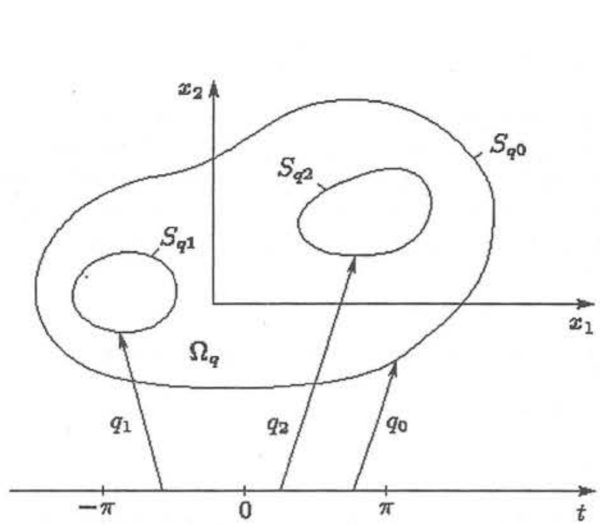

Figure 1. The domain Ω_q and the maps of its boundary.

are satisfied, $\tilde{C}^{m+1}(-\pi,\pi)$ is a subspace of periodic functions in $C^{m+1}([-\pi,\pi])$. The periodicity of a function $u \in C^k([-\pi,\pi])$ means that if $\tilde{u}$ is a periodic continuation of u on R with the period $[0,2\pi]$, then $\tilde{u} \in C^k([a,b])$ for an arbitrary $[a,b] \subset R$.

In the sequel we consider all periodic functions as being given on R or on the unit circle, in this case, points $t + 2\pi k$, $k \in Z$, (Z is the set of integers) are identified. So we consider periodic functions on $R/2\pi Z$, where $R/2\pi Z$ is a factor-group consisting of classes $\dot{t} = t + 2\pi Z$ containing a point t.

The set M is supplied with the topology generated by the topology of $\tilde{C}^{m+1}(-\pi,\pi)^{2(p+1)}$. We mark that the mapping q_i is a homomorphism of $(-\pi,\pi]$ onto S_{qi} for arbitrary $q \in M$, $i = 0,1,\ldots,p$.

The operator A_q of the theory of elasticity is defined by

$$A_q = -\mu\Delta u - (\lambda+\mu)\text{grad div}\, u \quad \text{in} \quad \Omega_q \,. \tag{2.2}$$

Here $u = (u_1, u_2)$ is a vector function of displacements, λ, μ are positive constants. We denote by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\sigma(u) = (\sigma_{ij}(u))$ the strain and stress tensors

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

$$\sigma_{ij}(u) = \lambda \text{div} u\, \delta_{ij} + 2\mu\varepsilon_{ij}(u), \quad i,j = 1,2\,, \tag{2.3}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$. The traction operator T_q is defined on S_q by

$$T_q u = ((T_q u)_1, (T_q u)_2),$$

$$(T_q u)_i = \sigma_{ij}(u)\nu_{qj} \quad \text{on} \quad S_q \quad i,j = 1,2 \; . \tag{2.4}$$

Here and below the summation over repeated index is implied, ν_{qj} are the components of the unit outward normal ν_q to S_q. Various formulations of problems of theory of elasticity may be considered, in particular, displacement, traction, mixed and other ones (see e. g. Kupradze, Gegelia, Bashelishvili and Burchuladze, 1979). We will be engaged in the traction formulation

$$T_q u = \mathcal{F}_q \quad \text{on} \quad S_q. \tag{2.5}$$

So we consider the problem: *Find a function u_q satisfying*

$$A_q u_q = 0 \quad in \;\; \Omega_q, \tag{2.6}$$

$$T_q u_q = \mathcal{F}_q \;\; on \;\; S_q. \tag{2.7}$$

The case when a function of body forces, i.e. the right hand side of (2.6), is not equal to zero, may be reduced to problem (2.6), (2.7). Further we suppose that the boundary S_{q0} is fixed, i.e.

$$S_{q0} = S \quad \forall q \in M, \tag{2.8}$$

the surface forces $\mathcal{F}_q$ are not equal to zero only on S, and they are fixed and self-balanced:

$$\mathcal{F}_q = 0 \;\; \text{on} \;\; S_{qi} \quad i = 1, \ldots, p \; , \quad \mathcal{F}_q \Big|_S = (\mathcal{F}_1, \mathcal{F}_2) \in H^{m-3/2}(S)^2,$$

$$\int_S \mathcal{F}_i ds = 0 \quad i = 1,2 \; , \quad \int_S (\mathcal{F}_1 x_2 - \mathcal{F}_2 x_1) ds = 0. \tag{2.9}$$

We introduce the spaces

$$V_{mq} = H^m(\Omega_q)^2 \; , \quad H_{mq} = H^{m-2}(\Omega_q)^2 \times H^{m-3/2}(S_q)^2. \tag{2.10}$$

Then the operator $G_q = (A_q, T_q)$ is a linear continuous mapping from V_{mq} into H_{mq}, i.e. $G_q \in \mathcal{L}(V_{mq}, H_{mq})$, and by known results, Litvinov (1990), Agmon, Douglis, Nirenberg (1964), Michlin (1973), Roitberg (1975) we obtain

THEOREM 2.1 *Let the set M be defined by (2.1), and (2.8), (2.9) hold. Then for each $q \in M$ the following representations are valid*

$$V_{mq} = \hat{V}_{mq} \oplus \check{V}_{mq} \; , \quad H_{mq} = \hat{H}_{mq} \oplus \check{H}_{mq} \; ,$$

where $\hat{V}_{mq} = \ker G_q$, $\check{H}_{mq} = G_q(V_{mq})$, $G_q = (A_q, T_q)$, $\oplus$ is the sign of the direct sum. The subspaces $\hat{V}_{mq}$ and $\hat{H}_{mq}$ are three-dimensional, and $\varphi_1 = (1,0)$, $\varphi_2 = (0,1)$, $\varphi_3 = (x_2, -x_1)$ is a basis in $\hat{V}_{mq}$, $\psi_1 = ((1,0),(1,0))$, $\psi_2 = ((0,1),(0,1))$, $\psi_3 = ((x_2,-x_1),(x_2,-x_1))$ is a basis in $\hat{H}_{mq}$.

