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Sensitivity in mathematical programming: a review

by

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In this paper we present basic results concerning first order sensitivity analysis of parametric mathematical programming problems. The idea behind this work is to present guidelines to the subject rather than to go into details.

1. Introduction

The aim of this review is to present basic problems, results and bibliographical notes for sensitivity analysis in mathematical programming problems. We consider the problem

$$\begin{aligned}&\inf\ \mathcal{J}(x)\\ &\text{subject to}\quad x\in\Omega;\end{aligned}\tag{1}$$

where Ω is a subset of the Euclidean space R^m. We refer to this problem as an original or an unperturbed one.

In the sequel we consider mainly problems where

$$\Omega=\{x\in R^m|\ \psi^i(x)=0,\ i\in J\ \varphi^i(x)\le 0\quad\text{for}\quad i\in I\}$$

and the functions $\psi^i,\varphi^i: R^n\to R$ are continuously differentiable.

Changes in data are taken into account by introducing a parameter u in the functions $\mathcal{J}$, ψ^i, φ^i.

The resulting parametric optimization problem is of the form

$$\begin{aligned}&\inf\ \mathcal{J}(u,x)\\ &\text{subject to}\quad x\in\Omega(u)\\ &\qquad\qquad\quad\ u\in\omega\,.\end{aligned}\tag{2}$$

In what follows we assume that

$$\Omega(u) = \{x \in R^m | \ \psi^i(u,x) = 0, \ i \in J \ \varphi^i(u,x) \leq 0 \ \ i \in I\},$$

$u \in R^n$, $x \in R^m$, $\mathcal{J} : R^n \times R^m \to R$ is continuously differentiable, $\Omega : R^n \rightrightarrows R^m$ is a closed–valued multifunction, and the functions $\psi^i, \varphi^i : R^n \times R^m \to R$ are continuously differentiable.

The *marginal function* $p : R^n \to R$ is defined as

$$p(u) = \inf_{x \in \Omega(u)} \mathcal{J}(u,x).$$

For problem (2) in general spaces the differential properties of the marginal function have been investigated by, e.g., Borisenko and Minchenko (1983), Minchenko (1984,1986), Hiriart–Urruty (1978), Rubinow (1985), Outrata (1990).

Let us fix $u_0 \in \omega$, and let us denote by $\mathcal{M}$ the multivalued mapping which assigns to any $u \in \omega$ the set of solutions to the problem

$$\begin{aligned} &\inf \ \mathcal{J}(u,x) \\ &\text{subject to} \ \ \psi^i(u,x) = 0, \ i \in J \\ &\qquad\qquad\quad \varphi^i(u,x) \leq 0, \ i \in I, \end{aligned} \tag{3}$$

$$\begin{aligned} \mathcal{M}(u) &= \operatorname{argmin}\{\mathcal{J}(u,x) : \ x \in \Omega(u)\}, \\ \mathcal{M}(u_0) &:= M_0, \ \ p(u_0) = p_0, \ \ \Omega(u_0) = \Omega. \end{aligned}$$

In particular, we are interested in results concerning differentiability of the optimal value function and the solutions to problem (3). There exist numerous papers and books concerning these problems e.g. Bank et al. (1982), Fiacco (1983), Levitin (1992).

It was observed by Rockafellar (1984) that parametric problem (3) is equivalent to the problem with perturbations appearing linearly in the r.h.s. only,

$$\begin{aligned} &\inf \ \mathcal{J}(v,x) \\ &\text{subject to} \ \ \psi^i(v,x) = 0, \ i \in J \\ &\qquad\qquad\quad \varphi^i(v,x) \leq 0, \ i \in I \\ &\qquad\qquad\quad v^i = u^i, \ i = 1,2,\ldots,n \end{aligned} \tag{4}$$

If we let (μ, λ, ν), $\mu \in R^p$, $\lambda \in R^s$, $\nu \in R^m$ to be multipliers corresponding to the constraints $\psi^i(v,x)$, $i \in J$, $\varphi^i(v,x)$, $i \in I$, $v^i = u^i$, $i = 1,2,\ldots,n$, respectively, then the Lagrangian for problem (4) is of the form

$$\mathcal{L}(x,v,\mu,\lambda,\nu) = \mathcal{L}(x,v,\mu,\lambda) + \nu^T(v-u),$$

where

$$\mathcal{L}(x,u,\mu,\lambda) = \mathcal{J}(u,x) + \sum_{i=1}^{p} \mu_i \psi^i(u,x) + \sum_{i=1}^{s} \lambda^i \varphi^i(u,x).$$

is the Lagrangian for problem (3).

