

## Contribution to dual variational principles for nonlinear elastic beams

by

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The aim of this contribution is the formulation of the complementary energy principle for two models of non linear compressed elastic beams. It is shown, that the dual problem for a simply supported moderately thick beam has the form of the anomalous dual principle (*inf sup* principle). Two dual problems are formulated for a compressed thin beam clamped at one end. General setting for a class of anomalous dual variational principles is also proposed. The study performed is entirely based on appropriate choice of Lagrangians.

**Key words:** anomalous dual variational principles, non-convex problems, nonlinear beam.

## Introduction

The search for the formulation of the complementary energy principle for elastic solids and structures like beams, plates and shells undergoing finite displacements is still an intriguing and far from being completely resolved problem of nonlinear elasticity. The state of the art in that domain is discussed in the papers by Bielski, Telega (1986), Gałka, Telega (1992), Gao (1992). Within the theory of duality expounded by Ekeland and Temam (1976) the dual problem is always convex. Consequently, by applying this theory of duality to the formulation of the complementary energy principle one obtains it in the form of a convex extremum problem. Such an approach was employed by Bielski,

Telega (1985A), Bielski, Telega (1985B), Telega (1989) to von Kármán plates and in Bielski, Telega (1992) to the nonlinear model of moderately thick plates. Three-dimensional problems were investigated in Bielski, Telega (1986), Bielski, Telega (1985C), Telega (1989). The series of papers Bielski, Galka, Telega (1988, 1989), Galka, Telega, Bielski (1989), Telega, Bielski, Galka (1988) deals with various models of nonlinear thin elastic shells. The duality approach to the formulation of the complementary energy principle for Bařar's five-parameter shell model, Bařar (1987), was proposed and developed in Galka, Telega (1992, 1994A,B). An important conclusion drawn from our duality studies is that nonlinear strain measures significantly restrict the applicability of the dual approach to the formulation of the complementary energy principle. For instance, the membrane force tensor has to be positive semi-definite (in a fixed coordinate system). Despite the restrictions, our studies of duality permit to get a deeper insight into the complementary energy principle. It is worth noting that one may neglect these restrictions by formulating the complementary energy principle as a variational principle and not as an extremum principle, i.e. not as a maximum principle. For instance, relaxing the restrictions rendering the density of the complementary energy a convex function one can formulate such a variational principle. Then the density of the complementary energy function is no longer a convex function. The contribution by the first two authors (Galka, Telega, 1994A,B) offers a new possibility of formulation of the complementary energy principle for a class of earlier precluded loadings. The approach employed is based on Auchmuty's notion of anomalous dual variational principles, Auchmuty (1983).

The objective of the present contribution is mainly to derive the anomalous dual variational principle for a compressed, moderately thick nonlinear elastic beam. The plan of the paper is as follows. In Section 1 two primal minimization problems are formulated: problem ( $P_1$ ) for a moderately thick simply supported beam and problem ( $P_2$ ) for a thin beam clamped at one of its ends. Next, in Section 2, Auchmuty's (1983) setting for anomalous dual variational principles derived from Lagrangians of type I is generalized so as to include *nonlinear* operators  $\Lambda$ . The extensions proposed allow also for the formulation of anomalous dual variational principles in the form of *inf sup* principles. Such a form assumes the complementary energy principle for a simply supported, moderately thick compressed beam, studied in Section 3. In Section 4 two dual principles are proposed for a thin beam clamped at one end. These two principles are derived from two different Lagrangians.

## 1. Preliminaries: basic relations and primal problems

Let the axis  $x$  be directed along the mid-line of a nonlinear elastic beam of length  $\ell$ . The generalized displacement field is denoted by  $(u, w, \varphi)$ , where  $u$  is the horizontal displacement along  $x$ -axis,  $w$  is the transverse displacement (deflection) and  $\varphi$  stands for the rotation of the beam transverse cross-section.

