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Topology optimal design of thermoelastic structures using a homogenization method

by

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This paper describes the development of a computational model for the topology optimization problem of a 2–D thermoelastic solid, with compliance objective function and an isoperimetric constraint on volume. The model is based on the optimal distribution of a material with variable "density", simulated through the introduction of a quasi periodic microstructure characterized by the introduction of small voids at the material microstructure level. The mechanical properties of this material are obtained using a homogenization method. Defining formally the Lagrangian associated with the optimization problem, the optimality conditions are derived. The results of analysis are implemented in a computer code to produce numerical solutions for the optimal topology, considering the temperature distribution dependent on design. The design optimization problem is solved through a sequence of linearized sub–problems. The influence of the temperature on the optimal solution obtained is analyzed in illustratory problems.

1. Introduction

As a subarea of structural optimization, topology optimization distinguishes itself by its design variables, the topological characteristics of the structure. Type of elements in a structure, number of members in a truss or frame, number of joints and number of holes are examples of this class of variables (Bendsøe and Mota Soares, 1993; Bendsøe and Kikuchi, 1993).

Until recently it has been very difficult to obtain a general formulation including such a broad number and type of variables. For example, the existing models did not permit, at least in a consistent way, the continuous variation from material to hole or from a truss element into a beam element.

These problems were overcome through generalization of the topology optimization problem by introduction of a new model based on the optimal distribution of a material with variable "density", simulated by the introduction of a quasi periodic microstructure characterized by the introduction of very small

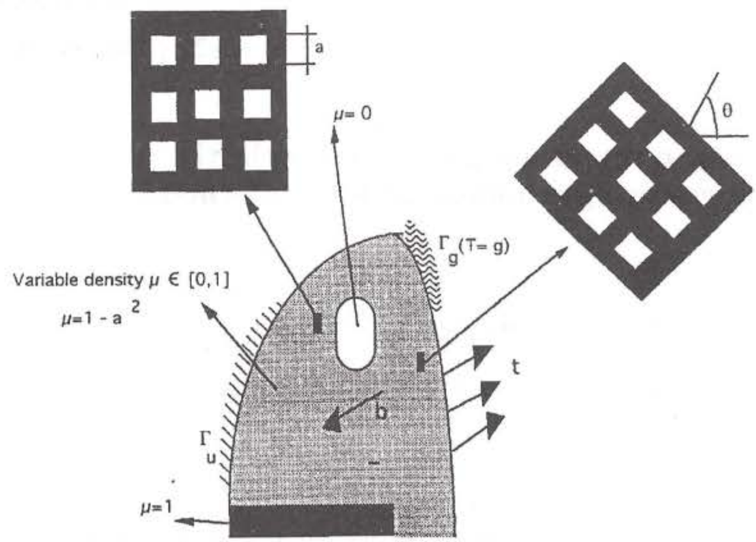

Figure 1. Topology optimization problem. Notation

voids, and formulating it in the common framework of a linear elastic continuum, Bendsøe and Kikuchi (1988).

This paper describes an extension of this model to structures subject to thermal loads, introducing and studying its effects in the topology design. It is an extension of recent works, (e.g., Rodrigues and Fernandes (1992, 1993)), where the simpler situation of prescribed temperature variation was considered, to the case of a design-dependent temperature distribution solution of a steady state heat conduction problem. The equivalent (homogenized) material constants, elasticity, thermal conductivity and thermal expansion, are computed using the homogenization method.

The problem is discretized using a virtual displacement based finite element method and the optimal solution is obtained using a first order augmented Lagrangian algorithm to solve the optimality conditions derived analytically.

Numerical examples are presented.

2. Analytical model

2.1. The topology optimization problem

Consider a structural component, occupying the structural domain Ω, subject to applied body forces $\boldsymbol{b}$, boundary tractions $\boldsymbol{t}$ on Γ_t, given temperature T on Γ_g and flux $\boldsymbol{h}$ on Γ_h (see Figure 1).