For each $q \in M$, there exists a unique $u_q \in \check{V}_{mq}$ satisfying (2.6), (2.7).

We introduce the following functionals on M.

$$\begin{aligned}
\Psi_0(q) &= \int_{\Omega_q} dx \ , \\
\Psi_1(q) &= \max_{x\in\bar{\Omega}_q} |u_q(x)| - c_1 \quad u_q \in \check{V}_{mq} \ , \\
\Psi_2(q) &= \max_{x\in\bar{\Omega}_q} \sum_{i,j=1}^{2} \left[\sigma_{ij}(u_q)(x) - \frac{1}{2}(\sigma_{11}(u_q)(x) + \sigma_{22}(u_q)(x))\delta_{ij}\right]^2 - c_2, \\
\Psi_3(q) &= \int_{\Omega_q} \sigma_{ij}(u_q)\varepsilon_{ij}(u_q)dx - c_3 \ , \qquad (2.11)
\end{aligned}$$

where c_1, c_2, c_3 are positive constants. Note that other functionals on M may also be considered. Now let

$$M_1 \ \textit{be a compact subset in } M \ . \qquad (2.12)$$

In particular, the set M_1 may be defined by

$$\begin{aligned}
M_1 \ = \ \Big\{ \ & q = (q_0, q_1, \dots, q_p) \in M, \ q_i(t) \in \tilde{Q}_i \ \forall t \in (-\pi, \pi], \\
& \tilde{Q}_i \text{ are closed subsets in } Q_i \ , \\
& \|q_i\|_{C^{m+1,\alpha}([-\pi,\pi])} \le c, \alpha \in (0,1], \ i = 1, \dots, p \Big\} \ .
\end{aligned}$$

We remind that q_0 is considered to be fixed (see (2.8)), $C^{k,\alpha}$ denotes a Hölder space with the norm

$$\|u\|_{C^{k,\alpha}([-\pi,\pi])} = \|u\|_{C^k([-\pi,\pi])} + \sup_{t,t'\in[-\pi,\pi]} \left| \frac{d^k u}{dt^k}(t) - \frac{d^k u}{dt^k}(t') \right| / |t-t'|^\alpha \ .$$

We define a set of admissible controls M_∂ as follows

$$M_\partial = \Big\{ \ q \in M_1, \ \Psi_i(q) \le 0 \quad i = 1, 2, 3 \Big\} \ , \qquad (2.13)$$

and consider the optimization problem: *Find* $\tilde{q}$ *satisfying*

$$\tilde{q} \in M_\partial \ , \quad \Psi_0(\tilde{q}) = \inf_{q\in M_\partial} \Psi_0(q). \qquad (2.14)$$

From the physical point of view, problem (2.14) corresponds to the minimization of the area (weight) of an elastic solid under the restrictions on displacements, stresses and strain energy. Other restrictions of the form $\Psi_k(q) \le 0$ may also be considered.

2.2. State equations on boundaries and on the unit circle

Let $G(x,y) = (G_{ij}(x,y))$ be a tensor of fundamental solutions of the equation

$$Au = -\mu\Delta u - (\lambda + \mu)\mathrm{grad\,div}u = 0, \quad u = (u_1, u_2). \qquad (2.15)$$

$G(x,y)$ is a symmetric tensor defined by

$$G_{ij}(x,y) = c_1\left(c_2\delta_{ij}\ln r - \frac{(x_i-y_i)(x_j-y_j)}{r^2}\right) \quad i,j=1,2\,, \tag{2.16}$$

where

$$r = \left[\sum_{i=1}^{2}(x_i-y_i)^2\right]^{1/2},$$

$$c_1 = -1/[8\pi\mu(1-\sigma)], \quad c_2 = 3-4\sigma, \quad \sigma = \lambda/[2(\lambda+\mu)]\,. \tag{2.17}$$

From the physical point of view, the function $G_{ij}(x,y)$ defines a displacement $u_i(x)$ engendered by the unit force $P_j(y)$ concentrated at a point y and directed along the coordinate axis x_j.

By $B_k(x,y) = (B_{ijk}(x,y))$ and $T_k(x,y) = (T_{ijk}(x,y))$ we denote the deformation and stress tensors at a point x that are engendered by the force $P_k(y)$. Due to (2.3) and (2.16), we obtain

$$B_{ijk}(x,y) = \frac{c_1}{r^2}\left[(1-2\nu)(\delta_{ik}\xi_j+\delta_{jk}\xi_i) - \delta_{ij}\xi_k + \frac{2}{r^2}\xi_i\xi_j\xi_k\right], \tag{2.18}$$

$$T_{ijk}(x,y) = \frac{c_3}{r^2}\left[c_4(\delta_{ik}\xi_j+\delta_{jk}\xi_i-\delta_{ij}\xi_k) + \frac{2}{r^2}\xi_i\xi_j\xi_k\right], \tag{2.19}$$

$$\xi_i = x_i - y_i, \quad c_3 = -1/[4\pi(1-\sigma)], \quad c_4 = 1-2\sigma.$$

The force $t(x) = (t_1(x), t_2(x))$ at a point x of a surface with a unit outward normal $\nu = (\nu_1,\nu_2)$ is defined by $t_i(x) = \sigma_{ij}(x)\nu_j(x)$. So denoting by $R_{ik}(x,y)$ the value at a point x of i component's of the surface force generated by $P_k(y)$, due to (2.19), we get

$$R_{ik}(x,y) = \frac{c_3}{r^2}\left[c_4(\nu_k\xi_i - \nu_i\xi_k) + \left(c_4\delta_{ik} + \frac{2\xi_i\xi_k}{r^2}\right)\xi_j\nu_j\right]. \tag{2.20}$$