The model studied is readily obtained from the nonlinear model of moderately thick plates, Bielski, Telega (1992), Reddy (1984). The strain-displacement relations are given by

$$\begin{aligned}\varepsilon(u, w) &= u_{,x} + \frac{1}{2}w_{,x}^2, \\ \rho(\varphi) &= \varphi_{,x}, \\ \eta(w, \varphi) &= -(w_{,xx} + \varphi_{,x}), \\ d(w, \varphi) &= w_{,x} + \varphi,\end{aligned}\tag{1.1}$$

where  $u_{,x} = \frac{du}{dx}$ ,  $w_{,xx} = \frac{d^2w}{dx^2}$ ; moreover we denote  $\kappa(w) = -w_{,xx}$ ,  $\eta(w, \varphi) = \kappa(w) - \rho(\varphi)$ . By  $N, M, S$  and  $T$  we denote the generalized stresses. The constitutive equations are

$$\begin{aligned}N &= A\varepsilon(u, w) = A(u_{,x} + \frac{1}{2}w_{,x}^2), \\ S &= \frac{1}{84}D\eta(w, \varphi) = \frac{1}{84}D(-w_{,xx} - \varphi_{,x}), \\ M &= D\rho(\varphi) = D\varphi_{,x}, \\ T &= Hd(w, \varphi) = H(w_{,x} + \varphi),\end{aligned}\tag{1.2}$$

where  $A, D$  and  $H$  are positive elastic beam coefficients. For a nonhomogeneous beam they depend on  $x \in [0, \ell]$ . We make the following assumption:

$$\begin{aligned}A &\in L^\infty(0, \ell), \quad D \in L^\infty(0, \ell), \quad \text{and} \quad H \in L^\infty(0, \ell), \\ \exists \lambda_0 > 0, \forall x \in (0, \ell) \quad A(x) &\geq \lambda_0, \quad D(x) \geq \lambda_0 \quad \text{and} \quad H(x) \geq \lambda_0.\end{aligned}$$

The constitutive relations (1.2) might be generalized in the following way. Suppose that  $W(a, b, c, d)$  is a convex, differentiable function such that

$$\begin{aligned}\exists c > 0, \forall (a, b, c, d) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \\ W(a, b, c, d) &\geq c(|a|^p + |b|^q + |c|^r + |d|^s),\end{aligned}$$

where  $\mathbf{R}$  stands for the space of reals and  $p, q, r, s \geq 2$ . In this case the constitutive equations are

$$\begin{aligned}N &= \frac{\partial W}{\partial \varepsilon}, \\ M &= \frac{\partial W}{\partial \rho}, \\ S &= \frac{\partial W}{\partial \eta}, \\ T &= \frac{\partial W}{\partial d}.\end{aligned}\tag{1.3}$$

If the elastic potential  $W$  is only subdifferentiable then (1.3) has to be replaced by

$$\begin{aligned}N &\in \partial_1 W(\varepsilon, \rho, \eta, d), \\ M &\in \partial_2 W(\varepsilon, \rho, \eta, d), \\ S &\in \partial_3 W(\varepsilon, \rho, \eta, d), \\ T &\in \partial_4 W(\varepsilon, \rho, \eta, d),\end{aligned}$$

where  $\partial_i W(\varepsilon, \eta, \rho, d)$ , ( $i = 1, 2, 3, 4$ ) denotes the subdifferential of the function  $W$  with respect to the  $i$ -th argument.

The nonlinear model of a thin beam is recovered provided that  $\varphi = -w_{,x}$ . Then we have:  $d(w, \varphi) \equiv 0$  and  $\eta(w, \varphi) \equiv 0$ . Now there are only two strain measures:

$$\begin{aligned}\varepsilon(u, w) &= u_{,x} + \frac{1}{2}w_{,x}^2, \\ \kappa(w) &= -w_{,xx}.\end{aligned}\tag{1.4}$$

The constitutive relationships reduce now to

$$\begin{aligned}N &= A\varepsilon(u, w) = A(u_{,x} + \frac{1}{2}w_{,x}^2), \\ M &= D\kappa(w) = -Dw_{,xx}.\end{aligned}\tag{1.5}$$

For a convex and differentiable elastic potential  $W(\varepsilon, \kappa)$ , such that

$$W(a, b) \geq c(|a|^p + |b|^q), c > 0; p, q \geq 2, \forall a, b \in \mathbf{R}$$

we have

$$\begin{aligned}N &= \frac{\partial W}{\partial \varepsilon}, \\ M &= \frac{\partial W}{\partial \rho}.\end{aligned}\tag{1.6}$$

