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The Navier–Stokes equations coupled with the heat equation: analysis and control*/

by

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In this paper we are concerned with the time-dependent Navier-Stokes equations coupled with the heat equation under the Boussinesq's approximation. We study the regularity of the strong solutions and we consider an optimal control problem associated to these equations. The problem consists in minimizing a functional involving the turbulence within the flow, the control being the heat flux through the boundary of the domain occupied by the fluid. We prove existence of optimal controls and derive some first order optimality conditions.

1. Introduction

In this paper we consider three-dimensional incompressible flows described by the following system

$$\begin{cases} \dfrac{\partial \vec{y}}{\partial t} - \nu\Delta_x \vec{y} + (\vec{y}\cdot\nabla_x)\vec{y} + \nabla_x p = \vec{f} + \vec{\beta}\tau \ \text{ in } \ \Omega_T = \Omega\times(0,T), \\ \dfrac{\partial \tau}{\partial t} - \kappa\Delta_x\tau + \vec{y}\cdot\nabla_x\tau = g \ \text{ in } \ \Omega_T, \\ \text{div}_x\vec{y} = 0 \ \text{ in } \ \Omega_T,\ \vec{y}(0) = \vec{\phi}_0 \ \text{ in } \ \Omega,\ \vec{y} = 0 \ \text{ on } \ \Sigma_T = \Gamma\times(0,T), \\ \tau(0) = \theta_0 \ \text{ in } \ \Omega,\ \tau = 0 \ \text{ on } \ \Sigma_T^0,\ \partial_n\tau = u \ \text{ on } \ \Sigma_T^1, \end{cases} \tag{1.1}$$

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where $T > 0$ is given; $\Omega \subset \mathbb{R}^3$ is an open and bounded set, with a boundary Γ of class C^2; $\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$; $\Sigma_T^0 = \Gamma_0 \times (0,T)$ and $\Sigma_T^1 = \Gamma_1 \times (0,T)$; $\nu > 0$ is the kinematic viscosity and $\kappa > 0$ is the thermal diffusion coefficient; $\vec{y}$ denotes the velocity of the flow and p the pressure; τ is the temperature in the fluid; $\vec{f} \in L^2([0,T], L^2(\Omega)^3)$ are the body forces; $g \in L^2(\Omega_T)$ is a heat source; $u \in L^2(\Sigma_T^1)$ is the heat flux through Γ_1; $\vec{\phi}_0 \in Y_0$ and $\theta_0 \in L^3(\Omega)$ are the initial velocity and temperature respectively; $\vec{\beta} \in L^\infty(\Omega_T)^3$ (it is a constant in the classical Bénard type problems). The space Y_0 is defined by

$$Y = \{\vec{y} \in H^1(\Omega)^3 : \operatorname{div} \vec{y} = 0\} \quad \text{and} \quad Y_0 = Y \cap H_0^1(\Omega)^3, \tag{1.2}$$

where div denotes the divergence operator. It is easy to check that Y and Y_0 are separable Hilbert spaces when they are endowed with the inner products

$$(\vec{y}, \vec{z})_Y = (\vec{y}, \vec{z})_{L^2(\Omega)^3} + a(\vec{y}, \vec{z})$$

and

$$(\vec{y}, \vec{z})_{Y_0} = a(\vec{y}, \vec{z}),$$

respectively, with

$$a(\vec{y}, \vec{z}) = \sum_{j=1}^{3} \int_\Omega \nabla y_j(x) \nabla z_j(x) dx \quad \forall \vec{y}, \vec{z} \in H^1(\Omega)^3. \tag{1.3}$$

In (1.1), $\vec{y}$ is the state and u is the control. The issue is to minimize the turbulence within the flow. A measure of the turbulence can be given by the norm of the vorticity $\nabla_x \times \vec{y}$

$$\nabla_x \times \vec{y} = (\partial_{x_2} y_3 - \partial_{x_3} y_2, \partial_{x_3} y_1 - \partial_{x_1} y_3, \partial_{x_1} y_2 - \partial_{x_2} y_1). \tag{1.4}$$

REMARK 1 *The $L^3(\Omega)$-regularity assumed for θ_0 will be used in Lemma 1 to deduce $L^8([0,T], L^4(\Omega))$-regularity of the temperature function τ.*

Before formulating the control problem properly, we need to analyze the system (1.1), which will be done in §2. In §3, we formulate the control problem, prove existence of a solution and derive the conditions for optimality.

The issue of controlling the turbulence of two-dimensional flows was first studied by Abergel and Temam (1990); Abergel and Casas (1993) considered the stationary case corresponding to three-dimensional flows; and Casas (1993A) studied the control by the body forces of three-dimensional flows governed by the evolution Navier-Stokes equations. The methods to treat the evolution problems are different from those used for the stationary case. For time-dependent two-dimensional flows the state equations are well posed, they have a unique strong solution, and consequently it is not difficult to derive the optimality conditions. For three-dimensional flows, there is no existence of strong solutions, in general, and it is necessary to carefully formulate the control problem to achieve the analogous result. In the stationary case, in dimension two or three, we do not have uniqueness of a solution. Two different techniques have been developed to overcome this difficulty: the first is due to Gunzburger et al. (1991) and the second is used in Abergel and Casas (1993). See also Casas (1993A).

There are some other papers dealing with the optimal control of Navier-Stokes equations, see, for instance, Fattorini and Sritharan (1993A), (1993B), Sritharan (1991), (1992).

2. Analysis of the state equation

In order to prove the existence of a solution of (1.1) in some suitable space, the following weak formulation for the velocity and temperature is usually suggested

$$\begin{cases} \text{Find } \vec{y} \in L^2([0,T], Y_0) \text{ and } \tau \in L^2([0,T], \mathcal{T}_0) \text{ such that} \\ \dfrac{d}{dt}(\vec{y}(t), \vec{\psi})_{L^2(\Omega)^3} + \nu a(\vec{y}(t), \vec{\psi}) + b(\vec{y}(t), \vec{y}(t), \vec{\psi}) \\ = (\vec{f}(t) + \vec{\beta}(t)\tau(t), \vec{\psi})_{L^2(\Omega)^3} \quad \forall \vec{\psi} \in Y_0, \ t \in (0,T), \\ \dfrac{d}{dt}(\tau(t), \zeta)_{L^2(\Omega)} + \kappa a_0(\nabla\tau, \nabla\zeta) + b_0(\vec{y}(t), \tau(t), \zeta) \\ = (g(t), \zeta)_{L^2(\Omega)} + (u(t), \zeta)_{L^2(\Sigma_T^1)} \quad \forall \zeta \in \mathcal{T}_0, \ t \in (0,T), \\ \vec{y}(0) = \vec{\phi}_0, \ \tau(0) = \theta_0, \end{cases} \tag{2.1}$$