Let $u=(u_1,u_2)$ be a smooth function satisfying (2.15) in Ω_q. By Betti's formula, Michlin (1962), Banerjee and Butterfield (1981), we obtain

$$u_i(x) = \int_{S_q}\left[(T_qu)_j(y)G_{ij}(x,y) - F_{qij}(x,y)u_j(y)\right]dS_y\,, \qquad x\in\Omega_q,\ i=1,2\,, \tag{2.21}$$

where $F_{qij}(x,y) = R_{ji}(y,x)$ for $\nu=\nu_q$, i.e.

$$\begin{aligned} F_{qij}(x,y) = {} & \frac{c_3}{r^2}\Big[c_4(\nu_{qi}(y)(y_j-x_j) - \nu_{qj}(y)(y_i-x_i)) + \\ & + \left(c_4\delta_{ij} + \frac{2(y_i-x_i)(y_j-x_j)}{r^2}\right)(y_k-x_k)\nu_{qk}(y)\Big]. \end{aligned} \tag{2.22}$$

The integral from the second addend in (2.21) should be understood in the sense of a Cauchy principal value. Because of the jump relation for $F_q(x,y) = (F_{qij}(x,y))$ on S_q, Michlin (1962), Banerjee and Butterfield (1981), the representation formula (2.21) yields the following expression

$$\frac{1}{2}u_i(x) = \int_{S_q} [(T_q u)_j(y) G_{ij}(x,y) - F_{qij}(x,y) u_j(y)]\, dS_y \ , \qquad x \in S_q,\ i = 1,2 \ . \tag{2.23}$$

Considering the problem (2.6), (2.7), we obtain the equation

$$N_q u(x) = u(x) + 2\int_{S_q} F_q(x,y) u(y) dS_y = f_q(x), \quad x \in S_q \ , \tag{2.24}$$

where

$$f_q(x) = 2\int_{S_q} G(x,y)\mathcal{F}_q(y) dS_y \ . \tag{2.25}$$

The boundary S_q is defined by the function $q = (q_0, q_1, \ldots, q_p) \in M$ (see Fig. 1). Denoting $v_{qi}(t) = u(q_i(t))$, $i = 0,1,\ldots,p$, $v_q = (v_{q0},\ldots,v_{qp})$, we transform problem (2.24) to the following one

$$\begin{aligned}(P(q)v_q)_i(t) &= v_{qi}(t) + 2\sum_{j=0}^{p}\int_{-\pi}^{\pi} F_q(q_i(t), q_j(\tau)) v_{qj}(\tau) D(q_j)(\tau) d\tau \\ &= g_i(t), \quad t \in [-\pi,\pi], \quad i = 0,1,\ldots,p \ .\end{aligned} \tag{2.26}$$

Here

$$D(q_j)(\tau) = \left[\left(\frac{dq_{j1}}{d\tau}(\tau)\right)^2 + \left(\frac{dq_{j2}}{d\tau}(\tau)\right)^2\right]^{1/2}, \quad g_i(t) = f_q(q_i(t)), \tag{2.27}$$

and g_i is independent of q because $\mathcal{F}_q$ is not equal to zero only on S_{q0}, and S_{q0} is fixed, see (2.8), (2.9). In the case of $i = j$, the integral in (2.26) should be understood as the limit of the integral

$$\int_{-\pi}^{\pi} F_{q\varepsilon}(t,\tau) v_{qi}(\tau) D(q_i)(\tau) d\tau,$$

$$F_{q\varepsilon}(t,\tau) = \begin{cases} F_q(q_i(t), q_i(\tau)) & \text{for} \quad |t-\tau| \geq \varepsilon \\ 0 & \text{for} \quad |t-\tau| < \varepsilon \end{cases}$$

as ε tends to zero.

3. Spaces and operators on $R/2\pi Z$, auxiliary statements

Define the space W_m as follows

$$W_m = \Big\{ u = (u_0, u_1, \dots, u_p),\ u_i = (u_{i1}, u_{i2}) \in \tilde{H}^{m-1/2}(-\pi,\pi)^2,\ i = 0,1,\dots,p,\ m \geq 3 \Big\} . \tag{3.1}$$

By $\tilde{H}^s(-\pi,\pi)$, $s > 0$ we denote a subspace of periodic functions in $H^s(-\pi,\pi)$. A norm in W_m is defined by

$$\|v\|_m = \left(\sum_{i=0}^{p} \sum_{k=1}^{2} \|u_{ik}\|^2_{\tilde{H}^{m-1/2}(-\pi,\pi)} \right)^{1/2} . \tag{3.2}$$

Here,

$$\|v\|_{\tilde{H}^{m-1/2}(-\pi,\pi)} = \left[a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) n^{2m-1} \right]^{1/2} , \tag{3.3}$$

where a_n, b_n are the Fourier coefficients of the function v, i.e.

$$v(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} v(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} v(t) \sin nt \, dt. \tag{3.4}$$

Let

$$\hat{W}_{mq} = (\gamma_q \hat{V}_{mq}) \circ q . \tag{3.5}$$

Here $\hat{V}_{mq}$ is $\ker(A_q, T_q)$ (see Theorem 2.1), γ_q is the trace operator on S_q, q is the function defined in (2.1). It follows from Theorem 2.1 that $\hat{W}_{mq}$ is a three-dimensional subspace in W_m with the basis $\{\varphi_{qi}\}_{i=1}^3$,

$$\varphi_{qi} = (\varphi_{qi0}, \varphi_{qi1}, \dots, \varphi_{qip}) \quad i = 1,2,3 ,$$

$$\varphi_{q1k} = (1,0),\ \varphi_{q2k} = (0,1),\ \varphi_{q3k} = (q_{k2}, -q_{k1}), \quad k = 0,1,\dots,p , \tag{3.6}$$

q_{ki} are defined in (2.1). We denote by J_q the operator of the simple layer potential

$$(J_q v)(x) = \int_{S_q} G(x,y) v(y) dS_y \quad x \in S_q , \tag{3.7}$$

and let

$$\psi_{qi} = (J_q \beta_i) \circ q \quad i = 1,2,3 ,$$

$$\beta_1 = (1,0) \text{ on } S_{qi},\ \beta_2 = (0,1) \text{ on } S_{qi},\ \beta_3 = (x_2, -x_1) \text{ on } S_{qi}, \quad i = 0,1,\ldots,p\,. \tag{3.8}$$