If $\bar{x}$ is an optimal solution to problem (3) for the parameter value u_0, then $(\bar{x}, v_0)$ is optimal to problem (4) with the first order optimality conditions

$$\nabla_x \mathcal{L}(\bar{x}, v_0, \mu, \lambda) = 0\,,$$
$$\nabla_y \mathcal{L}(\bar{x}, v_0, \mu, \lambda) + (\nu)^T = 0\,.$$

Because of this transformation many results are stated only for the parametric problem of the form

$$\begin{array}{ll} \inf\ \mathcal{J}(x) & \\ \text{subject to} & \psi^i(x) \doteq u^i\,,\ i = 1, 2, ..., p \\ & \varphi^i(x) \leq u^{p+i}\,,\ i = 1, 2, ..., s \end{array} \tag{5}$$

2. Basic problems of sensitivity theory

PROBLEM 1 (consistency of the constraint sets $\Omega(u)$ for $u \in \omega$, and small $\|u - u_0\|$) *When consistency of the constraint set $\Omega(u_0)$ for the original problem implies consistency of the sets $\Omega(u)$ for $u \in \omega$ and $\|u - u_0\|$ sufficiently small?*

When for a given $x_0 \in \Omega(u_0)$ one can find positive constants r_0, ρ_0, C_0 (depending upon u_0, and x_0) such that for all $u \in B(u_0, r_0)$ the sets $\Omega(u)$ are nonempty and

$$dist(x, \Omega(u))) \leq C_0 \Delta(u, x) \quad for \quad x \in B(x_0, \rho_0) \tag{$*$}$$

where $\Delta(u, x) = \Sigma_{i \in J} \psi^i_+(u, x) + \Sigma_{i \in I} |\varphi_i(u, x)|$ measures the violation of the constraints $x \in \Omega(u)$?

According to Levitin (1992) the system

$$\psi^i(u_0, x) = 0\,,\ i \in J\,,\ \varphi^i(u_0, x) \leq 0\,,\ i \in I$$

is *normal* with respect to a given perturbation at $x_0 \in \Omega(u_0)$ if the inequality $(*)$ holds. This problem was also investigated by Robinson (1976A,B).

PROBLEM 2 (stability of the optimal value; p_0-stability) *When the optimal value function $p(u)$ is continuous on ω at u_0?*

In particular, when the function $p(u)$ is Lipschitz continuous in a certain $B(u_0, r_0)$?

This problems is adressed in many papers on different levels of generality. The classical results of Berge (1963) should be mentioned here as well as the paper by Hogan (1973A) and their finite–dimensional versions by Martin (1975), Wets (1985) (for linear programming problems), Hogan (1973C) (for convex programming problems), Evans and Gould (1970), Greenberg and Pierskalla (1972).

PROBLEM 3 (stability of the solution set; $\mathcal{M}$–stability) *When, for a certain* $r > 0$, *the solution set* $\mathcal{M}(u)$ *is nonempty for* $u \in B(u_0, r_0)$ *and the multivalued mapping* $\mathcal{M} : R^n \rightrightarrows R^m$ *is upper semicontinuous and/or upper Lipschitzian?*

For linear programming this question was addressed by Robinson (1977), (1973B), Mangasarian (1982) and the book by Nozicka et al. (1974). For quadratic programming results we can refer to Klatte (1985). For more general problems, this question was investigated by Robinson (1976B, 1973A), Stern and Topkis (1976), Shapiro (1988A).

PROBLEM 4 (first order expansion of the optimal value function) *When, for a given sequence* $\{u_n\}$,

$$u_n = (u_0 + \varepsilon_n \bar{u} + \tilde{u}) \in \omega \tag{$**$}$$

where $\varepsilon_n \to +0$, $\bar{u} \in \omega$, $\tilde{u} \in \omega$, $\varepsilon_n^{-1}\|\tilde{u_n}\| \to 0$, *there exists*

$$\lim_{n\to\infty} \{\varepsilon_n^{-1}[p(u_n) - p(u_0)]\} = p^1(u_0; \bar{u}),$$

How to compute $p^1(u_0; \bar{u})$*?*

When the function $p(u)$ *is directionally differentiable and how to compute its directional derivatives?*

When the function $p(u)$ *is differentiable?*

The reference list concerning these problems is long. First of all, one should mention here the book of Fiacco (1983) where the problem of differentiability of the optimal value function is investigated via the implicit function theorem. One of the first who adressed this problem was probably Danskin (1967). In linear programming directional differentiability of the optimal value was investigated by Williams (1963). In convex programming directional derivatives were investigated by Hogan (1973A), Gol'stein (1971). For general nonlinear problems directional differentiability was investigated by Gauvin and Tolle (1977), Gauvin, Dubeau (1982), Rockafellar (1984, 1982), Gauvin (1993), Auslender and Cominetti (1990), Bonnans (1989), Bonnans, Ioffe, Shapiro (1992).

PROBLEM 5 (second order expansion of the optimal value function) *When, for* $\tilde{u_n} = \varepsilon_n^2 \bar{\bar{u}}/2 + o(\varepsilon_n^2)$, *there exists*

$$p^2(u_0; \bar{u}, \bar{\bar{u}}) = \lim_{n\to\infty} \{2\varepsilon_n^{-2}[p(u_n) - p(u_0) - \varepsilon_n p^1(u_0; \bar{u}]\}\ ?$$

How to compute $p^2(u_0; \bar{u}, \bar{\bar{u}})$ *?*

This problem was addressed in the papers of Shapiro (1985, 1988A,B), Bonnans (1992).

The above problems are related to differentiability properties of approximate (in particular, exact) solutions of perturbed problems.