where Y_0 and a are given by (1.2) and (1.3), respectively;

$$\mathcal{T}_0 = \{\zeta \in H^1(\Omega) : \zeta|_{\Gamma_0} = 0\}; \tag{2.2}$$

$$b(\vec{z}^1, \vec{z}^2, \vec{z}^3) = \int_\Omega (\vec{z}^1 \cdot \nabla)\vec{z}^2 \cdot \vec{z}^3 dx = \sum_{i,j=1}^{3} \int_\Omega z_i^1 \partial_{x_i} z_j^2 z_j^3 dx. \tag{2.3}$$

$$a_0 : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}, \quad a_0(\zeta_1, \zeta_2) = \int_\Omega \nabla\zeta_1 \cdot \nabla\zeta_2 dx; \tag{2.4}$$

$$b_0 : H^1(\Omega)^3 \times H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}, \quad b_0(\vec{z}, \zeta_1, \zeta_2) = \int_\Omega (\vec{z} \cdot \nabla\zeta_1)\zeta_2 dx. \tag{2.5}$$

The integral in (2.3) is well defined if $\vec{z}^1, \vec{z}^3 \in L^4(\Omega)^3$ and $\vec{z}^2 \in H^1(\Omega)^3$ or if $\vec{z}^1 \in L^4(\Omega)^3$, $\vec{z}^2 \in W^{1,4}(\Omega)^3$ and $\vec{z}^3 \in L^2(\Omega)^3$. Furthermore, by using the Hölder's inequality, we get

$$\int_\Omega |z_i^1 \partial_{x_i} z_j^2 z_j^3| dx \leq \|z_i^1\|_{L^4(\Omega)} \|\partial_{x_i} z_j^2\|_{L^2(\Omega)} \|z_j^3\|_{L^4(\Omega)} \tag{2.6}$$

and

$$\int_\Omega |z_i^1 \partial_{x_i} z_j^2 z_j^3| dx \leq \|z_i^1\|_{L^4(\Omega)} \|\partial_{x_i} z_j^2\|_{L^4(\Omega)} \|z_j^2\|_{L^2(\Omega)}. \tag{2.7}$$

These relations prove that b can be considered as a continuous trilinear form in the spaces where the previous norms are finite. Another fundamental property of b is the following: for every $\vec{y} \in L^2(\Omega)^3$ satisfying $\operatorname{div} \vec{y} = 0$ in Ω, we have that

$$b(\vec{y}, \vec{z}^1, \vec{z}^2) = -b(\vec{y}, \vec{z}^2, \vec{z}^1) \ \ \forall \vec{z}^1, \vec{z}^2 \in H_0^1(\Omega)^3. \tag{2.8}$$

In particular we deduce that

$$b(\vec{y}, \vec{z}, \vec{z}) = 0 \ \ \forall \vec{z} \in H_0^1(\Omega)^3. \tag{2.9}$$

On the other hand, it is clear that a_0 and b_0 are continuous bilinear and trilinear forms, respectively, in the spaces where they are defined. Moreover, analogously to (2.8) and (2.9), we have that

$$b_0(\vec{y}, \zeta_1, \zeta_2) = -b_0(\vec{y}, \zeta_2, \zeta_1) \ \ \forall \zeta_1, \zeta_2 \in H^1(\Omega) \text{ and } \forall \vec{y} \in Y_0 \tag{2.10}$$

and

$$b_0(\vec{y}, \zeta, \zeta) = 0 \ \ \forall \zeta \in H^1(\Omega) \text{ and } \forall \vec{y} \in Y_0. \tag{2.11}$$

We will say that $(\vec{y}, \tau)$ is a weak solution of (2.1) if $\vec{y} \in L^2([0,T], Y_0) \cap C_w([0,T], L^2(\Omega)^3)$; $\tau \in L^2([0,T], T_0) \cap C_w([0,T], L^2(\Omega))$; they satisfy the differential equations of (2.1) in the distribution sense and the initial conditions weakly in $L^2(\Omega)^3$ and $L^2(\Omega)$, respectively; and the following energy inequalities hold

$$\begin{aligned} \|\vec{y}(t)\|^2_{L^2(\Omega)^3} + 2\nu \int_0^t a(\vec{y}(s), \vec{y}(s))\, ds \ &\leq \ \|\vec{\phi}_0\|^2_{L^2(\Omega)^3} \\ &+2 \int_0^t (\vec{f}(s) + \vec{\beta}(s)\tau(s), \vec{y}(s))_{L^2(\Omega)^3}\, ds \quad \forall t \in [0,T] \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \|\tau(t)\|^2_{L^2(\Omega)} + 2\kappa \int_0^t a_0(\tau(s), \tau(s))\, ds \ &\leq \ \|\theta_0\|^2_{L^2(\Omega)} \\ &+2 \int_0^t [(g(s), \tau(s))_{L^2(\Omega)} + (u(s), \tau(s))_{L^2(\Sigma_1)}]\, ds \quad \forall t \in [0,T]. \end{aligned} \tag{2.13}$$

The existence of a weak solution can be proved by using the methods of Ladyzhenskaya (1969), Lions (1969) or Temam (1979); see also Foias et al. (1987). Once a solution of (2.1) has been found, the existence of the pressure $p \in D'(\Omega_T)$ can be proved, in such a way that $(\vec{y}, \tau, p)$ is a solution of (1.1), satisfying the partial differential equations in the distribution sense, the boundary condition in the trace sense and the initial condition in the way above mentioned. The uniqueness of a weak solution is an open question so far. This leads us to introduce a new class of solutions. We say that $(\vec{y}, \tau)$ is a strong solution of (2.1) if it is a weak solution and $\vec{y} \in L^8([0,T], L^4(\Omega)^3)$. We say that $(\vec{y}, \tau, p)$ is a strong solution of (1.1) if it is a solution in the above sense and $(\vec{y}, \tau)$ is a strong solution of (2.1). It is well known that (2.1) does not have more than one strong solution. Strong solutions satisfy the energy equalities instead of the inequalities (2.12) and (2.13). So they seem to be physically more significant than weak solutions. Unfortunately there is no existence result of strong solutions.

Now we state some regularity properties of strong solutions. First we introduce some notation

$$H^{2,1}(\Omega_T) = \left\{ y \in L^2(\Omega_T) : \frac{\partial y}{\partial x_i}, \frac{\partial^2 y}{\partial x_i x_j}, \frac{\partial y}{\partial t} \in L^2(\Omega_T),\ 1 \leq i,j \leq 3 \right\}$$

and

$$\|y\|_{H^{2,1}(\Omega_T)} = \left\{ \int_{\Omega_T} \left(|y|^2 + \left| \frac{\partial y}{\partial t} \right|^2 \right) dx dt \right.$$
$$\left. + \sum_{i=1}^{3} \int_{\Omega_T} \left| \frac{\partial y}{\partial x_i} \right|^2 dx dt + \sum_{i,j=1}^{3} \int_{\Omega_T} \left| \frac{\partial^2 y}{\partial x_i x_j} \right|^2 dx dt \right\}^{1/2}.$$

In Lions and Magenes (1968), Vol. 1, it is proved that every element of $H^{2,1}(\Omega_T)$, after a modification over a zero measure set, is a continuous function from $[0,T]$ to $H^1(\Omega)$, so we can consider $H^{2,1}(\Omega_T) \subset C([0,T], H^1(\Omega))$, moreover the inclusion is continuous.