If  $W$  is only subdifferentiable then

$$N \in \partial_1 W(\varepsilon, \kappa)$$

and

$$M \in \partial_2 W(\varepsilon, \kappa).$$

In Sections 3 and 4 dual principles will be derived for the introduced nonlinear beam models. For the problem  $(P_1)$  formulated below the following boundary conditions are imposed:

$$\begin{aligned}u(0) &= 0, \quad w(0) = w(\ell) = 0, \quad N(\ell) = q, \quad S(0) = s_0, \quad S(\ell) = s_1, \\ M(0) - S(0) &= r_0, \quad M(\ell) - S(\ell) = r_1.\end{aligned}$$

The quantities  $q, s_i$  and  $r_i$  are prescribed ( $i = 1, 2$ ). The space  $\mathcal{V}_1$  of kinematically admissible generalized displacement fields is defined by

$$\mathcal{V}_1 = \{ (u, w, \varphi) \in H^1(0, \ell) \times H^2(0, \ell) \times H^1(0, \ell) \mid u(0) = w(0) = w(\ell) = 0 \}.$$

The minimum principle of the total potential energy for the moderately thick beam is formulated as:

PROBLEM  $(\mathcal{P}_1)$

Find

$$J_1(\tilde{u}, \tilde{w}, \tilde{\varphi}) = \inf \{ J_1(u, w, \varphi) \mid (u, w, \varphi) \in \mathcal{V}_1 \},\tag{1.7}$$

where

$$J_1(u, w, \varphi) = \int_0^\ell W_1[\varepsilon(u, w), \rho(\varphi), \eta(w, \varphi), d(w, \varphi)]dx - F_1(u, w, \varphi). \quad (1.8)$$

Here

$$\begin{aligned} W_1[\varepsilon(u, w), \rho(\varphi), \eta(w, \varphi), d(w, \varphi)] &= \\ &= \frac{1}{2}A[\varepsilon(u, w)]^2 + \frac{1}{2}D[\rho(\varphi)]^2 + \frac{1}{2} \cdot \frac{1}{84}D[\eta(w, \varphi)]^2 + \frac{1}{2}H[d(w, \varphi)]^2, \end{aligned} \quad (1.9)$$

and

$$\begin{aligned} F_1(u, w, \varphi) &= \int_0^l (p(x)w(x) + m(x)\varphi(x))dx + qu(l) + \\ &\quad -r_0\varphi(0) + r_1\varphi(l) + s_0w_{,x}(0) - s_1w_{,x}(l). \end{aligned}$$

Let us pass to the formulation of the minimum principle of the total potential energy for the model of a thin beam. Now the space of kinematically admissible displacements is

$$\mathcal{V}_2 = \{(u, w) \in H^1(0, l) \times H^2(0, \ell) \mid u(0) = 0 = w(0) = w(\ell), w_{,x}(0) = 0\}.$$

The minimum principle of the total potential energy is formulated as

PROBLEM ( $\mathcal{P}_2$ )

Find

$$J_2(\tilde{u}, \tilde{w}) = \inf\{J_2(u, w) \mid (u, w) \in \mathcal{V}_2\}. \quad (1.10)$$

The functional  $J_2$  of the total potential energy is now given by

$$J_2(u, w) = \int_0^\ell W_2[\varepsilon(u(x), w(x)), \kappa(w(x))]dx - F_2(u, w), \quad (1.11)$$

where

$$W_2(\varepsilon(u, w), \kappa(w)) = \frac{1}{2}A[\varepsilon(u, w)]^2 + \frac{1}{2}D[\kappa(w)]^2, \quad (1.12)$$

$$F_2(u, w) = \int_0^l p(x)w(x)dx + qu(l) - mw_{,x}(l),$$

and  $p(x)$  denotes a distributed transverse loading. Usually, the elastic potential  $W(\varepsilon, \kappa)$  is assumed as a quadratic function; then

$$\begin{aligned} W_2[\varepsilon(u, w), \kappa(w)] &= \frac{A}{2}[\varepsilon(u, w)]^2 + \frac{D}{2}[\kappa(w)]^2 = \\ &= \frac{A}{2}(u_{,x} + \frac{1}{2}w_{,x}^2)^2 + \frac{D}{2}w_{,xx}^2. \end{aligned} \quad (1.13)$$

We note that the presence of the nonlinear strain measure  $\varepsilon(u, w)$  renders the functionals  $J_i (i = 1, 2)$ , given by (1.8), (1.11) and (1.9), (1.13) non-convex. Existence results for the problem  $(\mathcal{P}_2)$  were obtained by Aubert and Tahraoui (1982). The main result proved in Aubert and Tahraoui (1982) is stated as follows.