Consider sequences $\{u_n\}$, $\{x_n\}$, $\{\delta_n\}$ where $u_n \in \omega$, $u_n \to u_0$, $\delta \geq 0$, $\delta \to 0$, $x_n \in M_{\delta_n}(u_n)$, such that with respect to this perturbation the problem is M-stable, i.e.,

$$dist(x_n, M_0) \to 0.$$

where the set

$$\begin{aligned} M_\delta(u) &= \{x \in \Omega(u) : \mathcal{J}(u,x) \leq p_0 + \delta\}, \\ \Omega_\delta(u) &= \{x \in R^m : \varphi^i(u,x) \leq \delta,\ i \in I,\ |\psi^j(u,x)| \leq \delta,\ j \in J\}. \end{aligned}$$

PROBLEM 6 *Which points of M_0 can be limit points of the sequence $\{x_n\}$?*

This problem is considered in some papers cited above, e.g. in Gauvin (1993) in relation to the differentiability of the optimal value function.

PROBLEM 7 (differential properties of solutions) *When*

(a) $\|x_n - x_0\| \leq O(\|u_n - u_0\|)$,

(b) *for $\{u_n\}$ of the form $(**)$ the estimation $\|x_n - x_0\| \leq O(\varepsilon_n)$ holds,*

(c) *there exists $\bar{x}$ and a sequence $\{\tilde{x}_n\}$, $\|\bar{x}_n\| \to 0$, $\|\bar{x}_n\| = o(\varepsilon_n)$ such that for $\{u_n\}$ of the form $(**)$ the formula*

$$x_n = x_0 + \varepsilon_n \bar{x} + \tilde{x}_n$$

holds.

This problem was considered by Gauvin and Janin (1988A) and Jittorntrum (1984), Bonnans (1992), Bonnans, Joffe, Shapiro (1992), Shapiro (1988B), Auslender, Cominetti (1990), Malanowski (1987).

3. Preliminaries

DEFINITION 3.1 *The marginal function p is locally Lipschitz near u_0 if for some neighbourhood $\mathcal{N}(u_0)$ of u_0 there exists a constant $M > 0$ such that for any $u_1, u_2 \in \mathcal{N}(u_0)$,*

$$|p(u_2) - p(u_1)| \leq M\|u_2 - u_1\|.$$

Following Clarke (1983), a function p which is locally Lipschitz near u_0 possesses the gradient $\nabla p(u)$ at almost all points $u \in \mathcal{N}(u_0)$.

DEFINITION 3.2 (Clarke, 1983) *The* **generalized directional derivative** *of p at u_0 in the direction s, denoted by $D^0 p(u_0; s)$, is defined as*

$$D^0 p(u_0; s) = \lim \sup_{u \to u_0,\ t \to 0+} \frac{p(u + ts) - p(u)}{t}.$$

DEFINITION 3.3 (Clarke, 1983) *The* **generalized gradient** *of p at u_0, denoted by $\partial p(u_0)$, is the convex hull of the set of limits $\{\lim_n \nabla_u p(u_n)\}$, where $\nabla_u p(u_n)$ exists and $u_n \to u_0$ as $n \to \infty$.*

The generalized gradient $\partial p(u_0)$ is a nonempty convex compact set.

PROPOSITION 3.1 (Clarke, 1983) *If p is locally Lipschitz near u_0, then for any $s \in R^n$,*

$$D^0 p(u_0; s) = \max\{\xi \cdot s \mid \xi \in \partial p(u_0)\}.$$

i.e., $D^0 p(u_0; s)$ is the support function of $\partial p(u_0)$.

We write $I(u, x)$ for the set of active indices, i.e.,

$$I(u, x) = \{i \in I \mid \varphi^i(u, x) = 0\}.$$

DEFINITION 3.4 *We say that the Mangasarian–Fromowitz $(M - F)$ regularity condition holds at $\bar{x} \in \Omega(u_0)$, if*
(i) *there exists a direction $\eta \in R^m$ such that*

$$< \nabla_x \psi^i(u_0, \bar{x}), \eta > \; = 0 \quad for \quad i \in J$$

$$< \nabla_x \varphi^i(u_0, \bar{x}), \eta > \; < 0 \quad for \quad i \in I(u_0, \bar{x})$$

(ii) *the (partial) gradients $\nabla_x \psi^i(u_0, \bar{x})$ are linearly independent.*

It was proved by Gauvin and Tolle (1977) that if $\bar{x}$ is a local minimum of the problem, then the Mangasarian–Fromowitz condition is necessary and sufficient to have the set $K(\bar{x}) = K(u_0, \bar{x})$ of Lagrange multipliers nonempty and compact. Moreover, the Mangasarian–Fromowitz condition is preserved under small perturbations.