To introduce the material based formulation, consider the structural component made of a porous material with variable density μ. This material is simulated by a microstructure obtained by the periodic repetition of small square holes (Figure 1). The optimization goal is then to minimize, with respect to the material density and orientation, the compliance, equivalent to the energy norm of the total displacement, with an isoperimetric constraint on the total volume. Based on the previous description, the topology optimization problem can be stated as

$$\min_{(0\leq\mu(\boldsymbol{x})\leq 1,\ \theta(\boldsymbol{x}))} \int_\Omega b_i u_i d\Omega + \int_\Omega \beta_{ij}^H(\mu,\theta) T e_{ij}(\boldsymbol{x}) d\Omega + \int_{\Gamma_t} t_i u_i d\Gamma, \tag{1}$$

subject to volume constraint

$$\int_\Omega \mu(\boldsymbol{x}) d\Omega \leq \text{vol} \tag{2}$$

$$\int_\Omega \left[E_{ijkl}^H(\mu,\theta) e_{ij}(u) e_{kl}(w) - \beta_{ij}^H(\mu,\theta) e_{ij}(w) T - b_i w_i\right] d\Omega - \int_\Gamma t_i w_i d\Gamma = 0,$$
$$\forall \boldsymbol{w} \text{ admissible} \tag{3}$$

where T is the solution of the heat equation,

$$\int_\Omega k_{ij}^H(\mu,\theta) \frac{\partial T}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j} d\Omega - \int_\Omega f\bar{T} d\Omega - \int_{\Gamma_h} h\bar{T} d\Gamma = 0, \quad \forall \bar{T} = 0 \text{ on } \Gamma_g$$
$$\text{and } T = g \text{ on } \Gamma_g \tag{4}$$

The superscript H in the material coefficients denotes the homogenized material properties defined in the next section.

2.2. Homogenized material properties

For the porous material proposed, obtained by the periodic repetition of an unit cell (see Figure 2), the asymptotic homogenization method, relying on the microstructure local periodicity, is the natural model for the computation of the effective properties, Sanchez–Palencia (1980).

Assuming for the displacement $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ and temperature $T(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ an asymptotic expansion in terms of the cell parameter ε, where $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$, (see Figure 2),

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_0(\boldsymbol{x}) + \varepsilon \boldsymbol{u}_1(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 \boldsymbol{u}_2(\boldsymbol{x},\boldsymbol{y}) + \ldots \tag{5}$$

$$T^\varepsilon(\boldsymbol{x},\boldsymbol{y}) = T_0(\boldsymbol{x}) + \varepsilon T_1(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 T_2(\boldsymbol{x},\boldsymbol{y}) + \ldots \tag{6}$$

the homogenized solutions $\boldsymbol{u}_0(\boldsymbol{x})$ an $T_0(\boldsymbol{x})$, first terms of the asymptotic expansions, are then defined, as the limit $\varepsilon \to 0$, by the equilibrium equations (3) and the steady state heat conduction equation (4) respectively, with the porous periodic material substituted by an "equivalent" homogenized material.

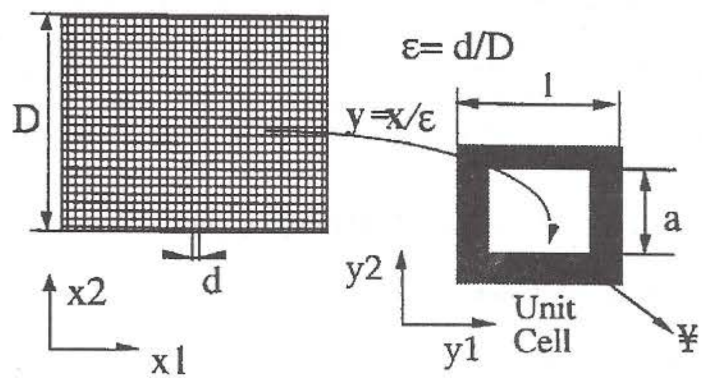

Figure 2. Unit cell

In the case of homogeneous base material, the equivalent homogenized material properties are defined by,

$$E^H_{ijkl} = \mu E_{ijkl} - \int_{¥} \left(E_{ijpm} \frac{\partial X^{kl}_p}{\partial y_m} \right) dy \tag{7}$$

$$k^H_{ij} = \mu k_{ij} - \int_{¥} \left(k_{im} \frac{\partial \Theta^j}{\partial y_m} \right) dy \tag{8}$$

$$\beta^H_{ij} = \mu \beta_{ij} - \int_{¥} \left(\beta_{ml} \frac{\partial X^{ij}_m}{\partial y_l} \right) dy \tag{9}$$