**THEOREM 1.1** *Let  $W_2$  be given by (1.13). Under the assumption*

$$q + \lambda_1^2 D > 0, \quad (1.14)$$

*where  $\lambda_1^2$  is the smallest eigenvalue of the problem*

$$f_{,xxxx} + \lambda^2 f_{,xx} = 0, \quad f(0) = f(\ell) = f_{,x}(0) = f_{,x}(\ell) = 0, \quad (1.15)$$

*the problem  $(\mathcal{P}_2)$  admits a unique solution  $(\tilde{u}, \tilde{w})$ .*

## 2. General setting for a class of anomalous dual variational principles

Auchmuty introduced the notion of Lagrangians of type I and II as well as anomalous dual variational principles (Auchmuty, 1983). At this point it is worth noting that much more complicated situations may arise than those studied in Auchmuty (1983). Auchmuty's results can be extended in various directions. As we have already noted, the operator  $\Lambda$  involved in Auchmuty's considerations is always linear and continuous. From the point of view of applications to geometrically nonlinear problems such assumption is insufficient. The aim of this Section is mainly a generalization of Auchmuty's Lemma 4.1 to the case when  $\Lambda$  is a nonlinear operator. The setting proposed here is suitable for a class of nonlinear structures. For instance, it applies to the nonlinear, simply supported beams, see the next Section. Let  $U = U_1 \times U_2$  and  $Y = Y_1 \times Y_2$  be locally convex topological vector spaces and  $U^* = U_1^* \times U_2^*$ ,  $Y^* = Y_1^* \times Y_2^*$  their topological duals (Laurent, 1972). By  $\langle \cdot, \cdot \rangle$  we denote the duality pairing between a space and its dual. Thus we have

$$\langle (u_1^*, u_2^*), (u_1, u_2) \rangle_{U^* \times U} = \langle u_1^*, u_1 \rangle_{U_1^* \times U_1} + \langle u_2^*, u_2 \rangle_{U_2^* \times U_2}. \quad (2.1)$$

In our subsequent developments duality pairings will be evident and only the simple notation  $\langle \cdot, \cdot \rangle$  will usually be used. Suppose that

$$J : U_1 \times U_2 \rightarrow \overline{\mathbf{R}} = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$$

is a functional, not necessarily convex. The primal problem means evaluating

$$(\mathcal{P}) \quad \inf \{J(u_1, u_2) \mid (u_1, u_2) \in U_1 \times U_2\}. \quad (2.2)$$

A functional  $L : (U_1, U_2) \times (Y_1^*, Y_2^*) \rightarrow \overline{\mathbf{R}}$  is said to be a Lagrangian of type I if, for any  $(u_1, u_2) \in U_1 \times U_2$

$$J(u_1, u_2) = \sup \{L(u_1, u_2, y_1^*, y_2^*) \mid (y_1^*, y_2^*) \in (Y_1^*, Y_2^*)\}. \quad (2.3)$$

The dual problem ( $\mathcal{P}^*$ ) is to find

$$\sup_{(y_1^*, y_2^*) \in Y^*} \inf_{(u_1, u_2) \in U} L(u_1, u_2, y_1^*, y_2^*). \quad (2.4)$$

An anomalous dual principle ( $\mathcal{P}^\otimes$ ) means evaluating

$$\inf \{K(y_1^*, y_2^*) \mid (y_1^*, y_2^*) \in Y^*\}, \quad (2.5)$$

where

$$K(y_1^*, y_2^*) = \sup_{(u_1, u_2) \in U} L(u_1, u_2, y_1^*, y_2^*). \quad (2.6)$$

We observe that the dual problem for the geometrically nonlinear beam is of the form (2.4) provided that  $q \geq 0$ . Simply supported compressed beams fall under (2.5), cf. Section 3. A clamped beam requires "mixed" Lagrangians, of the type I-II, not examined by Auchmuty, cf. Section 4.

Let

$$J(u_1, u_2) = J_1(\Lambda(u_1, u_2)) - J_2(u_1, u_2), \quad (2.7)$$

where  $\Lambda : U_1 \times U_2 \rightarrow Y_1 \times Y_2$  is a continuous Gâteaux differentiable operator, not necessarily linear. We assume, for the sake of simplicity, that  $D(\Lambda) = U$ , where  $D(\Lambda)$  denotes the domain of  $\Lambda$ .