THEOREM 3.1 (Gauvin, Tolle, 1977) *Assume that the Mangasarian–Fromowitz condition holds for some $\bar{x} \in \mathcal{M}(u_0)$. Let $\{u_k\}$ and $\{x_k\}$ be sequences such that $u_k \to u_0$ and $x_k \in \mathcal{M}(u_k)$, $x_k \to \bar{x}$. Then for k large enough the Mangasarian–Fromowitz condition is satisfied at x_k and there exist subsequences $\{\mu^l, \lambda^l\}$, $\{x^l\}$ with $(\mu^l, \lambda^l\} \in K(u^l, x^l)$ such that $(\mu^l, \lambda^l) \to (\bar{\mu}, \bar{\lambda})$ for some $(\bar{\mu}, \bar{\lambda}) \in K(u_0, \bar{x})$.*

COROLLARY 3.1 *If $\mathcal{M}(u_0)$ is nonempty and compact and if the Mangasarian-Fromowitz condition holds at each $\bar{x} \in \mathcal{M}(u_0)$, then $K(u_0, \bar{x})$ is compact.*

In deriving formulae for directional derivatives of the optimal value function we need second order optimality conditions.

For any feasible $\bar{x} \in \Omega$ we denote by $T(\bar{x})$ the *tangent cone* of Ω at $\bar{x}$, ie.,

$$T(\bar{x}) = \{y \in R^m \mid \; < \nabla \varphi^i(\bar{x}), y > \; \leq 0, \; i \in I(u_0, \bar{x}), \; < \nabla \psi^i(\bar{x}), y > = 0, \; i \in \mathcal{J}\}.$$

Moreover, by $C(\bar{x})$ we denote the cone of *critical directions*,

$$C(\bar{x}) = \{y \in T(\bar{x}) \mid \; < \nabla \mathcal{J}(\bar{x}), y > \; \leq 0\}.$$

Now we can formulate second order necessary conditions, see e.g., Gauvin (1993), Fiacco (1983).

Let $y \in C(\bar{x})$ be any critical direction for a $(M-F)$ regular local minimum $\bar{x}$. There exists $(\mu, \lambda) \in K(u_0, \bar{x})$ such that

$$y^T \nabla^2 \mathcal{L}(\bar{x}, \mu, \lambda)\, y \geq 0 \ .$$

Let $\bar{x}$ be a feasible point satisfying the first order necessary conditions with the set of multipliers $K(\bar{x})$.

THEOREM 3.2 *Given any critical direction $y \in C(\bar{x})$, $y \neq 0$, if there exists a $(\bar{\mu}, \bar{\lambda}) \in K(\bar{x})$ such that*

$$y^T \mathcal{L}(\bar{x}, \bar{\mu}, \bar{\lambda}) y > 0 \ ,$$

then $\bar{x}$ is a strict local minimum.

The above theorem provides the *weak version* of sufficient conditions. The *strong version* of sufficient conditions can be expressed by the formula

$$\inf\{y^T \nabla^2 \mathcal{L}(\bar{x}, \mu, \lambda) y |\ (\mu, \lambda) \in K(\bar{x})\} > 0 \, ,$$

which means that the Hessian of the Lagrangian is positive definite on $C(\bar{x}) \setminus \{0\}$ for any $(\mu, \lambda) \in K(\bar{x})$.

We denote by $K_1(\bar{x}, y)$ the set of second order multipliers satisfying the second order necessary conditions for the critical direction $y \in C(\bar{x})$, i.e.,

$$K_1(\bar{x}, y) = \{(\mu, \lambda) \in K(\bar{x}) |\ y^T \nabla^2 \mathcal{L}(\bar{x}, \mu, \lambda) y \geq 0\} \ .$$

4. Lipschitz continuity of the marginal function

There exists classical results of Berge and Hogan providing continuity results for lower and upper continuity of the marginal function. In these results the assumptions are expressed in terms of continuity of the feasible set multi–valued mapping.

The following result is proved by Gauvin and Dubeau (1982).

THEOREM 4.1 *Suppose that $\Omega(u_0) \neq \emptyset$, and Ω is uniformly compact around u_0 ($\bigcup_{u \in U_0} \Omega(u)$ is compact).*

If the Mangasarian–Fromowitz condition holds at some feasible $\bar{x} \in \Omega(u_0)$, then the marginal function p is continuous at u_0.

In this result the Mangasarian–Fromowitz condition is essential in proving the lower continuity of the marginal function. Without assuming this condition only upper continuity of p is guaranteed.

For parametric linear programming problems of the form

$$\begin{array}{ll} \min\, (c(u))^T x & \\ \text{subject to} & A(u)x = a(u) \\ & B(u)x \leq b(u) \end{array} \tag{6}$$

it is enough to assume the boundedness of the solution set of the dual to get lower continuity of the marginal function. Namely, we have the following result due to Martin (1975).

THEOREM 4.2 (Martin, 1975) *For any parameter value* $u_0 \in \omega$, *if the set of optimal solutions is bounded, then the marginal function* p *is upper continuous at* $u_0 \in \omega$. *If the set of optimal solutions to the dual problem is bounded, then* p *is lower continuous at* u_0.

THEOREM 4.3 (Gauvin, Dubeau, 1982) *If* $\Omega(u_0) \neq \emptyset$, Ω *is uniformly compact around* u_0 *and if the Mangasarian–Fromowitz condition is satisfied at each* $\bar{x} \in \mathcal{M}(u_0)$, *then the marginal function* p *is locally Lipschitz at* u_0.

Uniform compactness of Ω as a multivalued mapping can be expressed directly through the functions defining the problem. Namely the following result is due to Levitin (1992).