as a function of the density parameter μ (see Figs. 1, 2), where the set of periodic functions $\boldsymbol{X}^{kl}$ and $\boldsymbol{\Theta}^p$ are the solution of the equilibrium equations,

$$\int_{¥} E_{ijpm} \frac{\partial X^{kl}_p}{\partial y_m} \frac{\partial w_i}{\partial y_j} dy = \int_{¥} E_{ijkl} \frac{\partial w_i}{\partial y_j} dy, \quad \boldsymbol{X}^{kl} - \text{Y-Periodic},$$

$$\forall \boldsymbol{w} - \text{Y-Periodic}, \tag{10}$$

$$\int_{¥} k_{ij} \frac{\partial \Theta^p}{\partial y_j} \frac{\partial \bar{T}}{\partial y_i} dy = \int_{¥} k_{ip} \frac{\partial \bar{T}}{\partial y_i} dy, \quad \Theta^p - \text{Y-Periodic}, \forall \bar{T} - \text{Y-Periodic}, \tag{11}$$

defined on ¥ , the unit cell subdomain occupied with material (Figure 2). Equation (11) is solved considering flux equal to zero on the boundary hole. This implies that along the structure change boundary the flux will be zero. The reader is referred to the works by Francfort (1983) and Brahim–Otsmane et al. (1989) for a complete description of the homogenization method as it applies to problems in thermoelasticity.

2.3. Necessary conditions optimal solution

In the former sections the proposed topology optimization problem was formulated. To obtain the respective necessary conditions, let us introduce the augmented Lagrangian $L(\boldsymbol{u},\boldsymbol{v},T,T^*,\mu,\theta,\eta_1,\eta_2,\lambda)$ associated with the problem,

$$
\begin{aligned}
L = \int_\Omega \Big[& b_i u_i + \beta_{ij}^H(\mu,\theta) e_{ij}(\boldsymbol{u}) T + \beta_{ij}^H(\mu,\theta) e_{ij}(\boldsymbol{v}) T - E_{ijkl}^H(\mu,\theta) e_{kl}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) + \\
& + v_i b_i - k_{ij}^H(\mu,\theta) \frac{\partial T}{\partial x_i}\frac{\partial T^*}{\partial x_j} + fT^* + \eta_1(\mu-1) - \eta_2\mu \Big] d\Omega + \\
& + \frac{1}{2\varrho}\left\{ \left[\max(0,\lambda + \varrho\left(\int_\Omega \mu d\Omega - \mathrm{vol}\right)\right]^2 - \lambda^2 \right\} + \\
& + \int_{\Gamma_t} (t_i u_i + t_i v_i) d\Gamma + \int_{\Gamma_h} hT^* d\Gamma
\end{aligned}
\tag{12}
$$

where the Lagrange multipliers $\boldsymbol{v}, T^*, \eta_1, \eta_2$ and λ are constrained by the set of inequalities and equalities,

$$
\begin{array}{ll}
\eta_1(\boldsymbol{x}) \geq 0 & \forall \boldsymbol{x} \in \Omega \\
\eta_2(\boldsymbol{x}) \geq 0 & \forall \boldsymbol{x} \in \Omega \\
\lambda \geq 0 & \lambda \in \mathbb{R} \\
\boldsymbol{v} = 0 & \text{on } \Gamma_u \\
T^* = 0 & \text{on } \Gamma_g
\end{array}
$$

and $\varrho > 0$ is the penalty factor. We should note here that the adjoint variables $\boldsymbol{v}$ and T^* are introduced in the problem formulation in a natural way, as Lagrange multipliers associated with the equilibrium constraints (3-4). From the first variation of the augmented Lagrangian, the Karush–Kuhn–Tucker necessary conditions provide:

From stationarity with respect to displacement $\boldsymbol{u}$ and temperature T, the equilibrium equations characterizing the adjoint variables $\boldsymbol{v}$ and T^* are

$$
\begin{aligned}
& \int_\Omega [E_{ijkl}^H(\mu,\theta) e_{ij}(\boldsymbol{v}) e_{kl}(\delta\boldsymbol{u}) - \beta_{ij}^H e_{ij}(\delta\boldsymbol{u}) T - b_i \delta u_i] d\Omega - \\
& \qquad - \int_\Gamma t_i \delta u_i d\Gamma = 0, \qquad \forall \delta\boldsymbol{u} \ \text{ admissible}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
& \int_\Omega 2\beta_{ij}^H e_{ij}(\boldsymbol{u}) \delta T - k_{ij}^H \frac{\partial \delta T}{\partial x_i}\frac{\partial T^*}{\partial x_j} d\Omega = 0, \\
& \qquad \forall \delta T = 0 \text{ on } \Gamma_g \text{ and } T^* = 0 \text{ on } \Gamma_g
\end{aligned}
\tag{14}
$$

Comparing eqs. (13) and (3) we can conclude that $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})$.

Stationarity w.r.t. the design variables $\mu(\boldsymbol{x})$ and $\theta(\boldsymbol{x})$ provides the optimality conditions

$$2\frac{\partial \beta_{ij}^H}{\partial \mu} e_{ij}(\boldsymbol{u})T - \frac{\partial E_{ijkl}^H}{\partial \mu} e_{kl}(\boldsymbol{u}) e_{ij}(\boldsymbol{u}) - \frac{\partial k_{ij}^H}{\partial \mu}\frac{\partial T}{\partial x_i}\frac{\partial T^*}{\partial x_j} + \\ + \left[\max(0, \lambda + \varrho\left(\int_\Omega \mu d\Omega - \text{vol}\right)\right]) + \eta_1 - \eta_2 = 0, \qquad \forall \boldsymbol{x} \in \Omega \quad (15)$$

$$2\frac{\partial \beta_{ij}^H}{\partial \theta} e_{ij}(\boldsymbol{u})T - \frac{\partial E_{ijkl}^H}{\partial \theta} e_{kl}(\boldsymbol{u}) e_{ij}(\boldsymbol{u}) - \frac{\partial k_{ij}^H}{\partial \theta}\frac{\partial T}{\partial x_i}\frac{\partial T^*}{\partial x_j} = 0, \quad \forall \boldsymbol{x} \in \Omega \quad (16)$$

From stationarity w.r.t. the Lagrange multipliers η_1, η_2 and λ we obtain, respectively,

$$\begin{aligned} &\eta_1 \geq 0, \quad \mu - 1 \leq 0, \quad \eta_1(\mu - 1) = 0, \quad \forall \boldsymbol{x} \in \Omega, \\ &\eta_2 \geq 0, \quad \mu \geq 0, \qquad\quad \eta_2\mu = 0, \qquad\quad\ \forall \boldsymbol{x} \in \Omega \end{aligned} \quad (17)$$

and

$$\lambda = \max\left\{0, \lambda + \varrho\left(\int_\Omega \mu d\Omega - \text{vol}\right)\right\}. \quad (18)$$

These optimality conditions together with the equilibrium equations (3-4) provide not only a tool for the characterization of the optimal solution but also, by a recursive procedure, an iterative solution method.

3. Computational model

For the topology optimization with square holes, the dependence of the homogenized material coefficients on the density μ is computed for a number of discrete values of μ and then approximated by a polynomial interpolation on the whole interval $[0, 1]$.

The homogenized coefficients at the interpolation points are estimated by the program PREMAT, Guedes and Kikuchi (1990). This program uses adaptive finite element methods to interpolate the displacement functions $\boldsymbol{X}^{kl}$ and Θ^p.

Using the finite element method the discrete version of the problems (3-4, 13-14) is obtained through an interpretation of the domain of the problem by a finite number of finite elements, such that $\Omega = \cup_{e=1,n^o \text{ elements}} \Omega_e$, and using continuous polynomial approximations for the displacement variable within each finite element.

Once $E_{ijkl}^H(\mu, \theta)$, $k_{ij}^H(\mu, \theta)$ and $\beta_{ij}^H(\mu, \theta)$, the homogenized material properties interpolations, and the finite element solutions $\boldsymbol{u}^h$, T^h and T^{*h} are known, we are in position to solve the optimality conditions (15-16).