Having in mind applications to geometrically nonlinear problems, the form of  $\Lambda$  is inferred from the strain measures involved in a problem considered. For the nonlinear beam problem studied in Section 3, we may assume

$$\Lambda(u, w) = (\Lambda_1(u, w), \Lambda_2(w)) = (\varepsilon(u, w), \kappa(w)).$$

The functionals  $J_1$  and  $J_2$  are assumed to be lower semicontinuous; moreover  $J_1$  is a convex functional. Thus  $\tilde{J}_1(u_1, u_2) = J_1(\Lambda(u_1, u_2))$  is not necessarily a convex functional. The Lagrangian is defined by

$$L(u_1, u_2, y_1^*, y_2^*) = \langle \Lambda(u_1, u_2), (y_1^*, y_2^*) \rangle - J_1^*(y_1^*, y_2^*) - J_2(u_1, u_2). \quad (2.8)$$

Hence

$$J(u_1, u_2) = \sup_{(y_1^*, y_2^*) \in Y^*} L(u_1, u_2, y_1^*, y_2^*), \quad (u_1, u_2) \in U. \quad (2.9)$$

Denote by  $\tilde{\Lambda}_{(u_1, u_2)}$  the linearized operator of  $\Lambda$  at a point  $(u_1, u_2) \in U$ . Its adjoint  $\tilde{\Lambda}_{(u_1, u_2)}^*$  is defined by

$$\langle \tilde{\Lambda}_{(u_1, u_2)}(v_1, v_2), (y_1^*, y_2^*) \rangle_{Y \times Y^*} = \langle \tilde{\Lambda}_{(u_1, u_2)}^*(y_1^*, y_2^*), (v_1, v_2) \rangle_{U^* \times U}, \quad (2.10)$$

where  $(v_1, v_2) \in D(\tilde{\Lambda}_{(u_1, u_2)})$ ,  $(y_1^*, y_2^*) \in Y^*$ . Here our attention is mainly focussed on nonlinear operators  $\Lambda$  such that

$$\Lambda(u_1, u_2) = \tilde{\Lambda}_{(u_1, u_2)}(u_1, \frac{1}{2}u_2). \quad (2.11)$$

The Hamiltonian associated to the Lagrangian  $L$  is now defined by

$$H(u_1, u_2, y_1^*, y_2^*) = \langle \Lambda(u_1, u_2), (y_1^*, y_2^*) \rangle - L(u_1, u_2, y_1^*, y_2^*). \quad (2.12)$$

On account of (2.11) we have

$$H(u_1, u_2, y_1^*, y_2^*) = \langle \tilde{\Lambda}_{(u_1, u_2)}(u_1, \frac{1}{2}u_2), (y_1^*, y_2^*) \rangle - L(u_1, u_2, y_1^*, y_2^*). \quad (2.13)$$

The Lagrangian is not necessarily given by (2.8). Particularly, for  $L$  given by (2.8) one has

$$H(u_1, u_2, y_1^*, y_2^*) = J_1^*(y_1^*, y_2^*) + J_2(u_1, u_2). \quad (2.14)$$

A point  $(\hat{u}, \hat{y}^*) = \{(\hat{u}_1, \hat{u}_2), (\hat{y}_1^*, \hat{y}_2^*)\}$  is said to be an anomalous critical point of the Lagrangian  $L$  if

$$\Lambda(\hat{u}) \in \partial_2 H(\hat{u}, \hat{y}^*), \quad (2.15)$$

and

$$\tilde{\Lambda}_{\hat{u}}^*(\hat{y}^*) \in \partial_1 H(\hat{u}, \hat{y}^*), \quad (2.16)$$

where  $\partial_1$  and  $\partial_2$  stand for the partial subdifferentiations of the functions  $H(\cdot, \cdot, y_1^*, y_2^*)$  and  $H(u_1, u_2, \cdot, \cdot)$  respectively;  $\Lambda$  is not necessarily of the form (2.11). When the operator  $\Lambda$  is linear, the formulae (4.11) and (4.12) of Auchmuty (1983) are readily recovered. The main result of this Section is given by

**LEMMA 2.1** *Let  $H$  be the Hamiltonian associated with a Lagrangian of type I. Suppose that  $\Lambda$  satisfies (2.11). Then*