THEOREM 4.4 (Levitin, 1992) *If*

(a) the functions $\mathcal{J}(u,x)$, $\varphi^i(u,x)$, $i \in I$, $\psi^i(u,x)$, $i \in J$ *are continuous on* $\omega \times R^m$ *and continuously differentiable with respect to* x *at each point of the set* $\omega \times R^m$,

(b) for a certain $\delta > 0$ *the set* $\mathcal{M}_\delta(u_0)$ *is bounded,*

(c) for each $x_0 \in \mathcal{M}(u_0)$ *the Mangasarian-Fromowitz condition is satisfied,*

(d) for each $u_n \in \omega$, $u_n \to u_0$, $\delta_n \geq 0$, $\delta_n \to 0$, $x_n \in \Omega_{\delta_n}(u_n)$ *such that* $\overline{\lim}_{n\to\infty} \mathcal{J}(u_n, x_n) \leq p(u_0)$ *we have* $\overline{\lim}_{n\to\infty} \|x_n\| < +\infty$, *ie.,* $\{x_n\}$ *is bounded,*

then there exists $r > 0$, $L > 0$ *such that* $|p(u^1) - p(u^2)| \leq L\|u^1 - u^2\|$ *for* $u^1, u^2 \in B(u_0, r)$.

5. Directional derivatives of the marginal function

It is a classical result of Hogan (1973A), Gol'stein (1971) and Fiacco, Hutzler (1979) concerning the existence of the directional derivative of the marginal function in convex programming.

THEOREM 5.1 (Fiacco, Hutzler, 1979) *Suppose that the functions* $\mathcal{J}$, φ^i, $i \in I$ *are convex in* x *and the functions* ψ^i, $i \in J$ *are affine in* x. *Moreover, suppose that all the function are continuously differentiable with respect to* (u,x). *If* $\Omega(u_0) \neq \emptyset$, Ω *is uniformly compact around* u_0 *and the Mangasarian–Fromowitz condition holds at each* $x \in \mathcal{M}(u_0)$ *then the directional derivative* $p'(u_0; d)$ *exists for each* $d \in R^n$ *and*

$$p'(u_0; d) = \inf_{x \in M_0} \max_{(\mu,\lambda) \in K(u_0,x)} \nabla_u \mathcal{L}(x, u, \mu, \lambda) d.$$

In nonconvex case this assumption does not assure the existence of the directional derivative. The following result was proved by Gauvin, Dubeau (1982).

THEOREM 5.2 (Gauvin, Dubeau, 1982) *If $\Omega(u_0) \neq \emptyset$ and Ω is uniformly compact near u_0 and the Mangasarian–Fromowitz condition holds at each $x \in \mathcal{M}(u_0)$, then for any direction $d \in R^n$ we have*

$$\begin{aligned} &\sup\nolimits_{x\in\mathcal{M}(u_0)} \min\nolimits_{(\mu,\lambda)\in K(u_0,x)}\{\nabla_u\mathcal{L}(x,u,\mu,\lambda)d\} \\ &\qquad\le p^-(u_0;d) \le p^+(u_0;d) \\ &\qquad\le \max\nolimits_{x\in\mathcal{M}(u_0)} \max\nolimits_{(\mu,\lambda)\in K(u_0,x)}\{\nabla_u\mathcal{L}(x,u,\mu,\lambda)d\} \end{aligned}$$

We say that problem (5) is *stable* at u_0 if the set of feasible solutions $\Omega(u)$ is uniformly bounded in a neighbourhood of u_0. The following result has been proved by Gauvin (1993).

THEOREM 5.3 *Let problem (5) be stable at u_0 and let all the optimal points $\bar{x} \in \mathcal{M}(u_0)$ satisfy the (M-F) regularity condition. Moreover, assume that for all $\bar{x} \in \mathcal{M}(u_0)$ the weak version of the second order sufficient conditions is satisfied. Then for any direction d the optimal value function is directionally differentiable at u_0 and*

$$p'(u_0;d) = \min_{\bar{x}\in\mathcal{M}(u_0)} \inf_{y\in C(\bar{x})} \max_{(\mu,\lambda)\in K_1(\bar{x};y)} \{- < (\mu,\lambda), d >\} \ .$$

The next result due to Gauvin, Janin (1988A) shows that directional differentiability of the marginal function can be achieved provided there exists a Holder curve of solutions to perturbed problems.

This result is formulated without any regularity assumptions. It covers some cases when the set of Lagrange multipliers $K(\bar{x})$ is unbounded.