THEOREM 1 *Let us assume that $(\vec{y}, \tau, p)$ is a strong solution of system (1.1), then $\vec{y} \in H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$; $\tau \in L^2([0,T], H^1(\Omega)) \cap C([0,T], L^2(\Omega))$; p can be taken in $L^2([0,T], H^1(\Omega))$ and it is unique up to the addition of a distribution in $(0,T)$. Moreover*

$$\|\vec{y}\|_{H^{2,1}(\Omega_T)^3} + \|\tau\|_{L^2([0,T],H^1(\Omega))} + \|\tau\|_{C([0,T],L^2(\Omega))}$$
$$\leq \eta \left(\|\vec{\phi}_0\|_{Y_0} + \|\vec{f}\|_{L^2([0,T],L^2(\Omega)^3)} + \|\vec{y}\|_{L^8([0,T],L^4(\Omega)^3)} \right. \tag{2.14}$$
$$\left. \|\theta_0\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega_T)} + \|u\|_{L^2(\Sigma_T^1)} + \|\vec{\beta}\|_{L^\infty(\Omega_T)} \right),$$

where $\eta : [0,+\infty) \longrightarrow [0,+\infty)$ is an increasing function depending only on Ω, κ and ν.

The previous theorem is an immediate consequence of Theorem 2.1 and Corollary 2.1 of Casas (1993B). We have also the following result about differentiability of mapping $u \rightarrow \vec{y}_u$

THEOREM 2 *If system (1.1) has a strong solution $(\vec{y}_0, \tau_0, p_0)$ for some element u_0 of $L^2(\Sigma_T^1)$ and some $\vec{\phi}_0 \in Y_0$ and $\theta_0 \in L^2(\Omega)$, then there exists an open* *neighbourhood $\mathcal{U}$ of u_0 in $L^2(\Sigma_T^1)$ such that system (1.1) has a strong solution $(\vec{y}_u, \tau_u, p_u)$ for every $u \in \mathcal{U}$. Moreover the mapping $G : \mathcal{U} \longrightarrow H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$, defined by $G(u) = \vec{y}_u$, is of class C^∞. Finally, if $\vec{z} = DG(u) \cdot v$, for some $u \in \mathcal{U}$ and some $v \in L^2(\Sigma_T^1)$, then $\vec{z}$ is the unique strong solution of the problem*

$$\begin{cases} \dfrac{\partial \vec{z}}{\partial t} - \nu \Delta_x \vec{z} + (\vec{y}_u \cdot \nabla_x)\vec{z} + (\vec{z} \cdot \nabla_x)\vec{y}_u + \nabla_x \pi = \vec{\beta}\zeta \quad in \ \ \Omega_T, \\ \dfrac{\partial \zeta}{\partial t} - \kappa \Delta_x \zeta + \vec{z} \cdot \nabla_x \tau_u + \vec{y}_u \cdot \nabla_x \zeta = 0 \quad in \ \ \Omega_T, \\ \operatorname{div}_x \vec{z} = 0 \quad in \ \ \Omega_T, \ \vec{z}(0) = 0 \quad in \ \ \Omega, \ \vec{z} = 0 \quad on \ \ \Sigma_T, \\ \zeta(0) = 0 \quad in \ \ \Omega, \ \zeta = 0 \quad on \ \ \Sigma_T^0, \ \partial_n \zeta = v \quad on \ \ \Sigma_T^1. \end{cases} \tag{2.15}$$

for some $(\tau, \pi) \in L^2([0,T], H^1(\Omega)) \cap C([0,T], L^2(\Omega)) \times L^2([0,T], H^1(\Omega))$, which is unique up to the addition of a distribution in $(0,T)$.

PROOF. Let us denote by

$$F : \left[H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)\right] \times L^2(\Sigma_T^1) \longrightarrow L^2([0,T], L^2(\Omega)^3) \times Y_0$$

the mapping given by

$$F(\vec{y}, u) = \left(\frac{\partial \vec{y}}{\partial t} + \nu A\vec{y} + B\vec{y} - \vec{f} - \vec{\beta}\tau_{\vec{y},u}, \vec{y}(0) - \vec{\phi}_0 \right),$$

where $A : Y_0 \longrightarrow Y_0'$ and $B : Y_0 \longrightarrow Y_0'$ are defined as follows

$$\langle A\vec{y}, \vec{z} \rangle = (\vec{y}, \vec{z})_{Y_0} \quad \text{and} \quad \langle B\vec{y}, \vec{z} \rangle = b(\vec{y}, \vec{y}, \vec{z}), \tag{2.16}$$

and $\tau_{\vec{y},u} \in L^2([0,T], \mathcal{T}_0) \cap C([0,T], L^2(\Omega))$ is the unique solution of the equation

$$\begin{aligned} &\frac{\partial \tau}{\partial t} - \kappa \Delta_x \tau + \vec{y} \cdot \nabla_x \tau = g \quad \text{in} \ \ \Omega_T, \\ &\tau(0) = \theta_0 \quad \text{in} \ \ \Omega, \ \tau = 0 \quad \text{on} \ \ \Sigma_T^0, \ \partial_n \tau = u \quad \text{on} \ \ \Sigma_T^1. \end{aligned} \tag{2.17}$$

It is obvious that A and B are continuous. Moreover, for every $\vec{y} \in H^2(\Omega)^3 \cap Y_0$ we have

$$\langle A\vec{y}, \vec{z} \rangle = -\int_\Omega \Delta \vec{y} \cdot \vec{z} dx = -\sum_{j=1}^{3} \int_\Omega \Delta y_j z_j dx \quad \forall \vec{z} \in Y_0$$

and, with (2.7),

$$\begin{aligned} |\langle B\vec{y}, \vec{z} \rangle| &\le C_1 \|\vec{y}\|_{L^4(\Omega)^3} \|\vec{y}\|_{W_0^{1,4}(\Omega)^3} \|\vec{z}\|_{L^2(\Omega)^3} \\ &\le C_2 \|\vec{y}\|_{Y_0} \|\vec{y}\|_{H^2(\Omega)^3} \|\vec{z}\|_{L^2(\Omega)^3}. \end{aligned} \tag{2.18}$$