Using a constant interpolation of the design variables $\mu(x)$ and $\theta(x)$, in each finite element, and defining its values in the eth element by μ_e and θ_e the optimality conditions (15-16) are then defined for each finite element by,

$$2\frac{\partial\beta_{ij}^H}{\partial\mu_e}e_{ij}(\boldsymbol{u}^h)T - \frac{\partial E_{ijkl}^H}{\partial\mu_e}e_{kl}(\boldsymbol{u}^h)e_{ij}(\boldsymbol{u}^h) - \frac{\partial k_{ij}^H}{\partial\mu_e}\frac{\partial T^h}{\partial x_i}\frac{\partial T^{*h}}{\partial x_j} + \\ + \left[\max\left(0, \lambda + \varrho\left(\int_\Omega \mu d\Omega - \text{vol}\right)\right)\right] + \eta_1 - \eta_2 = 0, \tag{19}$$

$$2\frac{\partial\beta_{ij}^H}{\partial\theta_e}e_{ij}(\boldsymbol{u}^h)\Delta T - \frac{\partial E_{ijkl}^H}{\partial\theta_e}e_{kl}(\boldsymbol{u}^h)e_{ij}(\boldsymbol{u}^h) - \frac{\partial k_{ij}^H}{\partial\theta_e}\frac{\partial T}{\partial x_i}\frac{\partial T^{*h}}{\partial x_j} = 0. \tag{20}$$

Based on these optimality conditions, the design variables μ_e are updated iteratively using the first order augmented Lagrangian method,

$$\mu_{k+1} = \begin{cases} \max\{(1-\zeta)\mu_k; 0\} & \text{if } \mu_k + \eta D_k \le \max\{(1-\zeta)\mu_k; 0\} \\ \mu_k + \eta D_k & \text{if } \max\{(1-\zeta)\mu_k; 0\} \le \mu_k + \eta D_k \le \\ & \quad \le \min\{(1+\zeta)\mu_k; 1\} \\ \min\{(1+\zeta)\mu_k; 1\} & \text{if } \min\{(1+\zeta)\mu_k; 1\} \le \mu_k + \eta D_k \end{cases} \tag{21}$$

where the k th iteration descent direction vector D_k is defined as,

$$D_k = \frac{\partial E_{ijkl}^H}{\partial\mu_e}e_{ij}(\boldsymbol{u}^h)e_{kl}(\boldsymbol{u}^h) - 2\frac{\partial\beta_{ij}^H}{\partial\mu_e}e_{ij}(\boldsymbol{u}^h)T + \frac{\partial k_{ij}^H}{\partial\mu_e}\frac{\partial T^h}{\partial x_i}\frac{\partial T^{*h}}{\partial x_j} - \\ - \left[\max\left(0, \lambda_k + \varrho\left(\int_\Omega \mu d\Omega - \text{vol}\right)\right)\right].$$

In each iteration the Lagrange multiplier λ is updated by the projection formula, obtained directly from the optimality condition (18),

$$\lambda_{k+1} = \max\left\{0, \lambda_k + \varrho\left(\int_\Omega \mu d\Omega - \text{vol}\right)\right\}$$

The material optimal orientation θ is obtained by the direct solution of optimal equation (20) (see, e.g., Pedersen, 1989, 1990; Susuki, 1991).

4. Example

Example A. In this example consider the initial design shown in Fig. 3a. The dimensions are $72 \times 47.7 \times 1$ cm, the applied load has the value of 840 kg/cm^2 and the right and left sides are supported as shown in the figure. The volume constraint equals 30% of the total volume. The problem was solved using a 60×30 9–node isoparametric finite element mesh.

Assuming as design variables the density μ and the orientation θ, Figure 3.b shows the optimal topology obtained without temperature variation. Figures 3. c-f show the result obtained for constant and fixed temperature increments of $T = 4^0$ and 10^0.

EXAMPLE B. In this example let us consider the initial design shown in Fig. 4a. The dimensions are $72 \times 47.7 \times 1$ cm^3, the applied force has the value of 1000kg. The volume constraint equals 40% of the total volume, 1340 cm^3. The problem was solved using a 60×30, 9 node finite elements mesh. Assuming only density μ as design variable, Fig. 4.b shows the optimal topology obtained without temperature variation and Fig. 4.c the result obtained assuming the temperature distribution fixed and $T = 4^0$.