By (3.7), (3.8) we get

$$\psi_{qi} = (\psi_{qi0}, \psi_{qi1}, \ldots, \psi_{qip}) \quad i = 1,2,3\,,$$

$$\psi_{qij}(t) = \sum_{k=0}^{p} \int_{-\pi}^{\pi} G(q_j(t), q_k(\tau))\beta_i(q_k(\tau))D(q_k)(\tau)d\tau, \quad j = 0,1,\ldots,p\,, \tag{3.9}$$

$D(q_k)$ is defined by (2.27).

THEOREM 3.1 *Let the set M be defined by (2.1) and (2.8), (2.9) hold. Let also the operator $P(q)$ be defined by (2.26). Then $P(q) \in \mathcal{L}(W_m, W_m)$, and for each $q \in M$, the following representations are valid*

$$W_m = \hat{W}_{mq} \oplus \check{W}_{mq} = \hat{E}_{mq} \oplus \check{E}_{mq}\,, \tag{3.10}$$

where $\hat{W}_{mq} = \ker P(q)$, $\check{E}_{mq} = P(q)(W_m)$, $\hat{E}_{mq}$ is a three-dimensional subspace in W_m with a basis $\{\psi_{qi}\}_{i=1}^{3}$ defined by (3.9). There exists a unique $v_q \in \check{W}_{mq}$ satisfying (2.26).

PROOF. The operator J_q is an pseudodifferential operator of order -1 on S_q, and J_q is an isomorphism from $H^{m-3/2}(S_q)^2$ onto $H^{m-1/2}(S_q)^2$, Chudinovich (1991), Wendland (1985). It is obvious that if u_q is a solution of problem (2.6), (2.7) then $u = \gamma_q u_q$ is a solution of problem (2.24). On the contrary, if u is a solution of problem (2.24), then the function

$$u_q(x) = \int_{S_q} [G(x,y)\mathcal{F}_q(y) - F_q(x,y)u(y)]dS_y \quad x \in \Omega_q$$

is a solution of problem (2.6), (2.7). Problem (2.26) is obtained from (2.24) by a replacement of variables corresponding to the 1–1 mapping $q \in \tilde{C}^{m+1}(-\pi,\pi)^{2(p+1)}$. And so Theorem 3.1 follows from Theorem 2.1. ■

REMARK. The space $\check{W}_{mq}$ is defined non-uniquely by (3.10), and so we define $\check{W}_{mq}$ so that $\check{W}_{mq}$ be orthogonal to $\hat{W}_{mq}$ with respect to scalar product in $L_2(-\pi,\pi)^{2(p+1)}$.

We define the operator $\Gamma_{q1} \in \mathcal{L}(W_m, \hat{W}_{mq})$ by

$$\left.\begin{array}{l}\Gamma_{q1} \text{ is the operator of orthogonal projection} \\ \text{of } W_m \text{ onto } \hat{W}_{mq} \text{ with respect to scalar} \\ \text{product in } L_2(-\pi,\pi)^{2(p+1)}.\end{array}\right\} \tag{3.11}$$

We also define an operator $\Gamma_{q2} \in \mathcal{L}(\hat{W}_{mq}, \hat{E}_{mq})$ as follows

$$u = \sum_{i=1}^{3} c_i \varphi_{qi}\,, \quad \Gamma_{q2} u = \sum_{i=1}^{3} c_i \psi_{qi}\,. \tag{3.12}$$

THEOREM 3.2 *Let the set M and the operator P_q be defined by (2.1) and (2.26) accordingly. Then the operator $\mathcal{N}_q = P(q) + \Gamma(q)$, where $\Gamma(q) = \Gamma_{q2} \circ \Gamma_{q1}$ is an isomorphism from W_m onto W_m.*

Indeed, it is obvious that $\Gamma(q) \in \mathcal{L}(W_m, \hat{E}_{mq})$ and $\Gamma(q)(W_m) = \hat{E}_{mq}$. Therefore, by Theorem 3.1 the operator $P(q)+\Gamma(q)$ is a continuous bijection from W_m onto W_m, and so by the Banach theorem $P(q) + \Gamma(q)$ is an isomorphism from W_m onto W_m.

LEMMA 3.1 *Let the set M and the operator $\Gamma(q) = \Gamma_{q2} \circ \Gamma_{q1}$ be defined by (2.1), (3.11), (3.12), and $q^n \to q^0$ in M (M is supplied with the topology generated by the $\tilde{C}^{m+1}(-\pi,\pi)^{2(p+1)}$ topology). Then $\|\Gamma(q^n) - \Gamma(q^0)\|_{\mathcal{L}(W_m,W_m)} \to 0$ as $n \to \infty$.*

PROOF. Let $u = (u_0, u_1, \ldots, u_p) \in W_m$. By (3.11) we have

$$\|\Gamma_{q^n1}u - u\|^2_{L_2(-\pi,\pi)^{2(p+1)}} = \min_{c_i} \sum_{k=0}^{p} \|u_k - \sum_{i=1}^{3} c_i \varphi_{q^n ik}\|^2_{L_2(-\pi,\pi)^2}. \tag{3.13}$$

We denote by c_{in} a solution of problem (3.13), i.e. c_{in} minimize the norms in (3.13). Then we get

$$c_{in} \sum_{i=1}^{3} \sum_{k=0}^{p} (\varphi_{q^n ik}, \varphi_{q^n jk})_{L_2(-\pi,\pi)^2} = \sum_{k=0}^{p} (u_k, \varphi_{q^n jk})_{L_2(-\pi,\pi)^2}$$
$$j = 1, 2, 3\,. \tag{3.14}$$

The matrix of the system (3.14) is non-degenerate because the functions $\{\varphi_{q^n i}\}_{i=1}^{3}$ are a basis in $\hat{W}_{mq^n}$. Therefore c_{in} are defined uniquely. Let now $q^n \to q^0$ in M. Then $\varphi_{q^n ik} \to \varphi_{q^0 ik}$ in $\tilde{C}^{m+1}(-\pi,\pi)^2$, and so $c_{in} \to c_{i0}$ as $n \to \infty$, where c_{i0} is a solution of problem (3.14) for $n = 0$. As the embedding of $\tilde{C}^{m+1}(-\pi,\pi)$ into $L_2(-\pi,\pi)$ is compact, we obtain that $c_{in} \to c_{i0}$ uniformly for all $u \in K$, where K is an arbitrary bounded set in M.