Thus we have that $A\vec{y}, B\vec{y} \in L^2([0,T], L^2(\Omega)^3)$ for every $\vec{y} \in H^{2,1}(\Omega)^3 \cap Y_0$. Also it is immediate to check that F is of class C^∞ and

$$\frac{\partial F}{\partial \vec{y}}(\vec{y}, u) \cdot \vec{z} = \left(\frac{\partial \vec{z}}{\partial t} + \nu A\vec{z} + B'(\vec{y})\vec{z} - \vec{\beta}\zeta, \vec{z}(0) \right),$$

where

$$\langle B'(\vec{y})\vec{z}, \vec{\psi} \rangle = b(\vec{y}, \vec{z}, \vec{\psi}) + b(\vec{z}, \vec{y}, \vec{\psi}) \quad \forall \vec{\psi} \in Y_0, \tag{2.19}$$

and $\zeta \in L^2([0,T], \mathcal{T}_0) \cap C([0,T], L^2(\Omega))$ is the unique solution of the equation

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} - \kappa \Delta_x \zeta + \vec{y} \cdot \nabla_x \zeta + \vec{z} \cdot \nabla_x \tau = 0 \text{ in } \Omega_T, \\ \tau(0) = 0 \text{ in } \Omega, \ \tau = 0 \text{ on } \Sigma_T^0, \ \partial_n \tau = 0 \text{ on } \Sigma_T^1. \end{cases} \tag{2.20}$$

By using Lemma 2 proved below we deduce that $\dfrac{\partial F}{\partial \vec{y}}(\vec{y}_0, u_0)$ is an isomorphism from $H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$ onto $L^2([0,T], L^2(\Omega)^3) \times Y_0$. Moreover we have that $F(\vec{y}_0, u_0) = (0,0)$. Therefore we can make use of the implicit function theorem and obtain the existence of an open neighbourhood $\mathcal{U}$ of u_0 in $L^2(\Sigma_T^1)$ and a mapping $G : \mathcal{U} \longrightarrow H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$ such that $F(G(u), u) = (0,0)$ for every $u \in \mathcal{U}$. This means that $\vec{y}_u = G(u)$, together with some $\tau_u \in L^2([0,T], \mathcal{T}_0) \cap C([0,T], L^2(\Omega))$ and $p_u \in L^2([0,T], H^1(\Omega))$, is a strong solution of (1.1) corresponding to the data u and $\vec{\phi}_0$. Moreover G is also of class C^∞ and

$$\frac{\partial F}{\partial \vec{y}}(\vec{y}_u, u) \circ DG(u) \cdot v + \frac{\partial F}{\partial u}(\vec{y}_u, u) \cdot v = (0,0) \quad \forall v \in L^2(\Sigma_T^1).$$

Then, setting $\vec{z} = DG(u) \cdot v$, we deduce that $\vec{z}$ satisfies together with some $\zeta \in L^2([0,T], \mathcal{T}_0) \cap C([0,T], L^2(\Omega))$ the system

$$\begin{cases} \dfrac{\partial \vec{z}}{\partial t} + \nu A\vec{z} + B'(\vec{y}_u)\vec{z} = \vec{\beta}\zeta \text{ in } \Omega_T, \\ \dfrac{\partial \zeta}{\partial t} - \kappa \Delta_x \zeta + \vec{y}_u \cdot \nabla \zeta + \vec{z} \cdot \nabla \tau_u = 0 \text{ in } \Omega_T \\ \operatorname{div}_x \vec{z} = 0 \text{ in } \Omega_T, \ \vec{z}(0) = 0, \ \zeta(0) = 0, \ \zeta = 0 \text{ on } \Sigma_T^0, \ \partial_n \zeta = v \text{ on } \Sigma_T^1, \end{cases}$$

or equivalently

$$\begin{cases} \vec{z} \in H^{2,1}(\Omega)^3 \cap C([0,T], Y_0), \ \zeta \in L^2([0,T], \mathcal{T}_0) \cap C([0,T], L^2(\Omega)) \\ \dfrac{d}{dt}(\vec{z}(t), \vec{\psi})_{L^2(\Omega)^3} + \nu(\vec{z}(t), \vec{\psi})_{Y_0} + b(\vec{y}_u(t), \vec{z}(t), \vec{\psi}) \\ +b(\vec{z}(t), \vec{y}_u(t), \vec{\psi}) = (\vec{\beta}(t)\zeta(t), \vec{\psi})_{L^2(\Omega)^3} \ \ \forall \vec{\psi} \in Y_0, \ t \in (0,T), \\ \dfrac{\partial \zeta}{\partial t} - \kappa \Delta_x \zeta + \vec{y}_u \cdot \nabla \zeta + \vec{z} \cdot \nabla \tau_u = 0 \text{ in } \Omega_T \\ \vec{z}(0) = 0, \ \zeta(0) = 0, \ \zeta = 0 \text{ on } \Sigma_T^0, \ \partial_n \zeta = v \text{ on } \Sigma_T^1, \end{cases}$$

which implies that $(\vec{z}, \zeta)$ satisfies (2.15); see Temam (1979), pp. 267–269 for the existence of the pressure $p \in L^2([0,T], H^1(\Omega))$ allowing to pass from the above system to (2.15). ■

LEMMA 1 *Given $\vec{y} \in C([0,T], Y_0)$, the problem*

$$\begin{aligned} &\frac{\partial \tau}{\partial t} - \kappa \Delta_x \tau + \vec{y} \cdot \nabla_x \tau = g \quad in \quad \Omega_T, \\ &\tau(0) = \theta_0 \quad in\ \Omega, \ \ \tau = 0 \quad on \quad \Sigma_T^0, \ \ \partial_n \tau = u \quad on \quad \Sigma_T^1, \end{aligned} \tag{2.21}$$

has a unique solution $\tau \in C([0,T], L^2(\Omega)) \cap L^2([0,T], \mathcal{T}_0) \cap L^8([0,T], L^4(\Omega))$.

PROOF. The existence and uniqueness of a solution in $C([0,T], L^2(\Omega)) \cap L^2([0,T], \mathcal{T}_0)$ is standard. Let us prove the $L^8([0,T], L^4(\Omega))$-regularity. First note that the trace mapping $\gamma : W^{1,3/2}(\Omega) \longrightarrow L^2(\Gamma)$ is continuous, Nečas (1967). Then

$$\zeta \in W^{1,3/2}(\Omega) \longrightarrow \int_{\Gamma_1} u(x,t)\zeta(x) d\sigma(x)$$

is a continuous mapping for almost all $t \in (0,T)$. Then we can take functions $f_i \in L^2([0,T], L^3(\Omega))$, $0 \le i \le 3$, such that for every $\zeta \in W^{1,3/2}(\Omega)$

$$\int_{\Gamma_1} u(x,t)\zeta(x) d\sigma(x) = \int_\Omega f_0(x,t)\zeta(x) + \sum_{i=1}^{3} \int_\Omega f_i(x,t) \partial x_i \zeta(x) dx,$$

and

$$\sum_{i=0}^{3} \|f_i\|_{L^2([0,T],L^3(\Omega))} \le 4 \|u\|_{L^2(\Sigma_T^1)};$$

see, for instance, Adams (1975). Therefore the following identity holds

$$\begin{aligned} &\frac{d}{dt}(\tau(t), \zeta)_{L^2(\Omega)} + \kappa a_0(\tau(t), \zeta) + b_0(\vec{y}(t), \tau(t), \zeta) \\ &\qquad = \int_\Omega (g(t) + f_0(t))\zeta dx + \sum_{j=1}^{3} \int_\Omega f_i(t) \partial_{x_i} \zeta dx \end{aligned}$$

for every $\zeta \in \mathcal{T}_0$.

