Control and Cybernetics

vol. **23** (1994) No. 4

Controllable approximations of certain boundary control problems

by

R. Datko

Department of Mathematics
Georgetown University
Washington, DC 20057
USA

We describe a Galerkin approximation scheme for a general class of conservative infinite dimensional control systems with controls distributed on the boundary of a compact set in R^n. Each approximate system is controllable, has a simple structure and is free of certain anomalies which are present in the original systems.

Introduction

There is a general class of elastic control problems which is represented by linear hyperbolic partial differential equations with nonhomogeneous boundary values. The setting is a bounded open set, Ω, in R^n and the control action occurs on the boundary, Γ, of Ω. These problems purport to model the control of structures such as strings and beams. The theory is mathematically complete in the context in which these problems are posed (see e.g. Lions, 1988). However vis à vis other control systems it has several flaws which militate against its use in practical design. We mention three: (i) The uncontrolled mathematical models generate C_0-groups on the spaces of their initial conditions. However, equally, and in many instances more appropriate models, have uncontrolled structures which generate analytic semigroups on the spaces of their initial conditions (see e.g. Sakawa, 1984). The mathematical consequence of this is that the first class of control problems is completely controllable, Lions (1988), whereas the second class is only approximately controllable (Triggiani, 1975). (ii) There is no theory of complete controllability for systems with bounded control inputs. In fact such a theory is impossible. The best one can hope for in this direction is an approximate control theory which may be inferred from the results of this paper. (iii) The practical goal of most elastic control problems is some form of stabilization (MacMartin, Hall, 1991). In the case of the systems under discussion this is achieved by linear feedback controls which involve velocity components (Lions, 1988). Unfortunately the resulting dynamical systems are unstable for any time delay in the feedback (Datko, 1991). It is true that these instabilities occur in the very high frequencies (see e.g. Datko, You, 1991) where the mathematical models probably do not represent physical reality. However, the mathematical theory depends on the infinite dimensional nature of the models and is intrinsic to the development of the theory (Lions, 1988). For example, if one were interested only in controlling a finite number of frequencies the standard finite dimensional theory would suffice.

Indeed, practical stabilization of flexible structures is described in the language of finite dimensional linear control theory (see e.g. Balas, 1982; MacMartin, Hall, 1991). A major reason for this is that, although flexible structures are often mathematically infinite dimensional, they are nonlinear and reasonable infinite dimensional models are not available. Moreover in practice it usually suffices to control only a finite number of frequencies, which can often be described by systems of the type (2.4) in this paper (see e.g. Balas, 1982). Consequently the principal goal of this paper is to indicate how elastic systems of the type described by the system (2.13) in Section 2 may be projected onto finite dimensional models of the form (2.4) which are controllable.

The resulting projected system has a relatively simple structure and we show in Section 4 how to exploit this to compute ϵ–approximate controllers with arbitrarily small L_∞ or L_2 bounds. In the same section we also show the extent to which a common feedback stabilizer for the system (2.4) is robust with respect to small time delays. This is an important property since in practice the active (as opposed to passive) stabilizers are implemented by microprocessors and this may result in small delays in these stabilizers.

The paper has the following structure. Section 1 is devoted to notational conventions and some properties of Finite Laplace Transforms, which are needed in Section 4. Section 2 develops some controllability properties for second–order conservative control systems. We also show how a specific projected distributed parameter control system with distributed controls on the boundary can be reduced to a lumped conservative system, and that the number of controls is determined by the largest multiplicity of any repeated eigenvalues in the corresponding homogeneous system. Section 3 contains three examples of control systems which are amenable to the treatment in Section 2. The contents of Section 4 have been alluded to above. Section 5 is a brief discussion of some aspects of vibrational control systems governed by second–order partial or ordinary differential equations.

1. Preliminaries

1. N will denote the set of positive integers.
2. R^+ will denote the set of nonnegative real numbers, R^n the set of real n-vectors, C^n the set of complex n-vectors and C the complex numbers.

3. The inner product on C^n is given by $\langle \cdot, \cdot \rangle$.
4. (a) $M(n,m)$ will denote the set of real matrices with n rows and m columns.
   (b) I_n will denote the identity matrix in $M(n,n)$.
   (c) $diag[\lambda_1 I_{k_1}, \ldots, \lambda_\ell I_{k_\ell}]$ will denote a diagonal matrix whose first k_1 diagonal elements are λ_1, whose second k_2 diagonal elements are λ_2, etc.
   (d) If A is an $n \times n$ matrix, $\det A$ will denote its determinant and $r(A)$ its spectral radius.
   (e) If B is an $m \times n$ matrix, B^* will denote its conjugate transpose. If b and c are n-vectors we may sometimes denote the inner product by $\langle b, c \rangle = b^* c$.
   (f) If $B \in M(n,m)$ we denote

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_\ell \end{pmatrix}$$

a partition of B where $B_j \in M(k_j, m)$ and $\sum_{j=1}^{\ell} k_j = n$.

5. (a) Ω will denote an open, bounded set in R^n and Γ its boundary. We assume Ω is a Green–type region. That is if v and w are defined on $\Omega \cup \Gamma$ and have second partial derivatives on Ω and first partial derivatives on Γ then

$$\int_\Omega [v(x)\Delta w(x) - w(x)\Delta v(x)]dx = \int_\Gamma [v(\sigma)\frac{\partial w}{\partial \nu}(\sigma) - w(\sigma)\frac{\partial v}{\partial \nu}(\sigma)]d\sigma,$$

where $d\sigma$ denotes the Lebesgue surface measure on Γ, dx the Lebesgue measure on Ω and $\frac{\partial w}{\partial \nu}$ is the outward normal derivative on Γ.
   (b) The sets of real L_2-integrable functions on Ω and on Γ form Hilbert spaces. We respectively denote their inner products by

$$\langle \phi, \psi \rangle = \int_\Omega \phi(x)\psi(x)dx$$

and

$$\langle v, q \rangle = \int_\Gamma v(\sigma)q(\sigma)d\sigma.$$

(Notice the same symbol for inner product is used for these spaces as for C^n. However this should cause no difficulty in the sequel, since the context will dictate which space is being considered).