It may be shown that $\psi_{q^n ij} \to \psi_{q^0 ij}$ in $\tilde{H}^m(-\pi,\pi)^2$ as $q^n \to q^0$ in M. Therefore $\Gamma(q^n) \to \Gamma(q^0)$ in $\mathcal{L}(W_m, W_m)$. ■

In the space W_m we define the operator $\mathcal{N}(q)$ as follows

$$v = (v_0, v_1, \ldots, v_p) \in W_m\,, \quad \mathcal{N}(q)v = \{(\mathcal{N}(q)v)_i\}_{i=0}^{p}\,,$$
$$(\mathcal{N}(q)v)_i(t) = \sum_{j=0}^{p} \int_{-\pi}^{\pi} F_q(q_i(t), q_j(\tau)) v_j(\tau) D(q_j)(\tau) d\tau \tag{3.15}$$
$$t \in [-\pi,\pi], \quad i = 0, 1, \ldots, p\,.$$

By (2.26) we have

$$P(q) = I + 2\mathcal{N}(q), \tag{3.16}$$

where I is the unit operator in W_m.

LEMMA 3.2 *Let the set M be defined by (2.1) and the operator $\mathcal{N}(q)$ be defined by (3.15), (2.22), (2.27). Then $\mathcal{N}(q) \in \mathcal{L}(W_m, W_m)$ and the function $\mathcal{N} : q \to \to \mathcal{N}(q)$ is a continuous mapping from M into $\mathcal{L}(W_m, W_m)$.*

PROOF. By (2.22) we have

$$F_q(q_i(t), q_j(\tau)) = (F_{qkn}(q_i(t), q_j(\tau)))_{k,n=1}^{2} ,$$

$$\begin{aligned} F_{qkn}(q_i(t), q_j(\tau)) &= \frac{c_3}{\rho(\tau,t)}\Big\{c_4\big[\ \nu_{qk}(q_j(\tau))(q_{jn}(\tau) - q_{in}(t)) - \\ &\quad -\nu_{qn}(q_j(\tau))(q_{jk}(\tau) - q_{ik}(t))\big]\ + \\ &\quad + \left[c_4\delta_{kn} + \frac{2(q_{jk}(\tau) - q_{ik}(t))(q_{jn}(\tau) - q_{in}(t))}{\rho(\tau,t)}\right] \times \\ &\quad \times \sum_{s=1}^{2}(q_{js}(\tau) - q_{is}(t))\nu_{qs}(q_j(\tau))\Big\}. \end{aligned} \tag{3.17}$$

Here

$$\begin{aligned} \rho(\tau,t) &= \sum_{s=1}^{2}(q_{is}(t) - q_{js}(\tau))^2, \\ \nu_q(q_j(\tau)) &= \left(a(\tau)\frac{dq_{j2}}{d\tau}(\tau), -a(\tau)\frac{dq_{j1}}{d\tau}(\tau)\right), \\ a(\tau) &= \left[\left(\frac{dq_{j1}}{d\tau}(\tau)\right)^2 + \left(\frac{dq_{j2}}{d\tau}(\tau)\right)^2\right]^{-1/2}. \end{aligned} \tag{3.18}$$

In the case when $i \neq j$, the elements $F_{qkn}(q_i(t), q_j(\tau))$ are non-singular and they generate smoothing operators. So we consider the case when $i = j$. As $q_i \in \tilde{C}^{m+1}(-\pi,\pi)^2$, by (3.18) and the Taylor formula we get

$$\left|\sum_{s=1}^{2}(q_{is}(\tau) - q_{is}(t))\nu_{qs}(q_i(\tau))\right| = |A(\tau,t)| \leq c(t-\tau)^2.$$

Therefore the terms containing $A(\tau,t)$ as a factor in (3.17) generate a non-singular operator. The two remained addends in (3.17) are singular and similar. So it is sufficient to show that the operator $\mathcal{R}(q)$ defined by

$$\left.\begin{array}{l} u \in \tilde{H}^{m-1/2}(-\pi,\pi), \\ (\mathcal{R}(q)u)(t) = \displaystyle\int_{-\pi}^{\pi} \frac{1}{\rho(\tau,t)}\nu_{qk}(q_i(\tau))(q_{in}(\tau) - q_{in}(t))D(q_i)(\tau)u(\tau)d\tau, \\ \text{is a continuous mapping from } \tilde{H}^{m-1/2}(-\pi,\pi) \text{ into} \\ \tilde{H}^{m-1/2}(-\pi,\pi), \text{ and if } q^\mu \to q^0 \text{ in } M, \text{ then } \mathcal{R}(q^\mu) \to \mathcal{R}(q^0) \text{ in} \\ \mathcal{L}(\tilde{H}^{m-1/2}(-\pi,\pi), \tilde{H}^{m-1/2}(-\pi,\pi)). \end{array}\right\} \tag{3.19}$$