Assuming now the temperature solution of the heat conduction problem (4), with the structure subjected to $T = 4^0$ on the left side boundary, $T = 0^0$ on the right side boundary and flux equal to zero on the remaining boundaries, Fig. 4.d shows the material distribution obtained and Figs. 4.d-e the respective temperature and flux distribution. From this solution we can observe a tendency to concentrate material in the region with lower temperature, so minimizing the contribution of the thermal loads to the total compliance, and also, as the temperature varies across the domain, a good agreement with the results obtained with constant temperature distributions of $T = 0^0$ and $T = 4^0$ (Figs. 4 b-c).

5. Concluding remarks

The development presented in this work extends the generalized topology optimization model to include temperature variation effects. Following the approach outlined in this report, the associated optimality conditions are easily obtained without resorting to highly elaborate mathematical developments.

The problem is stated as a material distribution problem, considering the temperature dependent of design and is solved by finite element modeling and mathematical programming.

In the numerical examples a strong dependence of the optimal topology on temperature distribution is observed. However, the homogenization model used to compute the effective thermal conductivity constants limits the optimization problem to cases of flux equal to zero along the structure design boundary, so that further work is required to consider other types of thermal boundary conditions.

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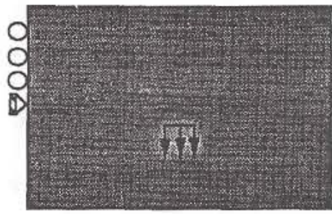
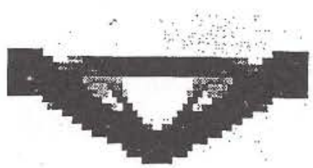
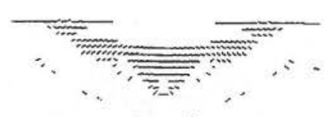


a) Initial Design.

b) Without temperature.
Final Vol. = 999.5 cm^3
Initial Compliance = 1116. Kg cm
Final Compliance=56.65 Kg cm



c) $T = 4^0$. Final Vol. = 992.6 cm^3
Initial Compliance = 1131. Kg cm
Final Compliance=96.70 Kg cm

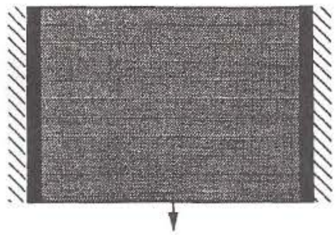


d) $T = 4^0$. Orientation

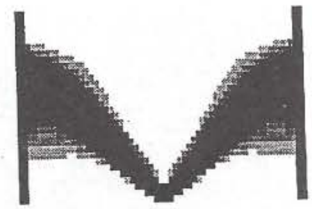
e) $T = 10^0$. Final Vol. = 879 cm^3
Initial Compliance = 1174. Kg cm
Final Compliance=162.3 Kg cm

f) $T = 10^0$. Orientation

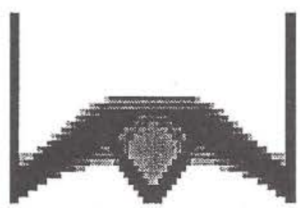
Figure 3. Example A



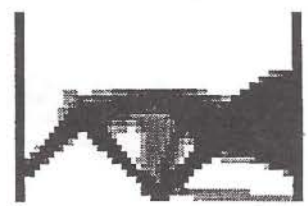
a) Initial Design.



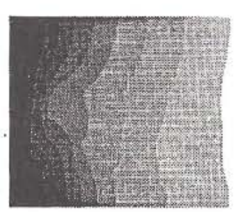
b) Without temperature.
Final Vol. = 1319 cm^3
Initial Compliance = 50 Kg cm
Compliance = 3.4 Kg cm



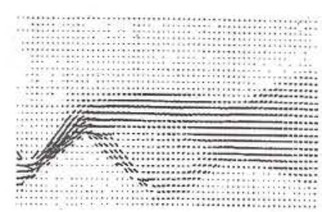
c) Optimal solution for temperature given and constant, $T = 4^0$.



d) Temp. $T = 4^0$ on left boundary - $T = 4^0$ on right.
Final Volume = 1319 cm^3
Initial Compliance = 53.6 Kg cm
Final Compliance = 6.15 Kg cm



e) Temperature distribution at the optimal solution



f) Flux distribution at the optimal design

Figure 4. Example B

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