6. (a) Let $f : \Omega \cup \Gamma \times R^+ \to R$. Then

$$\hat{f}(x,\lambda) = \int_0^\infty e^{-\lambda t} f(x,t)\, dt, \quad x \in \Omega \cup \Gamma, \tag{1.1}$$

provided there exists λ_0 in R such that (1.1) converges absolutely for $Re\,\lambda > \lambda_0$. The inverse Laplace transform of $\hat{f}(x,\lambda)$ is

$$f(x,t) = \mathcal{L}^{-1}(\hat{f}(x,\lambda))(t). \tag{1.2}$$

(b) If $0 < T < \infty$ then the Finite Laplace Transform (F.L.T.) of $f(x,t)$, $x \in \Omega \cup \Gamma$, $t \in R^+$, is defined by

$$\hat{f}_T(x,\lambda) = \int_0^T e^{-\lambda t} f(x,t)\, dt. \tag{1.3}$$

In the sequel we shall sometimes omit the subscript T on the left hand side of (1.3) if we believe the meaning is obvious.

The following theorem may be found in Doetsch (1956).

THEOREM 1.1 *The F.L.T. $\hat{f}_T(x,\lambda)$ is for fixed $x \in \Omega \cup \Gamma$ an entire analytic function of λ and is L_2-integrable in λ over the imaginary axis.*

2. Controllability considerations

The following is a variant of the Kalman controllability condition for linear autonomous control systems. The proof is omitted since it is an immediate consequence of the original version.

LEMMA 2.1 *Let $A \in M(n,n)$ and $B \in M(n,m)$, then the system*

$$\dot{x} = Ax + Bu \tag{2.1}$$

is controllable if and only if for any polynomial function

$$f(\lambda) = \sum_{j=0}^{r} a_j \lambda^j \tag{2.2}$$

the condition

$$f(A)B = 0 \ \ \textit{implies} \ \ f(A) = 0 \tag{2.3}$$

where $f(A)$ is the matrix polynomial associated with $f(\lambda)$.

The next lemma may be found in Zabczyk (1991), Zadanie 1.8, p. 25. However since no reference to its proof is given we supply one.

LEMMA 2.2 *Let $A \in M(n,n)$ and $B \in M(n,m)$. Then the system*

$$\ddot{x} = Ax + Bu \tag{2.4}$$

is controllable if and only if the system

$$\dot{z} = Az + Bu \tag{2.5}$$

is controllable.

PROOF. System (2.4) has the first order representation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0, & A \\ I_n, & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ Bu \end{pmatrix} = \hat{A} \begin{pmatrix} x \\ y \end{pmatrix} + \hat{B}u. \tag{2.6}$$

Any polynomial $g(\lambda)$ may be expressed in the form

$$g(\lambda) = g_1(\lambda^2) + \lambda g_2(\lambda^2), \tag{2.7}$$

where g_1 and g_2 are polynomials. A simple induction argument then shows that $g(\hat{A})$, where

$$\hat{A} = \begin{pmatrix} 0, & A \\ I_n & 0 \end{pmatrix}, \tag{2.8}$$

has the structure

$$\begin{pmatrix} g_1(A)\,, & g_2(A) \\ Ag_2(A)\,, & g_1(A) \end{pmatrix} = g(\hat{A}) \tag{2.9}$$

then,

$$g(\hat{A})\hat{B} = \begin{pmatrix} g_2(A)B \\ g_1(A)B \end{pmatrix}. \tag{2.10}$$

(i) Thus, suppose (2.4) is controllable, but (2.5) is not. Then we can find an even polynomial
$$g(\lambda) = g_1(\lambda^2)$$
such that
$$g(A) \neq 0, \qquad g(A)B = 0 \tag{2.11}$$
which by (2.9) and (2.11) implies $g(\hat{A}) \neq 0$, but $g(\hat{A})\hat{B} = 0$, a contradiction.

(ii) Assume (2.5) is controllable, but (2.4) is not. Then there exists polynomials g, g_1 and g_2 such that
$$g(\lambda) = g_1(\lambda^2) + \lambda g_2(\lambda^2)$$
and
$$g(\hat{A}) \neq 0, \text{ but.} g(\hat{A})\hat{B} = 0$$
This implies by (2.11) that either $g_1(A) \neq 0$ or $g_2(A) \neq 0$ and $g_1(A)B = g_2(A)B = 0$ which contradicts the controllability of (2.5).

We consider a control system of the form (2.4), where

$$\begin{bmatrix} A & = & diag[-\lambda_1^2 I_{k_1}, \dots, -\lambda_r^2 I_{k_r}] \\ B & = & \begin{pmatrix} B_1 \\ \vdots \\ B_r \end{pmatrix}, B_j \in M(k_j, m) \end{bmatrix}, \tag{2.12}$$

$\sum_{j=1}^{r} k_j = n$ and $k_j \leq m$ for all j.

ASSUMPTION 2.3 *We assume that matrices A and B in (2.12) have the following properties*

(i) $\lambda_j \neq \lambda_k$ if $j \neq k$ and $\lambda_j \neq 0$ for all j

(ii) Each B_j can be partitioned as follows

$$B_j = [B_{j1}, B_{j2}],$$

where $B_{j1} \in M(k_j, k_j)$ and $\det(B_{j1}) \neq 0$.

THEOREM 2.4 *Let Assumption 2.3 be satisfied, then the system (2.4) is controllable.*

PROOF. By Lemma 2.2 we only need to prove the system (2.5) is controllable. Thus assume (2.5) is not controllable. Then there exists a polynomial $f(\lambda)$ such that $f(-\lambda_j^2) \neq 0$ for at least one λ_j and such that

$$f(-\lambda_j^2)B_j = f(-\lambda_j^2)[B_{j1}, B_{j2}] = 0.$$

This implies $\det(B_{j1}) = 0$ which contradicts (ii) in Assumption 2.3.