- (a)  $\Lambda(u_1, u_2) \in \partial_2 H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*)$  implies
 
$$J(u_1, u_2) = L(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \leq K(\tilde{y}_1^*, \tilde{y}_2^*).$$
- (b)  $\tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}^*(y_1^*, y_2^*) \in \partial_1 H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*)$  implies
 
$$K(y_1^*, y_2^*) = L(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \leq J(\tilde{u}_1, \tilde{u}_2).$$

**PROOF.**

- (a) If  $\Lambda(u_1, u_2) \in \partial_2 H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*)$  then

$$H(u_1, u_2, y_1^*, y_2^*) - H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \geq \langle \Lambda(u_1, u_2), (y_1^* - \tilde{y}_1^*, y_2^* - \tilde{y}_2^*) \rangle$$

$$\forall (y_1^*, y_2^*) \in Y_1^* \times Y_2^*.$$

Hence

$$\begin{aligned} L(u_1, u_2, y_1^*, y_2^*) &= \langle \Lambda(u_1, u_2), (y_1^*, y_2^*) \rangle - H(u_1, u_2, y_1^*, y_2^*) \leq \\ &\leq \langle \Lambda(u_1, u_2), (\tilde{y}_1^*, \tilde{y}_2^*) \rangle - H(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) = L(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \leq \\ &\leq K(\tilde{y}_1^*, \tilde{y}_2^*) \end{aligned}$$

for each  $(y_1^*, y_2^*) \in Y_1^* \times Y_2^*$ .

Taking the supremum over  $(y_1^*, y_2^*) \in Y^*$  we obtain

$$J(u_1, u_2) = L(u_1, u_2, \tilde{y}_1^*, \tilde{y}_2^*) \leq K(\tilde{y}_1^*, \tilde{y}_2^*).$$

When



$$(b) \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}^*(y_1^*, y_2^*) \in \partial_1 H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*),$$

then one has

$$H(u_1, u_2, y_1^*, y_2^*) - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \geq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}^*(y_1^*, y_2^*), (u_1 - \tilde{u}_1, u_2 - \tilde{u}_2) \rangle \\ \forall (u_1, u_2) \in U_1 \times U_2.$$

Hence

$$\langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}^*(y_1^*, y_2^*), (u_1, u_2) \rangle - H(u_1, u_2, y_1^*, y_2^*) \leq \\ \leq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}^*(y_1^*, y_2^*), (\tilde{u}_1, \tilde{u}_2) \rangle - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*).$$

Thus

$$\langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(u_1, u_2), (y_1^*, y_2^*) \rangle - H(u_1, u_2, y_1^*, y_2^*) \leq \\ \leq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(\tilde{u}_1, \tilde{u}_2), (y_1^*, y_2^*) \rangle - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*).$$

Because

$$\tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(u_1, u_2) = \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(u_1, \frac{1}{2}u_2) + \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}u_2),$$

therefore

$$\langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}u_2), (y_1^*, y_2^*) \rangle + \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(u_1, \frac{1}{2}u_2), (y_1^*, y_2^*) \rangle - \\ - H(u_1, u_2, y_1^*, y_2^*) \leq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle + \\ + \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(\tilde{u}_1, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) = \\ = \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle + L(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \leq \\ \leq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle + J(\tilde{u}_1, \tilde{u}_2), \\ \forall (u_1, u_2) \in U_1 \times U_2.$$

Taking now the supremum over  $(u_1, u_2) \in U$ , we obtain

$$\langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle + \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(\tilde{u}_1, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle - \\ - H(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \leq \langle \tilde{\Lambda}_{(\tilde{u}_1, \tilde{u}_2)}(0, \frac{1}{2}\tilde{u}_2), (y_1^*, y_2^*) \rangle + J(\tilde{u}_1, \tilde{u}_2).$$

By virtue of (2.13) one has

$$K(y_1^*, y_2^*) = L(\tilde{u}_1, \tilde{u}_2, y_1^*, y_2^*) \leq J(\tilde{u}_1, \tilde{u}_2).$$

Thus the lemma is proved. ■

COROLLARY 2.1 From (2.15), (2.16) and the above lemma we conclude that if  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$  is an anomalous critical point of a Lagrangian of the type I then

$$J(\hat{u}_1, \hat{u}_2) = K(\hat{y}_1, \hat{y}_2) = L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*). \quad (2.17)$$

Under stronger assumptions Lemma 2.1 can be reversed.