Now we can apply the method of Ladyzenskaya, Solonnikov and Ural'ceva (1968), pp. 194–201, with $\theta = 2/3$, $r_3 = 4/3$, $q_3 = 2$, $r_4 = 1$, $q_4 = 3/2$, $r = 8$ and $q = 4$ to deduce the desired regularity. ■

LEMMA 2 *For every* $\vec{z}_0 \in Y_0$ *and* $\vec{f} \in L^2(\Omega_T)^3$ *there exists a unique solution* $\vec{z} \in H^{2,1}(\Omega)^3 \cap C([0,T], Y_0)$ *and* $\zeta \in L^2([0,T], \mathcal{T}_0) \cap C([0,T], L^2(\Omega))$ *of the system*

$$\begin{cases} \dfrac{\partial \vec{z}}{\partial t} + \nu A\vec{z} + B'(\vec{y}_u)\vec{z} = \vec{f} + \vec{\beta}\zeta \quad in \quad \Omega_T, \\ \dfrac{\partial \zeta}{\partial t} - \kappa\Delta_x\zeta + \vec{y}_u \cdot \nabla\zeta + \vec{z}\cdot\nabla\tau_u = 0 \quad in \quad \Omega_T \\ \mathrm{div}_x \vec{z} = 0 \quad in \quad \Omega_T, \ \ \vec{z}(0) = \vec{z}_0, \\ \zeta(0) = 0, \ \ \zeta = 0 \ on \ \ \Sigma_T^0, \ \ \partial_n\zeta = 0 \ on \ \ \Sigma_T^1. \end{cases} \tag{2.22}$$

Moreover the solution depends continuously on $(\vec{f}, \vec{z}_0) \in L^2([0,T], L^2(\Omega)^3) \times Y_0$.

PROOF. We are going to obtain some a priori estimates, which is the difficult part of the proof. To conclude the proof it is enough to make the usual discretization of the space and to pass to the limit with the help of the a priori estimates; see Casas (1993B) for a detailed exposition corresponding to the time-dependent Navier–Stokes equations.

Let us begin by multiplying the first partial differential equation of (2.22) by $\vec{z}$

$$\frac{1}{2}\frac{d}{dt}\|\vec{z}(t)\|^2_{L^2(\Omega)^3} + \nu\|\vec{z}\|^2_{Y_0} + \langle B'(\vec{y}_u(t))\vec{z}(t), \vec{z}(t)\rangle$$
$$= (\vec{f}(t) + \vec{\beta}(t)\zeta(t), \vec{z}(t))_{L^2(\Omega)^3}. \tag{2.23}$$

From (2.19), (2.9), (2.6) and the inequality

$$\|\vec{\psi}\|_{L^4(\Omega)} \le \sqrt{2}\, \|\vec{\psi}\|^{1/4}_{L^2(\Omega)}\, \|\vec{\psi}\|^{3/4}_{H^1_0(\Omega)^3}, \tag{2.24}$$

see Temam (1979), we deduce

$$|\langle B'(\vec{y}_u(t))\vec{z}(t), \vec{z}(t)\rangle| \le |b(\vec{y}_u(t), \vec{z}(t), \vec{z}(t))| + |b(\vec{z}(t), \vec{y}_u(t), \vec{z}(t))|$$
$$\le C_1\|\vec{y}_u(t)\|_{Y_0}\|\vec{z}(t)\|^2_{L^4(\Omega)^3} \le C_2\|\vec{z}(t)\|^{1/2}_{L^2(\Omega)^3}\|\vec{z}(t)\|^{3/2}_{Y_0}$$
$$\le C_3\|\vec{z}(t)\|^2_{L^2(\Omega)^3} + \frac{\nu}{4}\|\vec{z}(t)\|^2_{Y_0}.$$

Using this inequality in (2.23), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\vec{z}(t)\|^2_{L^2(\Omega)^3} + \frac{3}{4}\nu\|\vec{z}(t)\|^2_{Y_0}$$
$$\le (C_3 + \frac{1}{2})\|\vec{z}(t)\|^2_{L^2(\Omega)^3} + \|\vec{f}(t)\|^2_{L^2(\Omega)^3} + \|\vec{\beta}\|^2_{L^\infty(\Omega_T)^3}\|\zeta(t)\|^2_{L^2(\Omega)}. \tag{2.25}$$

On the other hand, multiplying the second equation of (2.22) by ζ and taking into account (2.11) and (2.10) we get

$$\frac{1}{2}\frac{d}{dt}\|\zeta(t)\|^2_{L^2(\Omega)} + \kappa\|\zeta(t)\|^2_{H^1_0(\Omega)}$$
$$= -\int_\Omega \vec{z}(t)\nabla_x\tau_u(t)\zeta(t)dx = \int_\Omega \vec{z}(t)\nabla_x\zeta(t)\tau_u(t)dx. \tag{2.26}$$

Now, using Hölder's inequality and (2.24), we have

$$\left|\int_\Omega \vec{z}(t)\nabla_x\zeta(t)\tau_u(t)dx\right| \le \|\vec{z}(t)\|_{L^4(\Omega)}\|\tau_u(t)\|_{L^4(\Omega)}\|\zeta(t)\|_{H^1_0(\Omega)}$$
$$\le C_4\|\vec{z}(t)\|^2_{L^4(\Omega)}\|\tau_u(t)\|^2_{L^4(\Omega)} + \frac{\kappa}{2}\|\zeta(t)\|^2_{H^1_0(\Omega)}$$
$$\le C_5\|\vec{z}(t)\|^{1/2}_{L^2(\Omega)}\|\vec{z}(t)\|^{3/2}_{Y_0}\|\tau_u(t)\|^2_{L^4(\Omega)} + \frac{\kappa}{2}\|\zeta(t)\|^2_{H^1_0(\Omega)}$$
$$\le C_6\|\vec{z}(t)\|^2_{L^2(\Omega)}\|\tau_u(t)\|^8_{L^4(\Omega)} + \frac{\nu}{4}\|\vec{z}(t)\|^2_{Y_0} + \frac{\kappa}{2}\|\zeta(t)\|^2_{H^1_0(\Omega)}.$$