We now consider an infinite dimensional control system of the type

$$\ddot{x}_j(t) + \lambda_j^2 x_j(t) = \int_\Gamma q_j(\sigma)\mu(\sigma, t)\, d\sigma, \quad j \in N, \tag{2.13}$$

where $\lambda_j^2 > 0$ and the functions $q_j, q_{j+1}, \dots, q_{j+r}$ are linearly dependent on Γ if $\lambda_j^2 = \lambda_{j+1}^2 = \dots = \lambda_{j+r}^2$.

REMARK 2.5 *Systems of the type (2.13) arise from many standard boundary value problems in elastic control involving vibrating structures (see e.g. Lions, 1988). The original system is described by partial differential equations. The form (2.13) is arrived at by considering Fourier expansions in appropriate Sobolev spaces, where solutions of (2.13) are considered only in some weak sense. Specific examples are supplied in Section 3. In this section we are only concerned with a finite dimensional Galerkin approximation of (2.13). Thus the underlying Sobolev space is irrelevant and need not be explicitly specified.*

We consider only the first n equations in (2.13). This reduces the system to one of the form

$$\ddot{x}(t) = Ax(t) = \int_\Gamma B(\sigma)\mu(\sigma, t)d\sigma, \tag{2.14}$$

where A satisfies (i) in Assumption 2.3 and

$$B(\sigma) = \begin{pmatrix} q_1(\sigma) \\ \vdots \\ q_n(\sigma) \end{pmatrix}. \tag{2.15}$$

Unfortunately the controls in (2.14) are distributed and hence lie in an infinite dimensional space, unless of course Γ consists of discrete points in which case the measure $d\sigma$ in (2.14) is atomic. This last condition occurs in elastic systems such as one-dimensional strings or Euler–Bernoulli beams. In this case the system (2.14) can easily be shown to satisfy Assumption 2.3 and hence by Theorem 2.4 is controllable. We wish to consider the more complex situation which is covered by the following assumption.

ASSUMPTION 2.5 *(i) The measure, $d\sigma$, in (2.14) is nonatomic and if $\lambda_j = \lambda_{j+1} = \ldots \lambda_{j+r}$ in A then $q_j, \ldots, q_{j+r}$ are linearly independent on Γ.*
(ii) A in (2.15) has r distinct eigenvalues $-\lambda_1^2, \ldots, -\lambda_j^2$ with respective indices of multiplicity $k_1, \ldots, k_r$ and $k_1 \geq k_j$ for $j = 2, \ldots, r$.

THEOREM 2.6 *Let Assumption 2.5 be satisfied. There exist constants $\{a_{j\ell}\}$, $1 \leq j \leq k_1$, $1 \leq \ell \leq n$, such that the system (2.14) is controllable for controls of the form*

$$\mu(\sigma, t) = \sum_{j=1}^{k_1} \sum_{\ell=1}^{n} a_{j-\ell}\mu_j(t)q_\ell(\sigma), \tag{2.16}$$

Thus (2.14), with controls of the type (2.16) reduces to a system of the form (2.4) which satisfies Assumption 2.3.

PROOF. We proceed in two steps.

(i) $-\lambda_1^2$ has multiplicity k_1 and thus $q_1, \ldots q_{k_1}$, are linearly independent on Γ. Consequently we can find functions $v_1, \ldots, v_{k_1}$ on Γ of the form

$$v_j(\sigma) = \sum_{k=1}^{k_1} v_{kj} q_k(\sigma), \quad 1 \leq j \leq k_1, \tag{2.17}$$

such that

$$\langle v_j, q_k \rangle = \int_\Gamma v_j(\sigma) q_k(\sigma) d\sigma = \delta_{jk} \text{ (the Kronecker delta).} \tag{2.18}$$

We repeat this procedure for each repeated eigenvalue. That is, we can find $\{v_j\}$, $k_1 + 1 \leq j \leq k_1 + k_2$, which are linear combinations of $q_j, k_1 + 1 \leq j \leq k_1 + k_2$ such that

$$\langle v_j, q_k \rangle = \delta_{jk}, k_1 + 1 \leq j, k \leq k_1 + k_2, \tag{2.19}$$

etc.

(ii) We choose nonzero real numbers $\epsilon_1, \ldots, \epsilon_r$ as follows.
We consider controls $\mu(\sigma, t)$ of the form

$$\begin{aligned} \mu(\sigma,t) &= \epsilon_1 \sum_{j=1}^{k_1} \mu_j(t) v_j(\sigma) + \ldots + \\ & \dot{\epsilon}_r \sum_{j=k_{r-1}+1}^{k_{r-1}+k_r} \mu_{j-k_{r-1}}(t) v_j(\sigma). \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.13) we obtain a control system of the form (2.4) which satisfies (i) in Assumption 2.3, where

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_r \end{pmatrix} \tag{2.21}$$

and each B_j, $1 \le j \le r$, has the form

$$\begin{bmatrix} B_1 = \epsilon_1 I_{k_1} + \epsilon_2 \hat{B}_{12}, \ldots + \epsilon_r \hat{B}_{1r} \\ B_2 = [\epsilon_2 I_{k_1} + \epsilon \hat{B}_{21} + \ldots + \epsilon_r \hat{B}_{2r}, B_{12}] \\ \vdots \\ B_r = [\epsilon_r I_{k_r} + \epsilon_1 \hat{B}_{r1} + \ldots + \epsilon_{r-1} B_{r,r-1}, B_{r2}] \end{bmatrix}. \tag{2.22}$$

We now choose $\{\epsilon_j\}$, $1 \le j \le r$, such that the matrices in (2.22) satisfy (ii) in Assumption 2.3. That is

$$\begin{bmatrix} \det[\epsilon_1 I_{k_1} + \ldots + \epsilon_r \hat{B}_{1r}] \neq 0 \\ \vdots \\ \det[\epsilon_r I_{k_r} + \ldots \epsilon_{r-1} \hat{B}_{r,r-1}] \neq 0 \end{bmatrix}. \tag{2.23}$$

This completes the proof of the theorem.