If $\bar{x}$ is an optimal solution to problem (5) for $u_0 = 0$, then there exist numbers $\lambda_0, \mu_i, i \in J$, λ_i, $i \in I$, not all equal zero, such that

$$\begin{aligned} \lambda_0\nabla\mathcal{J}(\bar{x}) + \sum_{i\in J}\mu_i\nabla\psi^i(\bar{x}) + \sum_{i\in I}\lambda_i\nabla\varphi^i(\bar{x}) &= 0, \\ \lambda_0,\ \lambda_i \ge 0\,,\quad i \in I& \\ \lambda_i\varphi^i(\bar{x}) = 0\,,\quad i \in I&\ . \end{aligned}$$

These are Fritz–John type necessary optimality conditions. Let $\tilde{K}(\bar{x})$ be the set of all Lagrange multipliers $(\lambda_0, \mu, \lambda)$, $(\mu, \lambda) \in R^{I\cup J}$, satisfying the above necessary optimality conditions. The set $K(\bar{x})$ contains normal multipliers, i.e.,

$$K(\bar{x}) = \{(\mu,\lambda) \in R^{I\cup J} \,|\, (1,\mu,\lambda) \in \tilde{K}(\bar{x})\} \ .$$

Let $K_0(\bar{x})$ be the recession cone of $\tilde{K}(\bar{x})$, ie.,

$$K_0(\bar{x}) = \{(\mu,\lambda) \in R^{I\cup J} \,|\, (0,\mu,\lambda) \in \tilde{K}(\bar{x})\} \ .$$

THEOREM 5.4 (Gauvin, Janin, 1988A, see also Narayaninsamy, 1986) *Let $\bar{x}$ and $x(td)$ be optimal solutions to problems (5) for $u = u_0 = 0$ and $u = td$ respectively. If* *(i)* $K(\bar{x}) \neq \emptyset$ *and* d *satisfies* $< (\mu, \lambda), d >> 0$ *for each* $(\mu, \lambda) \in K_0(u_0, \bar{x})$, $(\mu, \lambda) \neq 0$,

(ii) $\lim sup_{t\downarrow 0}|x(td) - \bar{x}| < +\infty$,

then the marginal function has a directional derivative and

$$p'(u_0; d) = \min_{\bar{x}\in M_0} \inf_{y\in C(\bar{x})} \sup\{-(\mu, \lambda)^T d : \ (\mu, \lambda) \in K_1(\bar{x}, y)\}.$$

where as above $K_1(\bar{x}, y)$ *is the set of second order multipliers.*

In the formula above $C(\bar{x})$ denotes the convex cone of all critical directions at $\bar{x}$.

We say that problem (5) is *directionally stable* at a fixed parameter value u_0 and for a given direction d if the solution set $\mathcal{M}(u_0)$ is nonempty and for any sequence $\{x_l\}$ of optimal solutions to problem (5) corresponding to the parameter value $u_0 + t_l d$, there exists a subsequence $\{x_k\}$ converging to the optimal solution $\bar{x}$ of (5) corresponding to the parameter value u_0.

Directional stability is a consequence of the uniform compactness conditions of Ω around u_0, or of the inf-boundedness condition used by Rockafellar (1984).

If at every optimal solution $\bar{x} \in M_0$ the linear independence regularity condition is satisfied, the directional derivative of the optimal value function can be obtained without any second order conditions. The following result has been proved by Gauvin, Tolle (1977) and Gauvin, Janin (1988B).

THEOREM 5.5 *If for any optimal solution* $\bar{x} \in M_0$ *of the directionally stable problem (5) the family* $\{\nabla\psi^i(\bar{x}),\ i \in \mathcal{J},\ \nabla\varphi^i(\bar{x}),\ i \in I(\bar{x})\}$ *consists of linearly independent vectors, then the marginal function* p *has a directional derivative given by the formula*

$$p'(u_0; d) = \min_{\bar{x}\in M}\{-(\sum_{i=1}^{p} \mu^i(\bar{x})d^i + \sum_{i=1}^{s} \lambda^i(\bar{x})d^{i+p}\},$$

where $(\mu(\bar{x}), \lambda(\bar{x}))$ *is the unique normal multiplier vector associated with the optimal solution* $\bar{x}$.

6. Computational aspects of directional differentiability of the optimal value function

As we see the formulae for directional derivatives are minmax problems which are hard to solve. In the present section we consider convex problems with linear equality constraints. In this class of problems the above minmax formulae can be considerably simplified (see Baumgart and Beer, 1992).

We consider the problem

$$\begin{array}{ll} \inf\ \mathcal{J}(u, x) & \\ \text{subject to} & \psi^i(u, x) = 0,\ i \in J = \{1, 2, ..., p\} \\ & \varphi^i(u, x) \leq 0,\ i \in I = \{1, 2, ..., s\} \end{array} \tag{7}$$

where φ^i, $i \in I$ are convex with respect to x, ψ^i, $i \in J$, are affine with respect to x, $\psi^i(u_0, x) =< k^i(u_0), x > +c^i(u_0)$, the functions $\mathcal{J}$, ψ^i, φ^i are differentiable with respect to u for all x at u_0, and for the parameter value u_0 all the functions are differentiable with respect to x.

Observe that for convex problems the set of Lagrange multipliers $K(\bar{x}) = K$, i.e. $K(\bar{x}) = K(u_0)$ is independent of the solution point.

The following theorem has been proved by Baumgart and Beer (1992).