Combining this inequality with (2.26) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta(t)\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|\zeta(t)\|^2_{H^1_0(\Omega)}$$
$$\le C_6\|\vec{z}(t)\|^2_{L^2(\Omega)}\|\tau_u(t)\|^8_{L^4(\Omega)} + \frac{\nu}{4}\|\vec{z}(t)\|^2_{Y_0}. \tag{2.27}$$

Adding (2.25) and (2.27) we get

$$\frac{1}{2}\frac{d}{dt}\left\{\|\vec{z}(t)\|^2_{L^2(\Omega)^3} + \|\zeta(t)\|^2_{L^2(\Omega)}\right\} + \frac{\nu}{2}\|\vec{z}(t)\|^2_{Y_0} + \frac{\kappa}{2}\|\zeta(t)\|^2_{H^1_0(\Omega)} \le \|\vec{f}(t)\|^2_{L^2(\Omega)^3}$$
$$+ C_7\left(1 + \|\tau_u(t)\|^8_{L^4(\Omega)}\right)\|\vec{z}(t)\|^2_{L^2(\Omega)} + \|\vec{\beta}\|^2_{L^\infty(\Omega_T)^3}\|\zeta(t)\|^2_{L^2(\Omega)}. \tag{2.28}$$

By reference to $\tau_u \in L^8([0,T], L^4(\Omega))$, see Lemma 1, and Gronwall's lemma, (2.28) leads to

$$\|\vec{z}\|_{L^\infty([0,T],L^2(\Omega)^3)} + \|\zeta\|_{L^\infty([0,T],L^2(\Omega)^3)}$$
$$\le C_8\left(\|\vec{f}\|_{L^2([0,T],L^2(\Omega)^3)} + \|\vec{z}_0\|_{L^2(\Omega)^3}\right). \tag{2.29}$$

Integrating (2.28) in $[0,T]$ and using (2.29), it comes

$$\|\vec{z}\|_{L^2([0,T],Y_0)} + \|\zeta\|_{L^2([0,T],T_0)} \le C_9\left(\|\vec{f}\|_{L^2([0,T],L^2(\Omega)^3)} + \|\vec{z}_0\|_{L^2(\Omega)^3}\right) \tag{2.30}$$

Finally, the $H^{2,1}(\Omega)$-regularity of $\vec{z}$ is a consequence of Theorem 2.1 of Casas (1993B). The uniqueness is proved as usual. ■

3. Study of the control problem

As mentioned in §1, our aim is to control the turbulence within the flow by acting on the body forces. Now the question is how to formulate mathematically this control problem. Are we going to consider strong or weak solutions of (1.1)?. Though we can formulate a control problem in the same way as in Abergel and Temam (1990) and we can prove the existence of a solution for this problem, we do not know how to derive the optimality conditions satisfied by this optimal control. In Abergel and Temam (1990) the authors considered two-dimensional flows, therefore every weak solution was also a strong solution. By using the results stated in the previous section, we can obtain the conditions for optimality for a reasonable control problem assuming that the optimal state is a strong solution of (1.1). Thus we need to formulate a problem such that every optimal control has associated a strong solution of (1.1). This is achieved by modifying slightly the functional used in Abergel and Temam (1990) : we consider the cost functional

$$J(u,\vec{y}) = \frac{1}{6}\int_0^T \left(\int_\Omega |\nabla_x \times \vec{y}|^2 dx\right)^3 dt + \frac{N}{2}\int_0^T \int_{\Gamma_1} |u|^2 d\sigma dt,$$

where $\nabla_x \times \vec{y}$ is the vorticity within the flow given by (1.4).

Then the optimal control problem is formulated in the following way

$$\text{(P)} \quad \begin{cases} \text{Minimize } J(u,\vec{y}), \\ (u,\vec{y}) \in K \times H^{2,1}(\Omega_T)^3 \text{ satisfying (1.1) together with some } (\tau,p), \end{cases}$$

with $(\tau,p) \in C([0,T],L^2(\Omega)) \cap L^2([0,T],H^1(\Omega)) \times L^2([0,T],H^1(\Omega))$, and $K \subset L^2(\Sigma_T^1)$ being nonempty, convex and closed.

The first term of the cost functional gives a measure of the turbulence in the flow through the norm of the vorticity in the space $L^6([0,T],L^2(\Omega)^3)$. The reason of this choice is that any weak solution of (1.1) verifying $J(u,\vec{y}) < +\infty$ is a strong solution, which reduces the admissible states of (P) to strong solutions of (1.1). The following proposition proves this claim.

PROPOSITION 1 *Let $(\vec{y},\tau,p)$ be a weak solution of (1.1) verifying $J(u,\vec{y}) < +\infty$, then it is a strong solution. Moreover*

$$\|\vec{y}\|_{L^8([0,T],L^4(\Omega)^3)} \le M, \tag{3.1}$$

for some constant M depending on $J(u,\vec{y})$ and $\|u\|_{L^2(\Sigma_T^1)}$.

PROOF. We first note that

$$\|\vec{z}\|_{Y_0} = \|\nabla_x \times \vec{z}\|_{L^2(\Omega)^3} \quad \forall \vec{z} \in Y_0; \tag{3.2}$$

it is enough to make an integration by parts, using that $\vec{z} = 0$ on Σ_T and $\operatorname{div} \vec{z} = 0$. Then the inequality $J(u,\vec{y}) < +\infty$ implies that $\vec{y} \in L^6([0,T],Y_0)$. On the other hand, since $(\vec{y}, \tau, p)$ is a weak solution, we have $\vec{y} \in L^\infty([0,T], L^2(\Omega)^3)$. Therefore, from (2.24) we obtain that

$$\|\vec{y}(t)\|_{L^4(\Omega)^3} \leq \sqrt{2}\|\vec{y}\|^{1/2}_{L^\infty([0,T],L^2(\Omega)^3)}\|\vec{y}(t)\|^{3/4}_{Y_0} = C_1\|\vec{y}(t)\|^{3/4}_{Y_0},$$

which implies that

$$\|\vec{y}\|^8_{L^8([0,T],L^4(\Omega)^3)} \leq C_1^8\|\vec{y}\|^6_{L^6([0,T],Y_0)} < +\infty.$$

Finally, (3.1) follows from this inequality and (3.2). ■

Now we study the existence of a solution of (P). The convexity of J is essential in the proof of existence. For other optimal control problems of Navier-Stokes equations with nonconvex cost functionals, some relaxation is necessary to state existence of solutions; see Fattorini and Sritharan (1993B).