Theorem 2.6 has a tedious combinatorial statement and proof. In the example given below we shall present a specific model to illustrate its statement and proof.

EXAMPLE 2.7 *Consider the system*

$$\begin{bmatrix} \ddot{x}_1 + \lambda_1^2 x_1 = \int_\Gamma q_1(\sigma)\mu(\sigma,t)d\sigma \\ \ddot{x}_2 + \lambda_1^2 x_2 + \int_\Gamma q_2(\sigma)\mu(\sigma,t)d\sigma \\ \ddot{x}_3 + \lambda_3^2 x_3 + \int_\Gamma q_3(\sigma)\mu(\sigma,t)d\sigma \\ \ddot{x}_4 + \lambda_4^2 x_4 + \int_\Gamma q_4(\sigma)\mu(\sigma,t)d\sigma \end{bmatrix}, \tag{2.24}$$

where $\lambda_1^2 \neq \lambda_3^2$ and the pairs (q_1, q_2) and (q_3, q_4) are linearly independent over Γ. We wish to construct functions v_1, v_2 which are linear combinations of q_1

and q_2, and v_3, v_4 which are linear combinations of q_3 and q_4 such that

$$\langle v_1, q_1\rangle = \langle v_2, q_2\rangle = \langle v_3, q_3\rangle = \langle v_4, q_4\rangle = 1$$

$$\langle v_1, q_2\rangle = \langle v_2, q_1\rangle = \langle v_3, q_4\rangle = \langle v_4, q_3\rangle = 0$$

For example v_1 has the form

$$v_1 = \frac{\langle q_2, q_2\rangle q_1 - \langle q_1, q_2\rangle q_2}{\langle q_1, q_1\rangle\langle q_2, q_2\rangle - \langle q_1, q_2\rangle^2}\,.$$

Similar expressions hold for v_2, v_3 and v_4. We then look for ϵ_1 and ϵ_2 such that

$$\mu(\sigma, t) = \mu_1(t)[\epsilon_1 v_1(\sigma) + \epsilon_2 v_3(\sigma)] + \mu_2(t)[\epsilon_1 v_2(\sigma) + \epsilon_2 v_4(\sigma)]$$

leads to a system of the type (2.4) satisfying Assumption 2.3. This results in a system (2.4) where B has the form

$$\begin{pmatrix} \epsilon_1 + \epsilon_2\langle v_3, q_1\rangle & , & \epsilon_2\langle v_4, q_1\rangle \\ \epsilon_2\langle v_3, q_2\rangle & , & \epsilon_1 + \epsilon_2\langle v_4, q_2\rangle \\ \epsilon_2 + \epsilon_1\langle q_3, v_1\rangle & , & \epsilon_2\langle v_2, q_3\rangle \\ \epsilon_1\langle q_4, v_1\rangle & , & \epsilon_2 + \epsilon_1\langle v_2, q_4\rangle \end{pmatrix} \begin{pmatrix} \mu_1(t) \\ \mu_2(t) \end{pmatrix}.$$

Clearly one may choose ϵ_1 and ϵ_2 to satisfy Assumption 2.3.

3. Examples of projections

EXAMPLE 3.1 *Let $P : \Omega \to R^+$ be continuous. We consider*

$$w_{tt} = \Delta w - Pw, \quad x \in \Omega, \quad t > 0 \tag{3.1}$$

$$w(x, 0) = \phi(x), \quad w_t(x, 0) = \psi(x), \quad x \in \Omega \tag{3.2}$$

$$\frac{\partial w}{\partial \nu}(\sigma, t) = \mu(\sigma, t), \quad \sigma \in \Gamma, \quad t > 0. \tag{3.3}$$

The function $\mu(\cdot,\cdot)$ in (3.3) is the control and is assumed to be Laplace transformable with respect to t. The functions ϕ and ψ in (3.2) lie, respectively, in $H_0^1(\Omega)$ and $L_2(\Omega)$.

The Laplace transform of (3.1)-(3.3) transforms the system into

$$\lambda^2 \hat{w}(x, \lambda) = \Delta w(x, \lambda) - P(x)\hat{w}(x, \lambda) + \lambda\phi(x) + \psi(x), \tag{3.4}$$

$$\frac{\partial \hat{w}}{\partial \nu}(x, \lambda) = \hat{\mu}(\sigma, \lambda). \tag{3.5}$$

Let $\{q_j\}$ be the orthonormal sequence of functions defined on $\Omega \cup \Gamma$ satisfying the conditions

$$\Delta q_j(x) - P(x)q_2(x) = -\lambda_j^2 q_j(x), \quad x \in \Omega \tag{3.6}$$

$$\frac{\partial q_j}{\partial \nu}(\sigma) = 0, \quad \sigma \in \Gamma \tag{3.7}$$

$$\langle q_j, q_k \rangle = \int_\Omega q_j(x) q_k(x) dx = \delta_{jk} \quad \textit{(the Kronecker delta)} \tag{3.8}$$

Notice, that since $P(x) > 0$ on Ω, $\lambda_j^2 > 0$ for all j. Using functions $\{q_j\}$ defined by (3.6)–(3.8) we expand $\hat{w}(x,\lambda), \phi(x)$ and $\psi(x)$ in the Fourier series

$$\left[\begin{array}{lcl} \hat{w}(x,\lambda) & = & \sum_{j=1}^{\infty} \hat{a}_j(\lambda) q_j(x), \quad \phi(x) = \sum_{j=1}^{\infty} \phi_j q_j(x), \\ \psi(x) & = & \sum_{j=1}^{\infty} \psi_j q_j(x) \end{array}\right], \tag{3.9}$$

where the $\{\hat{a}_j(\lambda)\}$ are to be determined. Using (3.9) and Green's Theorem we obtain relationships for the $\{\hat{a}_j\}$ in the form

$$\int_\Omega [v_j(x)\Delta \hat{w}(x,\lambda) - \hat{w}(x,\lambda)\Delta v_j(x)]dx \tag{3.10}$$

$$= \int_\Omega v_j(x)[\lambda^2 \hat{w}(x,\lambda) - \lambda\phi(x) - \psi(x) + \lambda_j^2 \hat{w}(x,\lambda)]dx$$

$$= \lambda^2 \hat{a}_j(\lambda) - \lambda\phi_j - \psi_j + \lambda_j^2 \hat{a}_j(\lambda) = \int_\Gamma q_j(\sigma)\hat{\mu}(\sigma,\lambda)d\sigma.$$

