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On an optimization problem for Kirchhoff plate

by

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The paper concerns the optimization problem of an elastic plate governed by the Kirchhoff equation. It is assumed that the plate is simply supported on a part of its boundary and free on the remaining part. The static case with a random loading is considered. The first order necessary conditions of optimality are derived. Results of computations obtained for deterministic and random cases are compared.

1. Introduction

Deterministic optimization problems for the Kirchhoff plate are considered e.g. in Myśliński, Sokołowski (1985). Some related results on the existence of an optimal solution and the necessary optimality conditions (for random loadings) are given in Gątarek, Sokołowski (1988) and in Myśliński, Sokołowski (1985) in the deterministic case. The finite element method, Ciarlet (1978), Strang, Fix (1973) is applied to obtain the finite dimensional approximations of the problem under consideration. Standard numerical methods of optimization are used, Findeisen, Szymanowski, Wierzbicki (1980).

2. Control problems with loading as a random parameter

Let $\mathcal{O} \subset \mathbf{R}^2$ be the domain occupied by the plate and $\partial\mathcal{O}=\overline{\Gamma}_1\cup\overline{\Gamma}_2$, where $\Gamma_1\cap\Gamma_2 = \emptyset$. Let $W = \{v \in H^1(\mathcal{O})|\ v = 0 \text{ on } \Gamma_1\}$ and $V = L^2(\Omega; H^2(\mathcal{O})\cap W)$. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probabilistic space, where Ω is a discrete set of ω_i, $i \in \mathbf{N}$, $\mathcal{F}$ is a σ–algebra spanned on Ω, $\mathrm{P}:\mathcal{F} \to \langle\ 0,\ 1\ \rangle$ is a probabilistic measure on Ω, $\mathrm{P}(\{\omega_i\}) = p_i$.

In the case of a transversal force $f(\omega; x_1, x_2), (x_1, x_2) \in \mathcal{O}, \omega \in \Omega$, acting on the plate, the state equation for the Kirchhoff plate model is the same as in the deterministic case. It can be written as follows, Myśliński, Sokołowski (1985):

$$\sum_{i,j,k,l=1}^{2} \frac{\partial^2}{\partial x_j \partial x_i}\left(D_{ijkl}(x_1,x_2)\frac{\partial^2 w(\omega;x_1,x_2)}{\partial x_k\,\partial x_l}\right) = f(\omega;x_1,x_2) \tag{2.1}$$

$$\text{for a.e. } (x_1,x_2)\in\mathcal{O},\ \text{a.s. in } \Omega,$$

where $\mathbf{D} = (D_{ijkl}),\ i,j,k,l = 1,2$ is a tensor of plate stiffness, $w : \Omega\times\mathcal{O}\to\mathbf{R}$ is displacement, $w\in V$, $f(\omega;x_1,x_2)$ is given.

There exists tensor $\mathbf{b} = (b_{ijkl}),\ i,j,k,l = 1,2$ such that:

$$D_{ijkl}(x_1,x_2) = h^3(x_1,x_2)b_{ijkl},\quad i,j,k,l = 1,2 \tag{2.2}$$

where $h : \mathcal{O}\to\mathbf{R}$ denotes plate thickness, $h\in L^\infty(\mathcal{O})$. We assume that the tensor $\mathbf{b}$ is symmetric, i.e.

$$b_{ijkl} = b_{jikl} = b_{klij},\quad i,j,k,l = 1,2 \tag{2.3}$$

The following boundary conditions satisfied almost surely, for a simply supported plate on Γ_1 and free on Γ_2, Myśliński, Sokołowski (1985), are prescribed:

$$w(\omega) = 0 \quad\text{on}\quad \Gamma_1 \quad \text{a.s. in } \Omega, \qquad M_n(\omega) = 0 \quad\text{on}\quad \Gamma_1 \quad \text{a.s. in } \Omega, \tag{2.4}$$

$$M_n(\omega) = 0 \quad\text{on}\quad \Gamma_2 \quad \text{a.s. in } \Omega, \qquad M_n'(\omega) = 0 \quad\text{on}\quad \Gamma_2 \text{ a.s. in } \Omega, \tag{2.5}$$

where M_n is the so–called bending moment. M_n and M_n' can be expressed by the following formulae

$$\begin{aligned} M_n \ &= \ h^3\left[\nu\left(\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}\right)+(1-\nu)\left(\frac{\partial^2 w}{\partial x_1^2}n_1^2 + 2\frac{\partial^2 w}{\partial x_1\partial x_2}n_1 n_2\right.\right. \\ &\left.\left. +\frac{\partial^2 w}{\partial x_2^2}n_2^2\right)\right], \end{aligned} \tag{2.6}$$

$$M_n' = \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(h^3\left(\frac{\partial^2 w}{\partial x_i^2}n_i + \nu\frac{\partial^2 w}{\partial x_j^2}n_i + (1-\nu)\frac{\partial^2 w}{\partial x_i\partial x_j}n_j\right)\right), \tag{2.7}$$

where ν is Poisson ratio which characterizes plate material, $\nu\in(0,0.5)$; $j\in\{1,2\}$, $j\neq i$ and $n = (n_1,n_2)$ is a unit normal vector on $\partial\mathcal{O}$.