By the Taylor formula we obtain

$$q_{in}(\tau) - q_{in}(t) = q'_{in}(t)(\tau - t) + a_q(\tau, t), \tag{3.20}$$

$$\rho(\tau, t) = \sum_{s=1}^{2}(q_{is}(\tau) - q_{is}(t))^2 = \sum_{s=1}^{2} q'_{is}(t)^2(\tau - t)^2 + \beta_q(\tau, t), \tag{3.21}$$

$$|a_q(\tau, t)| \le c|\tau - t|^2, \quad |\beta_q(\tau, t)| \le c_1|\tau - t|^3. \tag{3.22}$$

Substituting (3.20), (3.21) into $\mathcal{R}(q)u$ we get

$$\mathcal{R}(q)u = \mathcal{R}_1(q)u + \mathcal{R}_2(q)u, \tag{3.23}$$

$$(\mathcal{R}_1(q)u)(t) = \frac{q'_{in}(t)}{\sum\limits_{s=1}^{2} q'_{is}(t)^2} \int_\Gamma \frac{w_q(\tau)u(\tau)}{\tau - t} d\tau \ , \tag{3.24}$$

$$(\mathcal{R}_2(q)u)(t) = \int_\Gamma H(\tau, t) w_q(\tau) u(\tau) d\tau \ . \tag{3.25}$$

Here Γ is the unit circle with the center at 0 on the plane x_1Ox_2, the origin of count of the curvilinear coordinate on Γ is a point of Γ lying on Ox_1, the direction of count is counter-clockwise

$$\begin{aligned} w_q(\tau) &= \nu_{qk}(q_i(\tau))D(q_i)(\tau), \quad H(\tau, t) = e_q(\tau, t)/f_q(\tau, t), \\ e_q(\tau, t) &= \left(\sum_{s=1}^{2} q'_{is}(t)^2\right) a_q(\tau, t)(\tau - t) - q'_{in}(t) b_q(\tau, t), \\ f_q(\tau, t) &= \left(\sum_{s=1}^{2} q'_{is}(t)^2\right)^2 (\tau - t)^3 + \left(\sum_{s=1}^{2} q'_{is}(t)^2\right) \beta_q(\tau, t)(\tau - t). \end{aligned} \tag{3.26}$$

We remark that the function $\tau \to \tau - t$ is discontinuous on Γ at the point $\tau_0 = t + \pi$. Define the operator $\mathcal{R}_3$ as follows

$$(\mathcal{R}_3 u)(t) = \int_\Gamma \frac{w_q(\tau)u(\tau)}{\tau - t} d\tau \ . \tag{3.27}$$

It is obvious that $\mathcal{R}_3 u$ is the convolution on Γ of the principal value of the distribution $1/t$ and $-w_q u$. We denote by $c_k(f)$ the Fourier coefficients of a function f for the complex form of a Fourier series

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \ .$$

Then we have, Schwartz (1961)

$$\sum_{k=-\infty}^{\infty} |c_k(\mathcal{R}_3 u)|^2 = 4\pi^2 \sum_{k=-\infty}^{\infty} |c_k(1/t) c_k(-w_q u)|^2 \ . \tag{3.28}$$

Since $|c_k(1/t)| \le$ const $\quad \forall k$ (see Edwards 1979) and $w_q \in \tilde{C}^m(-\pi,\pi)$, the Parseval equality and (3.28) yield

$$\|\mathcal{R}_3 u\|^2_{L_2(-\pi,\pi)} = 2\pi \sum_{k=-\infty}^{\infty} |c_k(\mathcal{R}_3 u)|^2 \le c\|u\|^2_{L_2(-\pi,\pi)}. \tag{3.29}$$

For $a \neq 0$, we get

$$\begin{aligned} & a^{-1}[(\mathcal{R}_3 u)(t+a) - (\mathcal{R}_3 u)(t)] = \\ & = \ a^{-1}\int_\Gamma \left[\frac{w_q(\tau)u(\tau)}{\tau - t - a} - \frac{w_q(\tau)u(\tau)}{\tau - t}\right] d\tau = \\ & = \ a^{-1}\int_\Gamma \frac{w_q(\tau+a)u(\tau+a) - w_q(\tau)u(\tau)}{\tau - t} d\tau \, . \end{aligned} \tag{3.30}$$

By (3.29), (3.30) we obtain that if $u \in \tilde{H}^1(-\pi,\pi)$ then

$$\left(\frac{d}{dt}(\mathcal{R}_3 u)\right)(t) = \int_\Gamma (\tau - t)^{-1} \frac{d}{d\tau}(w_q(\tau)u(\tau))d\tau$$

and $\|\mathcal{R}_3 u\|_{\tilde{H}^1(-\pi,\pi)} \le c_1\|u\|_{\tilde{H}^1(-\pi,\pi)}$. By analogy, we have $\|\mathcal{R}_3 u\|_{\tilde{H}^m(-\pi,\pi)} \le$ $\le c_1\|u\|_{\tilde{H}^m(-\pi,\pi)}$, and taking into account (3.24), (3.27) we get

$$\left.\begin{array}{l} \mathcal{R}_1(q)u \in \mathcal{L}(\tilde{H}^m(-\pi,\pi), \tilde{H}^m(-\pi,\pi)) \text{ and from} \\ \text{the condition } q^\mu \to q^0 \text{ in } M \text{ it follows that} \\ \mathcal{R}_1(q^\mu) \to \mathcal{R}_1(q^0) \in \mathcal{L}(\tilde{H}^m(-\pi,\pi), \tilde{H}^m(-\pi,\pi)) \end{array}\right\} \tag{3.31}$$

By interpolation, Lions and Magenes (1968), we obtain that (3.31) holds for m replaced with $m - 1/2$. The operator $\mathcal{R}_2$ is smoothing and satisfies also (3.31). Therefore (3.19) holds. ■

We remind that by Ω_q we denote a domain in R^2 with a boundary $S_q = \bigcup_{i=0}^{p} S_{qi}$, S_q are defined by a function $q = (q_0, q_1, \ldots, q_p)$ (see Section 2.1., Fig. 1).