That is,

$$\hat{a}_j(\lambda) = \frac{1}{\lambda^2 + \lambda_j^2}[\lambda\phi_j + \psi_j + \int_\Gamma q_j(\sigma)\hat{\mu}(\sigma,\lambda)d\sigma], \quad j \in N. \tag{3.11}$$

If

$$a_j(t) = \mathcal{L}^{-1}(\hat{a}_j(\lambda))(t), \quad j \in N, \tag{3.12}$$

then (3.11) is equivalent to the infinite set of second-order ordinary differential equations

$$\left[\begin{array}{l} \ddot{a}_j(t) + \lambda_j^2 a_j(t) = \int_\Gamma q_j(\sigma)\mu(\sigma,t)\, d\sigma, \\ a_j(0) = \phi_j, \quad \dot{a}_j(0) = \psi_j, \quad j \in N, \end{array}\right] \tag{3.13}$$

i.e. a system of the form (2.13).

EXAMPLE 3.2 *This is a multidimensional Euler–Bernoulli beam. Let $\Omega \subset R^n$, $n \geq 2$, and consider the system*

$$w_{tt} + \Delta^2 w = 0, \quad x \in \Omega, \quad t > 0, \tag{3.14}$$

$$w(x,0)=\phi(x), \quad w_t(x,0)=\psi(x), \quad x\in\Omega, \tag{3.15}$$

$$w(0,t)=0, \quad \frac{\partial w}{\partial\nu}(\sigma,t)=\mu(\sigma,t), \quad \sigma\in\Gamma, \quad t>0. \tag{3.16}$$

The functions ϕ and ψ in (3.15) are respectively assumed to lie in $L_2(\Omega)$ and $H^{-2}(\Omega)$. The Laplace transform of (3.14) leads to the equation

$$-\lambda^2\hat{w}(x,\lambda)+\lambda\phi(x)+\psi(x)=\Delta^2\hat{w}(x,\lambda), \quad x\in\Omega. \tag{3.17}$$

We expand $\hat{w}(\cdot,\lambda)$, ϕ and ψ in terms of the orthonormal sequence $\{q_j\}$ defined by

$$\Delta^2 q_j(x)=\lambda_j^2 q_j(x), \quad x\in\Omega, \tag{3.18}$$

$$q_j(\sigma)=\frac{\partial q_j}{\partial\nu}(\sigma)=0, \quad v\in\Omega \tag{3.19}$$

$$\langle q_j,q_k\rangle=\int_\Omega q_j(\sigma)q_k(\sigma)d\sigma=\delta_{jk}, \tag{3.20}$$

and obtain (3.9). We then apply Green's Formula to the integrals

$$\int_\Omega [q_j(x)\Delta^2\hat{w}(x,\lambda)-\Delta q_j(x)\Delta\hat{w}(x,\lambda)]\,dx$$

and

$$-\int_\Omega [\hat{w}(x,\lambda)\Delta^2 q_j(x)-\Delta q_j(x)\Delta\hat{w}(x,\lambda)]\,dx$$

and add the result to obtain the following expressions for the $\{\hat{a}_j\}$ in (3.9)

$$\hat{a}_j(\lambda)=\frac{1}{\lambda^2+\lambda_j^2}[\lambda\phi_j+\psi_j-\int_\Gamma(\Delta q_j(\sigma))\hat{\mu}(\sigma,\lambda)d\sigma], \quad j\in N. \tag{3.21}$$

The equations (3.21) are equivalent to the ordinary differential equations

$$\ddot{a}_j(t)+\lambda_j^2 a_j(t)=-\int_\Gamma(\Delta q_j(\sigma))\mu(\sigma,t)\,dv, \;\; a_j(0)=\phi_j, \;\; \dot{a}_j(0)=\psi_j \tag{3.22}$$

which again reduce to the type (2.13).

The next example is an explicit boundary control problem for the wave equation on R^2. In this example there are an infinite number of multiple eigenvalues. The purpose of this example is to show that the elaborate construction used in proving Theorem 2.4 can in practice often be bypassed.

EXAMPLE 3.3 *We consider the two-dimensional wave equation*

$$w_{tt}(x,y,t) = w_{xx}(x,y,t) + w_{yy}(x,y,t),\ 0 < x < \pi,\ 0 < y < \pi,\ t > 0, \quad (3.23)$$

$$w(x,y,0) = \phi(x,y), w_t(x,y,0) = \psi(x,y) \quad 0 < x < \pi, \quad 0 < y < \pi, \quad (3.24)$$

$$\left[\begin{array}{lcl} w(x,0,t) & = & w(0,y,t) = 0, \\ w_x(\pi,y,t) & = & \mu_1(y,t), w_y(x,\pi,t) = \mu_2(x,t) \end{array} \right]. \quad (3.25)$$

System (3.23)-(3.24) is completely controllable in the context of elastic control theory (see e.g. Lions, 1988). Applying the methods of the previous two examples we obtain the orthonormal sequence of functions defined on $[0,\pi] \times [0,\pi]$ *by*

$$\left[\begin{array}{l} q_{kn}(x,y) = \frac{2}{\pi}[\sin(k + \frac{1}{2})x][\sin(n + \frac{1}{2})y] \\ k \in N, \quad y \in N. \end{array} \right] \quad (3.26)$$