Let $a(\omega;\cdot,\cdot) : V\times V\to\mathbf{R}$ be the bilinear form defined for the equation (2.1):

$$a(\omega;w,\phi) = \int_{\mathcal{O}}\sum_{i,j,k,l=1}^{2} D_{ijkl}\frac{\partial^2 w}{\partial x_k\partial x_l}\frac{\partial^2\phi}{\partial x_i\partial x_j}dx_1 dx_2 \tag{2.8}$$

and $F(\omega)(\cdot) : L^2(\Omega; (H^2(\mathcal{O})\cap W)') \rightarrow \mathbf{R}$ be the functional defined by f in standard way. Then the equation (2.1) can be rewritten in the variational form:

$$a(\omega; w, \phi) = F(\omega)(\phi) \quad \forall \phi \in V \quad \text{a.s. in } \Omega \tag{2.9}$$

If $\omega = \omega_i$, for i fixed, the deterministic equation of the Kirchhoff plate is obtained. Therefore:

(a) the equation (2.1) with the boundary conditions (2.4)–(2.5) is equivalent to (2.9),

(b) the solution $w(\omega) \in H^2(\mathcal{O})\cap W$ is unique.

The above results (a) and (b) hold as well as for ω, a random variable.

The following control problem is considered:

$$\inf_{h \in \mathcal{U}_{ad}} J(h), \tag{2.10}$$

where J is defined by

$$J(h) = E\left[\int_{\mathcal{O}} w^2(\omega, h, x_1, x_2) dx_1 dx_2\right], \tag{2.11}$$

w is a solution to (2.1)–(2.5) for a given control h, $E[\cdot]$ denotes the mean value, $\mathcal{U}_{ad}$ is the set of admissible controls,

$$\begin{aligned} \mathcal{U}_{ad} = & \{h \in L^\infty(\mathcal{O}) \cap H^s(\mathcal{O}) | h_{min} \leq h(x_1, x_2) \leq h_{max} \text{ a.e. on } \mathcal{O}, \\ & 0 < h_{min} < h_{max}, \int_{\mathcal{O}} h(x_1, x_2) dx_1 dx_2 = c, \|h\|_{H^s(\mathcal{O})} \leq M\}, \end{aligned} \tag{2.12}$$

h_{min}, h_{max}, c, M, $s > 0$ are given constants. As a particular case, for Ω a finite set, we have:

$$J(h) = \sum_{i=1}^{N} (p_i \int_{\mathcal{O}} w^2(\omega_i, h, x_1, x_2) dx_1 dx_2). \tag{2.13}$$

3. The existence of optimal solutions and the necessary optimality conditions

LEMMA 1 *For any $s > 0$, there exists a solution $\tilde{h} \in \mathcal{U}_{ad}$ to (2.10).*

The necessary optimality conditions for the problem under consideration can be formulated as follows, Céa (1971), since

$$J(h_\star + t(v - h_\star)) \geq J(h_\star) \quad \forall t \in (0, 1) \quad \forall v \in \mathcal{U}_{ad} \tag{3.14}$$

for any local solution $h_\star$ to (2.10), thus

$$dJ(h_\star, v - h_\star) \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \tag{3.15}$$

where $dJ(h_\star, v)$ denotes the directional derivative of J at $h_\star$ in the direction v. From (2.11) we obtain

$$dJ(h_\star, v) = E\left[\int_{\mathcal{O}} 2w(\omega, h_\star, x_1, x_2)\frac{\delta w}{\delta h}(\omega, h_\star, x_1, x_2, v)dx_1 dx_2\right], \tag{3.16}$$

where $\frac{\delta w}{\delta h}$ denotes the derivative of $w(\omega, h)$ at $h_\star$ in the direction v.