THEOREM 3.3 *Let the set M be defined by (2.1) and $q^n \to q^0$ in M. Then for each sufficiently large n there exists a mapping P_n such that P_n is a C^m-diffeomorphism of $\bar{\Omega}_{q^0}$ onto $\bar{\Omega}_{q^n}$ and $P_n \to I$ in $C^m(\bar{\Omega}_{q^0})^2$, where I is the identity mapping in $\bar{\Omega}_{q^0}$, i.e. $I(x) = x$.*

PROOF. In the beginning we will prove the theorem for the case when Ω_{q^n} and Ω_{q^0} are simply connected. Let S_n, S_0 be boundaries of Ω_{q^n}, Ω_{q^0} accordingly, δ be a small positive number and

$$G_0 = \left\{x \in \Omega_{q^0}, \ \inf_{y \in S_0} \|x - y\| > \delta\right\}. \tag{3.32}$$

Let also T_0 be a boundary of G_0 and $F_0 = \Omega_{q^0} \setminus \bar{G}_0$, see Fig. 2. By s we denote

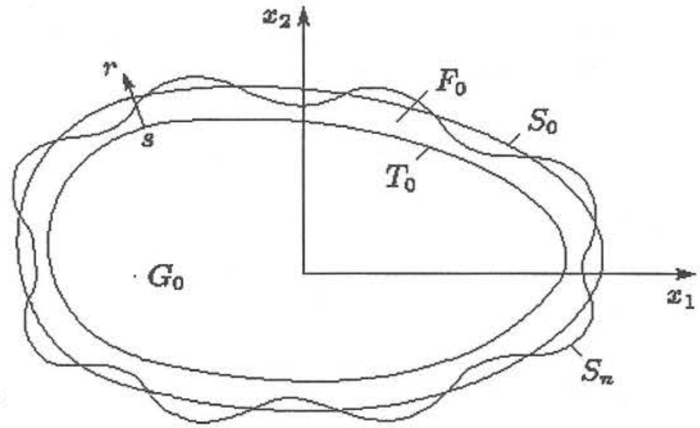

Figure 2. The domains Ω_{q^n}, Ω_{q^0} and their boundaries.

points of a parametrization of T_0. Then we can consider that $s \in R/2\pi Z$. Outside of G_0 we define the curvilinear coordinates (s, r), the axis r at point s is normal to T_0. Let now

$$q^n \to q^0 \ \text{ in } \ \tilde{C}^{m+1}(-\pi, \pi)^2 \ . \tag{3.33}$$

Then for sufficiently large n, we have $S_n \cap T_0 = \emptyset$, and we define the function $f_n : R/2\pi Z \times [0, \delta] \to R^2$ as follows

$$f_n(s, r) = (s, \beta_n(s, r)), \quad \beta_n(s, r) = \sum_{k=1}^{m+1} a_{nk}(s) r^k, \tag{3.34}$$

$$\frac{\partial \beta_n}{\partial r}(s, 0) = 1, \ \frac{\partial^k \beta_n}{\partial r^k}(s, 0) = 0 \ \ k = 2, \ldots, m \ , \ \ Q(s, \beta_n(s, \delta)) \in S_n. \tag{3.35}$$

Here Q is the transformation of the curvilinear coordinates (s, r) onto the Cartesian coordinates (x_1, x_2). By (3.34), (3.35) we get

$$a_{n1}(s) = 1, \ \ a_{nk}(s) = 0 \quad k = 2, \ldots, m \ , \tag{3.36}$$

and (3.33) yields $a_{n(m+1)}(s) \to 0$ uniformly with respect to s. Therefore f_n is a 1–1 mapping. Since $S_n \in C^{m+1}$ we obtain from (3.35) that

$$a_{n(m+1)} \in \tilde{C}^m(-\pi, \pi). \tag{3.37}$$

Now we define the mapping $P_n : \bar{\Omega}_{q^0} \to R^2$ as follows:

$$P_n(x) = \begin{cases} x & \text{for} \quad x \in \bar{G}_0 \\ (Q \circ f_n \circ Q^{-1})(x) & \text{for} \quad x \in \bar{\Omega}_{q^0} \setminus \bar{G}_0 \end{cases} . \tag{3.38}$$

By (3.34)–(3.38) we get that P_n is a C^m-diffeomorphism of $\bar{\Omega}_{q^0}$ onto $\bar{\Omega}_{q^n}$ and $P_n \to I$ in $C^m(\bar{\Omega}_{q^0})^2$.

In the case when Ω_{q^n} and Ω_{q^0} are multiply-connected, we introduce the curvilinear coordinates (s,r) in the vicinity of each component of the boundary of the domain Ω_{q^0}, the functions f_n are defined in each vicinity, and by analogy with (3.38), we define a C^m-diffeomorphism of $\bar{\Omega}_{q^0}$ onto $\bar{\Omega}_{q^n}$. ■

4. Optimization problem on $R/2\pi Z$

4.1. State equations and functionals

By Theorem 3.1 for each $q \in M$, there exists a unique v_q satisfying

$$v_q \in \check{W}_{mq}, \quad P(q)v_q = g, \tag{4.1}$$

where the operator P_q and $g = (g_0, g_1, \ldots, g)$ are defined by (2.25)–(2.27). So the function $M \ni q \to v_q \in W_m$ is defined.