Using (3.26) as an orthonormal set of vectors we easily reduce the system (3.23)–(3.25) to the infinite system of second-order ordinary differential equations

$$\left[\begin{array}{l} \ddot{x}_{jn}(t) + [(\frac{1}{2} + j)^2 + (\frac{1}{2} + k)^2]x_{jn}(t) \\ = \int_0^\pi q_{jn}(\pi,y,t)\mu_1(\pi,y,t)dy + \\ \int_0^\pi q_{jn}(x,\pi,t)\mu_2(x,\pi,t)dx, \quad j \in N, n \in N. \end{array} \right] \quad (3.27)$$

Notice that the controls μ_1 *and* μ_2 *in (3.27) are arbitrary functions defined on two edges of the boundary. We shall now specialize them to*

$$\mu_1(\pi,y,t) = v_1(t), \mu_2(x,\pi,t) = v_2(t). \quad (3.28)$$

The system (3.27) with the controls (3.28) reduces to the lumped system

$$\left[\begin{array}{l} \ddot{x}_{jn}(t) + [(\frac{1}{2} + j)^2 + (\frac{1}{2} + n)^2]x_{jn}(t) \\ = \frac{2}{\pi}[\frac{(-1)^j v_1(t)}{\frac{1}{2} + n} + \frac{(-1)^n v_2(t)}{\frac{1}{2} + j}] \end{array} \right]. \quad (3.29)$$

Notice that if $j \neq n$ *the repeated eigenvalues*

$$(\frac{1}{2} + j)^2 + (\frac{1}{2} + n)^2 = \lambda_{jn}^2 = \lambda_{nj}^2 \quad (3.30)$$

lead to the pairs of equations

$$\left(\begin{array}{c} \ddot{x}_{jn} \\ \ddot{x}_{nj} \end{array} \right) = -\lambda_{jn}^2 \left(\begin{array}{c} x_{jn} \\ x_{nj} \end{array} \right) + \frac{2}{\pi} \left(\begin{array}{cc} \frac{(-1)^j}{\frac{1}{2} + n}, & \frac{(-1)^n}{\frac{1}{2} + j} \\ \frac{(-1)^n}{\frac{1}{2} + j}, & \frac{(-1)^j}{\frac{1}{2} + n} \end{array} \right) \left(\begin{array}{c} v_1 \\ v_2 \end{array} \right) \quad (3.31)$$

which are controllable since the control matrix has a nonzero determinant. Thus the projections of (3.27) with controls of the type (3.28) lead to controllable systems.

4. Control and stabilization considerations

Let $A \in M(n,n)$ and $B \in M(n,m)$ satisfy (2.12) and assume

$$\ddot{x}(t) = Ax(t) + B\mu(t) \tag{4.1}$$

is controllable.

THEOREM 4.1 *If*

$$\mu = -B^*\dot{x}, \tag{4.2}$$

then the system

$$\ddot{x}(t) = Ax(t) - BB^*\dot{x}(t) \tag{4.3}$$

is uniformly exponentially stable.

PROOF. Each scalar component of (4.1) is of the form

$$\ddot{x}_j + \lambda_j^2 x_j = \sum_{k=1}^{m} b_{jk}\mu_k. \tag{4.4}$$

Thus, because of (4.2) we can write each component of (4.3) in the form

$$\ddot{x}_j + \lambda_j^2 x_j = -\sum_{k=1}^{m}(b_{jk})^2\dot{x}_j + \sum_{k\neq j=1}^{n} a_{jk}\dot{x}_k \tag{4.5}$$

If (4.3) is not uniformly exponentially stable there exists $x_0 \in C^n, x_0 \neq 0$, and $w \neq 0$ such that

$$x(t) = x_0 e^{iwt} \tag{4.6}$$

is a solution of (4.3). This implies

$$(w^2 I_n - A)x_0 = -iwBB^*x_0 \tag{4.7}$$

But then

$$\langle (w^2 I_n - A)x_0, x_0\rangle = -iw\langle B^*x_0, \quad Bx_0\rangle$$

which implies that

$$B^*x_0 = 0.$$

Thus $w^2 = \lambda_j^2$ for some j and x_0 is an eigenvector of the matrix A, which implies that all its components but the j^{th} must equal zero. But this in turn implies, when (4.6) is substituted into (4.5), that

$$\sum_{k=1}^{m}(b_{jk})^2 = 0.$$

Hence the system cannot be controllable, which is a contradiction.

We next look at some controllability properties for (4.1). Suppose

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \tag{4.8}$$

is a control which drives an initial value $(x(0), \dot{x}(0))$ of (4.1) to the origin in a finite time, T. Let $\hat{x}$ and $\hat{\mu}$ respectively denote the finite Laplace transforms of the solution $x(t)$ and the control $\mu(t)$. Because these are entire analytic functions a simple calculation shows that the components of these functions must satisfy the equations

$$\begin{bmatrix} i\lambda_j x_j(0) + \dot{x}_j(0) + \sum_{k=1}^{m} b_{jk}\hat{\mu}_k(i\lambda_j) = 0 \\ -i\lambda_j x_j(0) + \dot{x}_j(0) + \sum_{k=1}^{m} b_{jk}\hat{\mu}_k(-i\lambda_j) = 0 \end{bmatrix}. \tag{4.9}$$

Solving (4.9) for $x_j(0)$ and $\dot{x}_j(0)$ we obtain

$$\begin{bmatrix} x_j(0) = -\frac{1}{2_i\lambda_j}\sum_{k=1}^{m} b_{jk}(\hat{\mu}_k(i\lambda_j) - \hat{\mu}_k(-i\lambda_j)) \\ \dot{x}_j(0) = -\frac{1}{2}\sum_{k=1}^{m} b_{jk}(\hat{\mu}_k(i\lambda_j) + \mu_k(-i\lambda_j)) \end{bmatrix}. \tag{4.10}$$

But the right hand sides of (4.10) can be rewritten in the form

$$\begin{bmatrix} x_j(0) = \sum_{k=1}^{m}\int_0^T b_{jk}\frac{(\sin\lambda_j t)}{\lambda_j}\mu_k(t)dt \\ \dot{x}_j(0) = -\sum_{k=1}^{m}\int_0^T b_{jk}(\cos\lambda_j t)\mu_k(t)dt \end{bmatrix}. \tag{4.11}$$

Equations (4.11) are necessary and sufficient for the control of (4.1) to the origin in time T when the initial conditions are $(x(0), \dot{x}(0))$.