THEOREM 6.1 *Consider problem (7). Suppose that one element $\bar{x} \in M_0$ and one element $(\bar{\mu}, \bar{\lambda}) \in K(u_0)$ are available. Let*

$$\begin{aligned}
&I(u_0) = \{i \in I |\ \varphi^i(\bar{x}) < 0\}, \\
&\bar{I}(u_0) = I \setminus I(u_0) = I(\bar{x}) = I(u_0, \bar{x}), \\
&J(u_0) = \{i \in I |\ \bar{\lambda}^i > 0\}, \\
&\bar{J}(u_0) = I \setminus J(u_0).
\end{aligned}$$

Then

$$\begin{aligned}
p'(u_0; d) = \ & \min_{x,y \in R^m} \{< \nabla_u \mathcal{J}(u_0, x), d > - < \nabla_x \mathcal{J}(u_0, \bar{x}), y > | \\
& \nabla_x \mathcal{J}(u_0, x) + \textstyle\sum_{i \in J(u_0)} \bar{\lambda}^i \nabla_x \varphi^i(u_0, x) + \sum_{i \in J} \bar{\mu}^i k_i(u_0) = 0 \\
& < \nabla_u \varphi^i(u_0, x), d > \ \leq \ < \nabla_x \varphi^i(\bar{x}, u_0), y), i \in \bar{I}(u_0) \\
& < \nabla_u \psi^i(u_0, x), d > = < k_i(u_0), y >, \ i \in J \\
& \varphi^i(u_0, x) = 0, \ i \in J(u_0) \\
& \varphi^i(u_0, x) \leq 0, \ i \in \bar{J}(u_0) \\
& \psi^i(u_0, x) = 0, \ i \in J
\end{aligned}$$

7. Directional derivatives of the solution

Directional differentiability of solutions have been investigated by several authors, eg., by Auslender, Cominetti (1990), Gauvin, Janin (1988A), Shapiro (1988B). Here we present the approach proposed by Gauvin and Janin (1988A). Let us consider any solution $\bar{x}$ to problem (5) and a fixed direction $d \in R^n$. For an optimal solution $\bar{x}$ to problem (5) the linear approximation of (5) is given by the following linear programming problem

$$\begin{aligned}
&\min \ \nabla f(\bar{x}) y \\
&\text{subject to} \ \ \nabla \varphi^i(\bar{x}) y \leq d_i, \ \ i \in I(\bar{x}) = I(u_0, \bar{x}) = I(0, \bar{x}) \\
&\qquad\qquad\quad \nabla \psi^i(\bar{x}) y = d_{s+i}, \ \ i = 1, 2, ..., p
\end{aligned} \tag{8}$$

Let us denote the solution set of (8) by $Y(\bar{x}, d)$. The dual problem to (8) is of the form $\max\{-\lambda^T d |\ \ \lambda \in K(u_0, \bar{x})\}$, where $K(u_0, \bar{x})$ denotes the set of Lagrange multipliers corresponding to problem (5) for $u_0 = 0$. The solution set of the dual problem is $K_1(\bar{x}, d)$. This set is related to multipliers which are in some sense optimal for the direction d. Let

$$I(\bar{x}, d) = \{i \in I(\bar{x}) |\ \text{there exists } y \in Y(\bar{x}, d) \text{ such that } \nabla \varphi^i(\bar{x}) y = d_i\}$$

and let

$$I^*(\bar{x}, d) = \{i \in I(\bar{x}, d) |\ \sup\{\lambda_i |\ \lambda \in K_1(\bar{x}, d)\} > 0\}.$$

The tangent subspace at $\bar{x}$ corresponding to the set of indices $I^*(\bar{x}, d)$ is given by

$$E = \{z \in R^m |\ \nabla\psi^i(\bar{x})z = 0,\ i = 1, 2, ..., p\ \ \nabla\varphi^i(\bar{x})z = 0,\ i \in I^*(\bar{x}, d)\}.$$

The following result is due to Gauvin and Janin (1988A).

THEOREM 7.1 *Let $\bar{x}$ be a solution to problem (5) at u_0 and let d be a direction such that*
(i) *$K_1(\bar{x}, d) = \{\mu^*, \lambda^*\}$ is a singleton,*
(ii) *the family $\{\nabla\psi^i(\bar{x}),\ i = 1, 2, ..., p,\ \ \nabla\varphi^i(\bar{x}),\ i \in I(\bar{x}, d)\}$ is linearly independent,*
(iii) *$z^T\nabla^2\mathcal{L}(\bar{x}, \mu^*, \lambda^*)z > 0$ for $z \in E$, $z \neq 0$.*
Then for any local optimal solution $x(td)$ to problem (5) at td near $\bar{x}$, the function $t \to x(td)$ has the right derivative $x'(0^+) = z^*$, where z^* is the unique optimal solution of the corresponding quadratic programming problem

$$\begin{aligned} &\min\ z^T\mathcal{L}(\bar{x}, \lambda^*, \mu^*)z \\ &\textit{subject to}\ \ \nabla\varphi^i(\bar{x})z \leq d^i,\ i \in I(\bar{x}) \\ &\qquad\qquad\quad \nabla\psi^i(\bar{x})z = d^i,\ i = 1, 2, ..., p \\ &\qquad\qquad\quad \nabla\mathcal{J}(\bar{x})z = -(\mu^*, \lambda^*)^T d. \end{aligned} \tag{9}$$

8. Generalized gradients of the marginal function

THEOREM 8.1 (Gauvin, Dubeau, 1982) *If $\Omega(u_0) \neq \emptyset$, and Ω is uniformly compact around u_0, and if the Mangasarian-Fromowitz regularity condition holds at every $\bar{x} \in \mathcal{M}_0$, then*

$$\partial p(u_0) \subset conv\{\bigcup_{\bar{x}\in M_0}\ \bigcup_{(\mu,\lambda)\in K(u_0,\bar{x})} [\nabla_u L(\bar{x}, u_0; \mu, \lambda)]\},$$

where conv stands for convex hull.