THEOREM 4.1 *Let the set M be defined by (2.1), and (2.8), (2.9) hold. Then the function $q \to v_q$ defined by a solution of problem (4.1) is a continuous mapping from M into W_m.*

PROOF. We define the mapping $J : M \times W_m \to W_m$ by

$$J(q,u) = (P(q) + \Gamma(q))u - g, \tag{4.2}$$

and by λ we denote the implicit function defined by $\lambda(q) \in W_m$, $J(q, \lambda(q)) = 0$. Since $\Gamma(q)v_q = 0$ (see (3.11) and Remark in Section 3.), we obtain $J(q, v_q) = 0$, i.e. $\lambda(q) = v_q$. By Lemmas 3.1, 3.2 and (3.16) the function $q \to P(q) + \Gamma(q)$ is a continuous mapping from M into $\mathcal{L}(W_m, W_m)$. By Theorem 3.2 the operator $P(q) + \Gamma(q)$ is an isomorphism from W_m onto W_m. Now Theorem 4.1 follows from the implicit function theorem, Schwartz (1967). ■

THEOREM 4.2 *Let the set M be defined by (2.1), and (2.8), (2.9) hold. Let also the functionals Ψ_i $i = 0,1,2,3$ be defined by (2.11), where $u_q \in \check{V}_{mq}$ is a solution of problem (2.6), (2.7). Then Ψ_i are continuous on M functionals.*

PROOF. The continuity of the functional Ψ_0 is obvious. We prove the continuity of the functional Ψ_2. The proof for the other functionals is analogous. Let

$$q^n \to q^0 \text{ in } M. \tag{4.3}$$

We denote by u^n and u^0 the solutions of problem (2.6), (2.7) for $q = q^n$ and $q = q^0$ accordingly. By (2.21) we have

$$u^n(x) = -\sum_{j=0}^{p} \int_{-\pi}^{\pi} F_q(x, q_j^n(\tau)) v_{q^n j}(\tau) D(q_j^n)(\tau) d\tau + \mathcal{P}(x) \qquad x \in \Omega_{q^n}, \tag{4.4}$$

$$u^0(x) = -\sum_{j=0}^{p} \int_{-\pi}^{\pi} F_q(x, q_j^0(\tau)) v_{q^0 j}(\tau) D(q_j^0)(\tau) d\tau + \mathcal{P}(x)$$

$$x \in \Omega_{q^0}, \tag{4.5}$$

$$\mathcal{P}(x) = \int_{-\pi}^{\pi} G(x, q_0(\tau)) \mathcal{F}(q_0(\tau)) D(q_0)(\tau) d\tau. \tag{4.6}$$

Here q_0 is the first component of an element $q \in M$, q_0 is fixed and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, see (2.8), (2.9), $v_{q^n j}$, $v_{q^0 j}$ are the solutions of problem (2.26) for $q = q^n$ and $q = q^0$ accordingly.

By Theorem 3.3 for each large n, there exists a C^m-diffeomorphism P_n of $\bar{\Omega}_{q^0}$ onto Ω_{q^n}. So we define the function $\tilde{u}^n$ as follows:

$$\tilde{u}^n(x) = u^n(P_n(x)), \quad \tilde{u}^n \in H^m(\Omega_{q^0})^2. \tag{4.7}$$

For an arbitrary $q \in M$, the operator of the double layer potential B_q defined by

$$u \in W_m \quad (B_q u)(x) = \sum_{j=0}^{p} \int_{-\pi}^{\pi} F_q(x, q_j(\tau)) u_j(\tau) D(q_j)(\tau) d\tau, \quad x \in \Omega_q,$$

is a linear continuous mapping from W_m into $H^m(\Omega_q)^2$. So by (4.3) and Theorems 3.3, 4.1, we obtain

$$\tilde{u}^n \to u^0 \ \text{ in } \ H^m(\Omega_{q^0})^2. \tag{4.8}$$

By the replacement of variables corresponding to the C^m-diffeomorphism P_n we have

$$\Psi_2(q^n) = \max_{x \in \bar{\Omega}_{q^0}} \sum_{i,j=1}^{2} \Big[\tilde{\sigma}_{ij}(\tilde{u}^n)(x) - \frac{1}{2} \big(\tilde{\sigma}_{11}(\tilde{u}^n)(x) + \tilde{\sigma}_{22}(\tilde{u}^n)(x) \big) \delta_{ij} \Big]^2 - c_2,$$

$$\tilde{\sigma}_{ij}(\tilde{u}^n)(x) = \sigma_{ij}(u^n)(P_n(x)) \quad x \in \bar{\Omega}_{q^0}. \tag{4.9}$$

Due to (4.8), (4.9), we get $\Psi_2(q^n) \to \Psi_2(q^0)$. ■

THEOREM 4.3 *Let the set M be defined by (2.1) and (2.8), (2.9), (2.12) hold. Let also the functionals Ψ_i $i = 0, 1, 2, 3$ and the set M_∂ be defined by (2.11), (2.13), M_∂ being non-empty. Then there exists a solution of problem (2.14).*

Indeed, by (2.12) and Theorem 4.2 we get that M_∂ is a compact set in M, The functional Ψ_0 is continuous on M, and so there exists a solution of problem (2.14).

For the sensitivity analysis and for the construction of approximate solutions of problem (2.14) it is useful to calculate the derivative of the function $q \to v_q$, where v_q is a solution of problem (4.1). So we consider this problem.

We introduce the notation

$$\lambda(q) = v_q. \tag{4.10}$$

It follows from the proof of Theorem 4.1 that λ is the implicit function defined by $\lambda(q) \in W_m$, $J(q, \lambda(q)) = 0$, $J(q, u)$ is determined by (4.2). By applying (3.17) and representation (3.23)–(3.26), it may be shown that $q \to P(q)$ is a continuously Fréchet-differentiable mapping from M into $\mathcal{L}(W_m, W_m)$. It may also be shown that $q \to \Gamma_q$ is a continuously Fréchet-differentiable mapping from M into $\mathcal{L}(W_m, W_m)$. By $P'(q)$, $\Gamma'(q)$ we denote the derivatives of these mappings at a point q. So by applying the theorem on differentiability of an implicit function, Schwartz (1967), we obtain that λ is a continuously Fréchet-differentiable mapping from M into W_m and its derivative at point $q \in M$ is defined by

$$\lambda'(q)h = -(P(q) + \Gamma(q))^{-1} \left[(P'(q) + \Gamma'(q))h \right] \lambda(q), \tag{4.11}$$

where $h = (h_0, h_1, \ldots, h_p)$, $h_i = (h_{i1}, h_{i2})$, $h_i \in \tilde{C}^{m+1}(-\pi, \pi)^2$, h_0 is equal to zero since the component S_{q0} is fixed (see (2.8)).

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