Suppose we are not interested in exact control to the origin but only approximate or ϵ-control, i.e. to an ϵ-neighborhood of the origin. Moreover suppose the time of control is less important than some uniform bound on the controllers. We shall indicate how to find explicit controls which perform these tasks. To illustrate we assume the system (4.1) is rank one, i.e.

$$B = b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{4.12}$$

We also assume the control is of the form

$$\mu(t) = \frac{1}{T}\sum_{j=1}^{n}[\alpha_j \cos\lambda_j t + \beta_j \sin\lambda_j y]. \tag{4.13}$$

Substituting (4.13) into (4.11) we obtain the equations

$$\left[\begin{array}{l} x_j(0) = \dfrac{1}{T}\displaystyle\int_0^T \frac{b_j}{\lambda_j}\sin\lambda_j t\left(\sum_{k=1}^{n}\alpha_k\cos\lambda_k t + \beta_k \sin\lambda_k t\right)dt \\ \dot{x}_j(0) = -\dfrac{1}{T}\displaystyle\int_0^T b_j\cos\lambda_j t\left(\sum_{k=1}^{n}\alpha_k\cos\lambda_k t + \beta_k\sin\lambda_k t\right)dt \end{array}\right. \tag{4.14}$$

For T sufficiently large one can solve (4.14) explicitly for $\{\alpha_j\}$ and $\{\beta_j\}$ since in the first integral in (4.14) the term

$$\left[\frac{1}{T}\int_0^T \beta_j\frac{b_j}{\lambda_j}\sin^2\lambda_j t\,dt = \frac{\beta_j b_j}{2\lambda_j^2 T}[\lambda_j T - \sin\lambda_j T\cos\lambda_j T] \right. \tag{4.15}$$

$$\left. \longrightarrow \frac{\beta_j b_j}{2\lambda_j} \text{ as } T\to\infty\right]$$

and in the second integral in (4.14) the term

$$\left[-\frac{1}{T}\int_0^T \alpha_j b_j\cos^2\lambda_j t\,dt = \frac{-\alpha_j b_j}{2\lambda_j T}[\lambda_j T + \sin\lambda_j T\cos\lambda_j T]\right. \tag{4.16}$$

$$\left. \longrightarrow \frac{-\alpha_j b_j}{2} \text{ as } T\to\infty\right],$$

while the remaining terms in both equations tend to zero as $T\to\infty$.

Thus for T sufficiently large the solutions of (4.14) are asymptotic to

$$\alpha_j = \frac{-2\dot{x}_j(0)}{b_j}, \quad \beta_j = \frac{2\lambda_j x_j(0)}{b_j}, \quad j = 1,\ldots,n, \tag{4.17}$$

which is ϵ-controllability. Moreover, since (4.17) does not depend on T, $\mu(t)$ in (4.13) has a bound of the form

$$|\mu(t)| \le \frac{1}{T}M, \tag{4.18}$$

where M is independent of T. The L_2 bound is

$$\|\mu\| = \left(\int_0^T |\mu(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{M}{T^{\frac{1}{2}}}. \tag{4.19}$$

We now consider the robustness of the stabilized systems (4.3) with respect to time delays in the stabilizer, i.e. in place of (4.2) we assume

$$-B^* \dot{x}(t-h), \quad h > 0.$$

The system (4.1) then becomes

$$\ddot{x}(t) = Ax(t) - BB^* \dot{x}(t-h), \quad h > 0. \tag{4.20}$$

Since (4.20) is uniformly exponentially stable for $h = 0$, the question arises : for what values of $h > 0$ does (4.20) remain uniformly exponentially stable? The answer to this is almost trivial. This is because simple estimates are available to determine the range of h for which stability holds. It is well known that there exists a smallest $h_0 > 0$ for which (4.20) has a nontrivial periodic solution of the form $x(t) = x_0 e^{-i\omega t}$, w-real, $\omega \neq 0$. This leads to the eigenequation

$$[-\omega^2 I - A - i\omega e^{-i\omega h_0} BB^*]x_0 = 0 \tag{4.21}$$

Since A and BB^* are symmetric this implies that $i\omega e^{-i\omega h_0}$ is real. Hence

$$e^{-i\omega h_0} = \begin{bmatrix} e^{-\frac{\pi}{2} i} \\ \text{or} \\ e^{-\frac{3}{2}\pi i} \end{bmatrix}. \tag{4.22}$$

Thus

$$h_0 \geq \frac{\pi}{2|\omega|}. \tag{4.23}$$

On the other hand equation (4.21) implies that

$$r[(\omega(\omega^2 I + A)^{-1} BB^*] \geq 1 \tag{4.24}$$

($r(A)$ in the spectral radius of A)

Since

$$r[\omega(\omega^* I + A)^{-1} BB^1] \to 0 \text{ as } |\omega| \to \infty \tag{4.25}$$

there exists $\omega_0 > 0$ such that

$$r[\omega(\omega^2 I + A)^{-1} BB^1] < 1 \tag{4.26}$$

for all $|\omega| > \omega_0$. Thus if

$$h \in \left[0, \frac{\pi}{2\omega_0}\right) \tag{4.27}$$

the system (4.20) is uniformly exponentially stable, which is a stability margin.

REMARK 4.2 *Since A is (4.1) is a diagonal matrix and*

$$r[\omega(\omega^2 I + A)^{-1}BB^*] < |\omega|\,|(\omega^2 I + A)^{-1}|\,|B|^2,$$

where $|\cdot|$ denotes any convenient matrix norm, an estimate for $\frac{\pi}{2\omega}$ is easy to obtain.