THEOREM 8.2 (Gauvin, Dubeau, 1982) *Let $u_0 \in \omega$ be given. Suppose that $\Omega(u_0) \neq \emptyset$ and Ω is uniformly compact around u_0. Assume that at each $\bar{x} \in M_0$*
(LI) *the partial gradients $\nabla_x\psi^i(u_0, \bar{x})$, $i = 1, 2, ..., p$ and $\nabla_x\varphi(u_0, \bar{x})$, $i \in I(u_0, \bar{x})$ are linearly independent.*
Then, p is locally Lipschitz near u_0, $-p$ is regular (in the sense of Clarke) at u_0 and

$$\partial p(u_0) = conv\{\nabla_u\mathcal{L}(\bar{x}, u_0, \mu_{\bar{x}}, \lambda_{\bar{x}}) |\ \bar{x} \in M_0\},$$

where $\mu_{\bar{x}}$, $\lambda_{\bar{x}}$ are the unique multipliers corresponding to $(u_0, \bar{x})$ and $\mathcal{L}(x, u, \mu, \lambda) = \mathcal{J}(u, x) + \sum_{i=1}^p \mu^i\psi^i(u, x) + \sum_{i=1}^s \lambda^i\varphi^i(u, x)$ is the standard Lagrangian.

THEOREM 8.3 (Outrata, 1990) *Assume that* $\Omega(u_0) \neq \emptyset$, Ω *is uniformly compact around* u_0, *and the functions* $\mathcal{J}$, ψ^i, $i = 1, 2, ..., p$ φ^i, $i = 1, 2, ..., s$ *are twice continuously differentiable. Let* $u_0 \in \omega$, $x_0 \in M_0$ *and the Mangasarian–Fromowitz condition hold at* (u_0, x_0). *Suppose that the problem satisfies the second order optimality conditions at* $x_0 \in M_0$ *for all multipliers* $(\mu, \lambda) \in K(u_0, x_0)$. *Then* $M_0 = \{x_0\}$, p *is locally Lipschitz near* u_0, *and regular in the sense of Clarke at* u_0, *and*

$$\partial p(u_0) = \{\nabla_u \mathcal{L}(x_0, u_0, \mu, \lambda) | \, (\mu, \lambda) \in K(u_0, x_0)\}.$$

9. Parametric linear programming problems

Let us consider the parametric linear programming problems of the form

$$\begin{aligned} &\inf\ c^T x \\ &\text{subject to} \quad A(u)x = a, \\ &\qquad\qquad\quad\ B(u)x \leq b, \end{aligned}$$

where $A[R^n \to R^{p \times m}]$, and $B[R^n \to R^{s \times n}]$, are continuously differentiable, $p < m$, and $c \in R^m$, $a \in R^p$, $b \in R^s$.

We assume that for each $u \in \omega$ the polyhedron

$$\Omega(u) = \{x \in R^m | \ A(u)x = a, \ B(u)x \leq b\}$$

is nonempty and uniformly compact and that the Mangasarian–Fromowitz regularity condition is satisfied at all $x \in \Omega(u)$. This implies that the corresponding marginal function h is locally Lipschitz over ω.

We denote by $J(u_0, x_0)$ subset of $I(u_0, x_0)$ such that the vectors $A^i(u_0)$, $i = 1, 2, ..., p$, $B^i(u_0)$, $i \in J(u_0, x_0)$ are linearly independent. Moreover, we denote $L(u_0, x_0) = \{i \in J(u_0, x_0) | \ \lambda_{0i} = 0\}$.

PROPOSITION 9.1 (Ben-Tal, Eiger, Outrata, Zowe, 1992) *Assume that there exists a direction* $k \in R^n$ *such that for all sufficiently small* $\vartheta > 0$ *the perturbed programs*

$$\begin{aligned} &\inf\ c^T x \\ &\textit{subject to} \quad A(u_0 + \vartheta k)x = a \\ &\qquad\qquad\quad\ B(u_0 + \vartheta k)x \leq b \end{aligned}$$

have a solution x_ϑ *and multipliers* μ_ϑ, λ_ϑ *which satisfy the strict inequalities*

$$\begin{aligned} &\lambda_{\vartheta i} > 0 \quad \textit{for} \quad i \in L(u_0, x_0), \\ &< B^i(u_0 + \vartheta k), x_\vartheta > \ < \ b_i \quad \textit{for} \quad i \in I(u_0, x) \setminus J(u_0, x_0). \end{aligned}$$

Then the formula

$$\sum_{i=1}^{p} \mu_{0i} (\nabla A^i(u_0))^T x_0 + \sum_{i=1}^{s} \lambda_{0i} (\nabla B^i(u_0)^T x_0 \ \in \ \partial p(u_0)$$

holds.

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