THEOREM 3 *Let us assume that the following two hypotheses hold:*

1. *There exists a feasible pair* $(u, \vec{y}) \in K \times H^{2,1}(\Omega_T)^3$ *satisfying* (1.1) *together with some* $(\tau, p) \in L^2([0,T], H^1(\Omega)) \cap C([0,T], L^2(\Omega)) \times L^2([0,T], H^1(\Omega))$.
2. *Either* $N > 0$ *or* K *is bounded in* $L^2(\Sigma_T^1)$.

Then there exists at least one optimal solution $(u_0, \vec{y}_0)$ *of* (P).

PROOF. Let $\{(u_k, \vec{y}_k)\}_{k=1}^\infty \subset K \times H^{2,1}(\Omega_T)^3$ be a minimizing sequence of (P). The existence of such a sequence is a consequence of the first hypothesis. The second one implies that $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $L^2(\Sigma_T^1)$. Then we can take a subsequence, denoted in the same way, such that $u_k \to u_0$ weakly in $L^2(\Sigma_T^1)$. Moreover, noting that K is closed and convex, we deduce that $u_0 \in K$.

On the other hand, due to Proposition 1, we know that $\{\vec{y}_k\}_{k=1}^\infty$ is bounded in $L^8([0,T], L^4(\Omega)^3)$. Then Theorem 1 states that $\{\vec{y}_k\}_{k=1}^\infty$ is bounded in the space $H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$. Therefore we can assume, by taking a subsequence if necessary, that $\vec{y}_k \to \vec{y}_0$ weakly in $H^{2,1}(\Omega_T)^3$, with $\vec{y}_0$ also belonging to $C([0,T], Y_0)$. Using the compactness of the inclusion $H^{2,1}(\Omega_T)^3 \subset L^2([0,T], L^2(\Omega)^3)$ and noting that $H^{2,1}(\Omega_T) \subset C([0,T], H^1(\Omega))$, the inclusion being continuous, it is easy to pass to the limit in the system of equations satisfied by $(u_k, \vec{y}_k)$ and some (τ_k, p_k) and to conclude that $(u_0, \vec{y}_0)$ also satisfies (1.1) for some (τ, p). Hence $(u_0, \vec{y}_0)$ is a feasible pair for Problem (P).

Finally, the convexity of J allows to deduce that $(u_0, \vec{y}_0)$ is a solution of (P) arguing as follows

$$J(u_0, \vec{y}_0) \leq \liminf_{k\to+\infty} J(u_k, \vec{y}_k) = \inf(\mathrm{P}).$$

■

Our last theorem states the optimality conditions satisfied by the solutions $(u_0, \vec{y}_0)$ of (P).

THEOREM 4 *Let us assume that* $(u_0, \vec{y}_0)$ *is a solution of* (P) *and* τ_0 *and* p_0 *are the temperature and the pressure, respectively, corresponding to the velocity* $\vec{y}_0$. *Then there exist two unique elements* $\vec{\varphi}_0 \in H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$ *and* $\psi_0 \in C([0,T], L^2(\Omega)) \cap L^2([0,T], H^1(\Omega))$ *and a function* $\pi_0 \in L^2([0,T], H^1(\Omega))$, *unique up to the addition of a distribution in* $(0,T)$, *such that the following system is satisfied*

$$\begin{cases} \dfrac{\partial \vec{y}_0}{\partial t} - \nu \Delta_x \vec{y}_0 + (\vec{y}_0 \cdot \nabla_x)\vec{y}_0 + \nabla_x p_0 = \vec{f} + \beta \tau_0 \quad in \ \ \Omega_T, \\ \dfrac{\partial \tau_0}{\partial t} - \kappa \Delta_x \tau_0 + \vec{y}_0 \cdot \nabla_x \tau_0 = g \quad in \ \ \Omega_T, \\ \mathrm{div}_x \vec{y}_0 = 0 \quad in \ \ \Omega_T, \ \ \vec{y}_0(0) = \vec{\phi}_0 \quad in \ \ \Omega, \ \ \vec{y}_0 = 0 \quad on \ \ \Sigma_T, \\ \tau_0(0) = \theta_0 \quad in \ \ \Omega, \ \ \tau_0 = 0 \quad on \ \ \Sigma_T^0, \ \ \partial_n \tau_0 = u_0 \quad on \ \ \Sigma_T^1; \end{cases} \tag{3.3}$$

$$\begin{cases} -\dfrac{\partial \vec{\varphi}_0}{\partial t} - \nu \Delta_x \vec{\varphi}_0 - (\vec{y}_0 \cdot \nabla_x)\vec{\varphi}_0 + (\nabla_x \vec{y}_0)^T \vec{\varphi}_0 + \nabla_x \pi_0 \\ = \tau_0 \nabla_x \psi_0 + \|\nabla_x \times \vec{y}_0\|^4_{L^2(\Omega)^3} [\nabla_x \times (\nabla_x \times \vec{y}_0)] \quad in \ \ \Omega_T, \\ -\dfrac{\partial \psi_0}{\partial t} - \kappa \Delta_x \psi_0 - \vec{y}_0 \cdot \nabla_x \psi_0 = \vec{\beta} \vec{\varphi}_0 \quad in \quad \Omega_T, \\ \mathrm{div}_x \vec{\varphi}_0 = 0 \quad in \ \ \Omega_T, \ \ \vec{\varphi}_0(T) = 0 \quad in \ \ \Omega, \ \ \vec{\varphi}_0 = 0 \quad on \ \ \Sigma_T, \\ \psi_0(T) = 0 \quad in \ \ \Omega, \ \ \psi_0 = 0 \quad on \ \ \Sigma_T^0, \ \ \partial_n \psi_0 = 0 \quad on \ \ \Sigma_T^1; \end{cases} \tag{3.4}$$

$$\int_0^T \int_{\Gamma_1} (\psi_0 + N u_0)(u - u_0)\, d\sigma dt \geq 0 \quad \forall u \in K. \tag{3.5}$$

PROOF. From Theorem 2 we deduce the existence of an open neighbourhood $\mathcal{U}$ of u_0 in $L^2(\Sigma_T^1)$ such that (1.1) has a unique strong solution for every $u \in \mathcal{U}$. Moreover, the mapping $G : \mathcal{U} \longrightarrow H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$, given by $G(u) = \vec{y}_u$, is of class C^∞.