It is easy to show, applying the implicit function theorem, Maurin (1976), that:

$$dJ(h_\star, v) = -E\left[\sum_{i,j,k,l=1}^{2}\int_{\mathcal{O}}(3h_\star^2 vb)_{ijkl}\frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial^2 p}{\partial x_k \partial x_l}dx_1 dx_2\right]. \tag{3.17}$$

where p is a solution to the adjoint state equation, $p \in V$:

$$\sum_{i,j,k,l=1}^{2}\int_{\mathcal{O}} D^\star_{ijkl}\frac{\partial^2 p}{\partial x_i \partial x_j}\frac{\partial^2 \phi}{\partial x_k \partial x_l}dx_1 dx_2 = 2\int_{\mathcal{O}} w\phi dx_1 dx_2 \quad \forall \phi \in V. \tag{3.18}$$

4. Numerical examples

Let J_Δ denotes an approximation of J where Δ is a parameter of approximation. A pointwise transversal force acting at $n(\Delta)$ points on rectangular plate is considered and it is assumed that

$$\Gamma_1 = \{(x_1, d) | 0 < x_1 \leq b\} \cup \{(b, x_2) | 0 \leq x_2 < d\}, \tag{4.19}$$

where b, d are given. Computations are performed with the following values of parameters:

$\nu = 0.3,\ h_{min} = 0.8,\ h_{max} = 1.2,\ c = 0.25,\ b = d = 0.5.$

It is assumed that the plate is divided into 16 rectangles by partition of every side into 4 equal parts. Moreover initial values $h_1 = ... = h_{n(\Delta)} = 1$ are prescribed.

Computations are performed for the following cases:

1) for the deterministic loading $f_r = 1,\ r = 1, ..., n(\Delta)$;
2) for the deterministic loading $f_r = 1$ for all indexes r such that (z_1, z_2) is a node point, except the point (0.125,0.375) where $f_r = 50$;
3) for the deterministic loading $f_r = 1$ for all indexes r such that (z_1, z_2) is a node point, except the points (0.125,0.375) and (0.375,0.125) where $f_r = 50$;
4) for the random loading:
   - ω_1 takes place with probability $p_1 = 0.75$ and $f_r(\omega_1) = 1$ for every node different from (0.125,0.375), where $f_r(\omega_1) = 50$;

x_1 / x_2	0.000	0.125	0.250	0.375	0.500
0.500	1.01	1.01	1.06	1.11	1.07
0.375	1.00	1.00	1.09	1.19	1.11
0.250	0.95	0.93	1.01	1.09	1.06
0.125	0.88	0.82	0.93	1.00	1.01
0.000	0.92	0.88	0.95	1.00	1.01

Table 1. Plate thickness after optimization: *Case 1.* Value of J_Δ at the starting point: $J_\Delta(\mathbf{h_0}) = 1.825 \cdot 10^{-2}$. Value of J_Δ after optimization: $J_\Delta(\mathbf{h}^*) = 1.619 \cdot 10^{-2}$.

x_1 / x_2	0.000	0.125	0.250	0.375	0.500
0.500	1.11	1.10	1.13	1.13	1.16
0.375	1.10	1.00	1.01	1.02	1.12
0.250	1.07	0.95	0.96	0.81	1.09
0.125	0.94	0.90	0.92	0.94	1.07
0.000	0.93	0.92	1.00	1.06	1.04

Table 2. Plate thickness after optimization: *Case 2.* Value of J_Δ at the starting point: $J_\Delta(\mathbf{h_0}) = 2.153 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(\mathbf{h}^*) = 1.940 \cdot 10^{-1}$.

- ω_2 takes place with probability $p_2 = 0.25$ and $f_r(\omega_2) = 1$ for every node different from (0.375,0.125), where $f_r(\omega_2) = 50$.

5) for the random loading:
   - ω_1 takes place with probability $p_1 = 0.75$ and $f_r(\omega_1) = 1$ for every node different from (0.125,0.125), where $f_r(\omega_1) = 50$;
   - ω_2 takes place with probability $p_2 = 0.25$ and $f_r(\omega_2) = 1$ for every node different from (0.375,0.375), where $f_r(\omega_2) = 50$.

6) for the random loading:
   - ω_1 takes place with probability $p_1 = 0.75$ and $f_r(\omega_1) = 1$ for every node different from (0.375,0.375), where $f_r(\omega_1) = 50$;
   - ω_2 takes place with probability $p_2 = 0.25$ and $f_r(\omega_2) = 1$ for every node different from (0.125,0.125), where $f_r(\omega_2) = 50$.

The results of computations are presented in Tables 1-6.