5. Discussion

REMARK 5.1 *The projection scheme we have presented for infinite dimensional distributed parameter systems of the type (2.13) is one of many possible choices. These are traditionally termed Reduced-Order-Models (ROM). Other choices are Finite Element Methods (FEM) which are not necessarily Galerkin methods. However, whatever the method, the ordinary differential equation format for the control system most often has the structure (2.6) and more specifically satisfies Assumption 2.3 (see e.g. Balas, 1982 or MacMartin, Hall, 1991).*

Controllability is sometimes determined by tests on the modal data (Balas, 1982), but the result presented in Zabczyk (1991) is in our opinion more efficient and applies to more general systems of the same type.

However the purpose of this paper was not to present another ROM method, but to show how certain distributed parameter control systems can be reduced in a direct manner to manageable computational systems.

REMARK 5.2 *As mentioned in the Introduction elastic models such as those represented by (2.13) or Examples 3.1, 3.2 or 3.3 have serious flaws when considered in toto. Moreover they are at best approximations to actual physical phenomena. Thus it seems to us that no merit, except mathematical complexity, is attached to studying the usual controllability and stabilization properties for these systems since they offer no particular insights into practical design. Indeed in the case of stabilization the results are a "reductio ad absurdum" when small time delays are permitted in the controls (see e.g. Botsema, de Vries, 1988; Datko, 1991 or Datko, 1993) — a reasonable engineering assumption.*

On the other hand ROM models have practical advantages without the liabilities of distributed parameter models of the type given by (2.13). We mention two: (i) There is a computational utilizable theory of ϵ-control or complete control from uniformly bounded control sets; (ii) Feedback stabilization is simple and is robust with respect to uncertainties in the matrices A and B in (2.4), and this stabilization is also robust with respect to small time delays in the stabilizer. In fact, explicit estimates can be obtained for the extent of this robustness.

REMARK 5.3 *Vibrating elastic system possess some light damping. Often the purpose of control is to enhance this damping. Thus in place of the system (2.4), where A is negative definite, we might consider*

$$\ddot{x}(t) = Ax(t) + \epsilon D\dot{x}(t) + B\mu(t), \tag{5.1}$$

where D is a semi–negative definite matrix and $\epsilon > 0$. It can easily be shown that Theorem 2.4 holds for (5.1) when ϵ is sufficiently small. This follows from the fact that finite dimensional controllability is robust with respect to matrix representation. In practice the damping term in (5.1) is slight (see e.g. Balas, Chu, Doyle, 1989). It might be viewed as the projection of an internally damped distributed model. For instance, in Example 3.3 we could replace Equation (3.23) by

$$w_{tt} = \Delta w + \epsilon \Delta w_t, \quad 0 < x < \pi, \quad 0 < y < \pi, \tag{3.23'}$$

and Equation (3.24) by

$$\begin{bmatrix} w(x,y,0) = \phi(x,y), w_t(x,y_0,0) = \psi(x,y) \\ \Delta w(x,y,0) = \tau(x,y) \end{bmatrix}. \tag{3.24'}$$

The resulting projected systems would then assume the form (5.1) and conform to a lightly damped model. But in the context of distributed control (3.23)$'$, (3.24)$'$, (3.25) *is a different animal than* (3.23), (3.24), (3.25). *For one thing it is only ϵ-controllable, since the corresponding homogeneous system generates an analytic semigroup (see e.g. Triggiani, 1975). However it also lacks robustness with respect to small time delays in the boundary controls (3.25) (see e.g. Datko, 1991). This is yet another indication that the distributed theory of boundary control for elastic systems requires a serious review.*

References

BALAS G.L., CHU E.C. AND DOYLE J.C. (1989) Vibration damping and robust control of the JPL/AFAL experiment using μ-synthesis, *Proceeding IEEE Conf. on Dec. and Control*, Tampa, Dec. 1989, 2689–2694.

BALAS M.G. (1982) Trends in large space structure control theory: fondest hopes, wildest dreams, *IEEE Trans. on Automatic Control*, vol. **27**, **#3**, June 1982, 522–535.

BOTSEMA J. AND DE VRIES S.A. (1988) Robustness of flexible structures against small time delays, *Proceedings IEEE Conf. on Dec. and Control*, Austin, Dec. 1988, 1647–1648.

DATKO R. AND YOU Y. (1991) Some second–order vibrating systems cannot tolerate small time delays in their damping, *J. of Optimization Theory and Appl.* **70**, 521–536.

DATKO R. (1991) Two questions concerning the boundary control of certain elastic systems, *J. Diff. Eqs.* **92**, 27–44.

DATKO R. (1993) Two examples of ill–posedness with respect to small time delays in stabilized elastic systems, *IEEE Trans. on Automatic Control*, vol. **38**, **#1**, Jan. 1993, 163–166.

DOETSCH G. (1956) *Handbuch der Laplace Transformation*, Vol. III, Birkhäuser Verlag.

LIONS J.-L. (1988) Exact controllability, stabilization and perturbations for distributed parameter systems, *SIAM Review* **30**, 1–68.

MACMARTIN D.G. AND HALL S.R. (1991) Structural control experiments using an $\mathcal{H}_\infty$ power flow approach, *Journal of Sound and Vibration* **148**, **#2**, July 1991, 223–241.

ROSS C.F. (1988) Active isolation of ship machinery platforms, *Proceedings IEEE Conf. on Dec. and Control*, Austin, Dec. 1988, 2045–2046.

SAKAWA Y. (1984) Feedback control of second-order evolution equations with damping, *SIAM J. Control Optim.* **22**, 343–361.

TRIGGIANI R.T. (1975) On the stabilization problem in Banach space, *J.Math. Anal. and Appl.* **52**, pp. 383–403.

ZABCZYK J. (1991) *Zarys Matematycznej Teorii Sterowania*, Wydawnictwo Naukowe PWN.