Let us define $I : \mathcal{U} \longrightarrow \mathbb{R}$ by $I(u) = J(u, G(u))$. It is immediate that u_0 is a solution of the optimization problem

$$\begin{cases} \text{Minimize } I(u), \\ u \in \mathcal{U} \cap K. \end{cases}$$

Since K is convex and $\mathcal{U}$ is a neighbourhood of u_0, for every $u \in K$ we can find a number $\epsilon_u > 0$, depending on u, such that $u_\epsilon = u_0 + \epsilon(u - u_0) \in \mathcal{U} \cap K$ for all $\epsilon \in [0, \epsilon_u]$. Therefore, noting that I is of class C^∞, which follows easily from the chain rule, we obtain that

$$I'(u_0) \cdot (u - u_0) = \lim_{\epsilon \searrow 0} \frac{I(u_0 + \epsilon(u - u_0)) - I(u_0)}{\epsilon} \tag{3.6}$$

$$= \lim_{\epsilon \searrow 0} \frac{I(u_\epsilon) - I(u_0)}{\epsilon} \geq 0 \quad \forall u \in K.$$

Let us compute the derivate $I'(u_0) \cdot v$ for any element $v \in U$. Putting $\vec{z} = DG(u_0) \cdot v$, then, from Theorem 2, we have that $\vec{z}$ satisfies:

$$\begin{cases} \dfrac{\partial \vec{z}}{\partial t} - \nu \Delta_x \vec{z} + (\vec{y}_u \cdot \nabla_x)\vec{z} + (\vec{z} \cdot \nabla_x)\vec{y}_u + \nabla_x p_v = \vec{\beta}\zeta \text{ in } \Omega_T, \\ \dfrac{\partial \zeta}{\partial t} - \kappa \Delta_x \zeta + \vec{y}_u \cdot \nabla_x \zeta + \vec{z} \cdot \nabla_x \tau_u = 0 \text{ in } \Omega_T, \\ \text{div}_x \vec{z} = 0 \text{ in } \Omega_T, \ \vec{z}(0) = 0 \text{ in } \Omega, \ \vec{z} = 0 \text{ on } \Sigma_T, \\ \zeta(0) = 0 \text{ in } \Omega, \ \zeta = 0 \text{ on } \Sigma_T^0, \ \partial_n \zeta = v \text{ on } \Sigma_T^1. \end{cases} \tag{3.7}$$

for some $p_v \in L^2([0,T], \dot{H}^1(\Omega))$ and $\zeta \in L^2([0,T], H^1(\Omega)) \cap C([0,T], L^2(\Omega))$. By using the chain rule we deduce that

$$\begin{aligned} I'(u_0) \cdot v &= \frac{\partial J}{\partial u}(u_0, \vec{y}_0) \cdot v + \frac{\partial J}{\partial \vec{y}}(u_0, \vec{y}_0) \circ DG(u_0) \cdot v \\ &= \frac{\partial J}{\partial u}(u_0, \vec{y}_0) \cdot v + \frac{\partial J}{\partial \vec{y}}(u_0, \vec{y}_0) \cdot \vec{z} \\ &= N \int_{\Sigma_T^1} u_0(x,t) v(x,t) d\sigma dt \\ &+ \int_0^T \left(\int_\Omega |\nabla_x \times \vec{y}_0|^2 dx \right)^2 \left(\int_\Omega (\nabla_x \times \vec{y}_0) \cdot (\nabla_x \times \vec{z}) dx \right) dt \\ &= N \int_{\Sigma_T^1} u_0(x,t) v(x,t) d\sigma dt \\ &+ \int_0^T \left(\int_\Omega \|\nabla_x \times \vec{y}_0\|^4_{L^2(\Omega)^3} [\nabla_x \times (\nabla_x \times \vec{y}_0)] \cdot \vec{z} \right) dx dt. \end{aligned} \tag{3.8}$$

Since $\vec{y}_0 \in H^{2,1}(\Omega)^3$, it follows easily that the right hand side of evolution equation (3.4) belongs to $L^2([0,T], L^2(\Omega)^3)$. Then we deduce from Casas (1993B), Theorem 2.1, with $\vec{w} = \vec{e} = \vec{y}_0$, the existence of a unique $\vec{\varphi}_0 \in H^{2,1}(\Omega_T)^3 \cap C([0,T], Y_0)$ satisfying (3.4) for some $\tau_0 \in L^2([0,T], H^1(\Omega)) \cap C([0,T], L^2(\Omega))$ and some $\pi_0 \in L^2([0,T], H^1(\Omega))$, unique up to the addition of a distribution in $(0,T)$. Using now (3.4) and (3.7) and integrating by parts, we obtain from (3.8) that

$$I'(u_0) \cdot v = N \int_{\Sigma_T^1} u_0 v \, d\sigma \, dt - \int_0^T \int_\Omega \tau_0 \nabla_x \psi_0 \cdot \vec{z} \, dx \, dt$$

$$+ \int_0^T \int_\Omega \left[-\frac{\partial \vec{\varphi}_0}{\partial t} - \nu \Delta_x \vec{\varphi}_0 - (\vec{y}_0 \cdot \nabla_x)\vec{\varphi}_0 + (\nabla_x \vec{y}_0)^T \vec{\varphi}_0 \right] \cdot \vec{z} \, dx \, dt$$

$$= N\int_{\Sigma_T^1} u_0 v\, d\sigma\, dt + \int_0^T\int_\Omega \vec{z}\cdot\nabla_x\tau_0\psi_0\, dx\, dt$$

$$+\int_0^T\int_\Omega\left[\frac{\partial\vec{z}}{\partial t} - \nu\Delta_x\vec{z} + (\vec{y}_0\cdot\nabla_x)\vec{z} + (\vec{z}\cdot\nabla_x)\vec{y}_0\right]\cdot\vec{\varphi}_0\, dx\, dt$$

$$= N\int_{\Sigma_T^1} u_0 v\, d\sigma\, dt + \int_0^T\int_\Omega\left[\vec{\beta}\cdot\vec{\varphi}_0\zeta + \vec{z}\cdot\nabla_x\tau_0\psi_0\right]\, dx\, dt$$

$$= N\int_{\Sigma_T^1} u_0 v\, d\sigma\, dt + \int_0^T\int_\Omega \vec{z}\cdot\nabla_x\tau_0\psi_0\, dx\, dt$$

$$+\int_0^T\int_\Omega\left[-\frac{\partial\psi_0}{\partial t} - \kappa\Delta_x\psi_0 - \vec{y}_0\cdot\nabla_x\psi_0\right]\zeta\, dx\, dt$$

$$= N\int_{\Sigma_T^1} u_0 v\, d\sigma\, dt + \int_0^T\int_\Omega \vec{z}\cdot\nabla_x\tau_0\psi_0\, dx\, dt + \int_{\Sigma_T^1}\psi_0 v\, d\sigma\, dt$$

$$+\int_0^T\int_\Omega\left[\frac{\partial\zeta}{\partial t} - \kappa\Delta_x\zeta + \vec{y}_0\cdot\nabla_x\zeta\right]\psi_0\, dx\, dt = \int_{\Sigma_T^1}(Nu_0 + \psi_0)v\, d\sigma\, dt.$$

Finally, the expression obtained for $I'(u_0)\cdot v$, combined with (3.6), implies (3.5).

■

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