In *Cases 1, 3, 5, 6* the optimal plate thickness is symmetric because of the symmetric loading. In each *Case* significant concentration of material along simply supported part of boundary (Γ_1) is observed.

x_1 / x_2	0.000	0.125	0.250	0.375	0.500
0.500	1.05	1.11	1.06	1.09	1.17
0.375	1.08	1.00	1.07	1.14	1.09
0.250	1.00	0.90	0.94	1.07	1.06
0.125	0.87	0.83	0.90	1.00	1.11
0.000	0.90	0.87	1.00	1.08	1.05

Table 3. Plate thickness after optimization: *Case 3.* Value of J_Δ at the starting point: $J_\Delta(\mathbf{h_0}) = 6.271 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(\mathbf{h}^*) = 5.070 \cdot 10^{-1}$.

x_1 / x_2	0.000	0.125	0.250	0.375	0.500
0.500	1.10	1.08	1.14	1.15	1.19
0.375	1.10	1.00	1.03	0.95	1.16
0.250	1.06	0.94	0.94	1.01	1.11
0.125	0.94	0.90	0.95	0.81	1.09
0.000	0.93	0.93	1.04	1.08	1.06

Table 4. Plate thickness after optimization: *Case 4.* Value of J_Δ at the starting point: $J_\Delta(\mathbf{h_0}) = 2.153 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(\mathbf{h}^*) = 1.869 \cdot 10^{-1}$.

x_1 / x_2	0.000	0.125	0.250	0.375	0.500
0.500	1.13	1.07	1.09	1.10	1.15
0.375	1.07	0.94	0.96	0.98	1.10
0.250	1.06	0.92	0.94	0.96	1.09
0.125	1.04	0.90	0.92	0.94	1.07
0.000	0.96	1.04	1.06	1.07	1.13

Table 5. Plate thickness after optimization: *Case 5.* Value of J_Δ at the starting point: $J_\Delta(\mathbf{h_0}) = 8.360 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(\mathbf{h}^*) = 8.070 \cdot 10^{-1}$.

x_1 / x_2	0.000	0.125	0.250	0.375	0.500
0.500	1.14	1.07	1.12	1.19	1.15
0.375	1.05	0.93	1.03	1.13	1.19
0.250	1.03	0.88	0.98	1.03	1.12
0.125	0.95	0.82	0.88	0.93	1.07
0.000	0.89	0.95	1.03	1.05	1.14

Table 6. Plate thickness after optimization: *Case 6.* Value of J_Δ at the starting point: $J_\Delta(\mathbf{h_0}) = 3.209 \cdot 10^{-1}$. Value of J_Δ after optimization: $J_\Delta(\mathbf{h}^*) = 2.693 \cdot 10^{-1}$.

In *Case 2* significant concentration of plate material takes place around the node with coordinates (0.125,0.375) because of incremented loading value at this point.

In *Case 3* additional regions of concentration of material (except Γ_1) appear around points (0.125,0.375) and (0.375,0.125).

The optimal plate thickness in *Case 4* (random loading) differs from the results of *Case 1,2,3*. An additional amount of material around point (0.125,0.375) appears compared to the *Case 1*. The optimal plate thickness is not symmetric, as in *Case 3*, although loading values at points (0.125,0.375) and (0.375,0.125) are the same. It is caused by different probabilities of loading values at the points.

Case 5 and *Case 6* differ because of different probabilities for loading values at points (0.125,0.125) and (0.375,0.375).

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References

Björck A., Dahlquist G. (1979) *Numerische Methoden*, R. Oldenbourg Verlag, München.

Céa J. (1971) *Optimisation: Théorie et Algorithmes*, Dunod, Paris.

Ciarlet P. G. (1978) *The Finite Element Method for Elliptic Problems*, North- Holland, Amsterdam–New York–Oxford.

Findeisen W., Szymanowski J., Wierzbicki A. (1980) *Theory and Numerical Methods for Optimization*, Polish Scientific Publishers Warsaw, (in Polish).

Gątarek D., Sokołowski J. (1988) Parametric Optimization of Distributed Parameter Systems with Random Inputs, Systems Research Institute of the Polish Academy of Sciences, Technical Report 6/88, Warsaw (in Polish).

Maurin K. (1976) *Analysis. Part I. Elements*, D. Reidel Publishing Co., Dordrecht–Boston.

Myśliński A., Sokołowski J. (1985) Nondifferentiable optimization problems for eliptic systems, *SIAM Journal on Control and Optimization*, **23**, 632–648.

Sokołowski J. (1979) Convergence of solutions of partial differential equations of eliptic type in dependence on sequences of coefficients, Systems Research Institute of the Polish Academy of Sciences, Warsaw (in Polish).

Strang G., Fix O. J. (1973) *Analysis of the Finite Element Method*, Prentice–Hall, New Jersey.