LEMMA 2.2 Suppose that  $(\hat{u}_1, \hat{u}_2)$  solves the primal problem (2.2), whereas  $(\hat{y}_1^*, \hat{y}_2^*)$  is a solution to the anomalous dual problem (2.5), provided that  $J$  is given by (2.3). Let the Hamiltonian be defined by (2.12) where the operator  $\Lambda$  satisfies (2.11). If (2.17) is satisfied then  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$  is an anomalous critical point of the Lagrangian  $L$ .

PROOF. According to the assumptions we may write

$$L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) = J(\hat{u}_1, \hat{u}_2) = \inf_{(u_1, u_2) \in U} L(u_1, u_2, \hat{y}_1^*, \hat{y}_2^*) \leq L(u_1, u_2, \hat{y}_1^*, \hat{y}_2^*) \\ \forall (u_1, u_2) \in U,$$

and

$$L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) = K(\hat{y}_1^*, \hat{y}_2^*) = \\ = \inf_{(y_1^*, y_2^*) \in Y^*} L(\hat{u}_1, \hat{u}_2, y_1^*, y_2^*) \leq L(\hat{u}_1, \hat{u}_2, y_1^*, y_2^*) \\ \forall (y_1^*, y_2^*) \in Y^*.$$

Thus  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$  is a  $\partial$ -critical point of  $L$ , Auchmuty (1983). Hence we may write

$$0 \in \partial_1 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) \quad \text{and} \quad 0 \in \partial_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) \quad (2.18)$$

Since the operator  $\Lambda$  is Gâteaux differentiable therefore (2.18)<sub>2</sub> and (2.17) give

$$0 \in -\Lambda(\hat{u}_1, \hat{u}_2) + \partial_2 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*).$$

Similarly, (2.18)<sub>1</sub> and (2.17) yield

$$0 \in -\tilde{\Lambda}_{(\hat{u}_1, \hat{u}_2)}^*(\hat{y}_1^*, \hat{y}_2^*) + \partial_1 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*). \quad \blacksquare$$

The next lemma generalizes Auchmuty's Lemma 4.2. The nonlinear operator  $\Lambda$ , however, is not necessarily of the form (2.11).

LEMMA 2.3 Let  $L$  be a Lagrangian of type I and  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$  be an anomalous critical point of  $L$ . Suppose  $L$  is partially Gâteaux differentiable at  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$ , then  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$  is a critical point of  $L$ .

PROOF. Let us denote by  $D_1 L(\cdot, \cdot, \hat{y}_1^*, \hat{y}_2^*)$  and  $D_2 L(\hat{u}_1, \hat{u}_2, \cdot, \cdot)$  the partial Gâteaux derivatives, cf. Ioffe, Tihomirow (1979).

From (2.12) one has

$$\begin{aligned} \langle D_2 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2) \rangle &= \langle \Lambda(\hat{u}_1, \hat{u}_2), (h_1, h_2) \rangle - \\ &\quad - \langle D_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2) \rangle, \end{aligned}$$

for each  $(h_1, h_2) \in (Y_1, Y_2)$ . Since  $(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*)$  is an anomalous critical point, we write

$$\begin{aligned} \langle D_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2) \rangle &= \\ = \langle \Lambda(\hat{u}_1, \hat{u}_2) - D_2 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (h_1, h_2) \rangle &= 0, \quad \forall (h_1, h_2) \in Y. \end{aligned}$$

Hence

$$D_2 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) = 0.$$

Similarly

$$\begin{aligned} \langle D_1 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (z_1, z_2) \rangle &= \\ = \langle \tilde{\Lambda}(\hat{u}_1, \hat{u}_2)(z_1, z_2), (\hat{y}_1^*, \hat{y}_2^*) \rangle - \langle D_1 H(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*), (z_1, z_2) \rangle, \\ \forall (z_1, z_2) \in U, \end{aligned}$$

and consequently  $D_1 L(\hat{u}_1, \hat{u}_2, \hat{y}_1^*, \hat{y}_2^*) = 0$ . ■

**REMARK 2.1** *The last proof exploits Auchmuty's Lemma 3.6. It states that if  $f : X \rightarrow \mathbb{R}$  is Gâteaux differentiable at  $\hat{x}$  and  $\xi \in \partial f(\hat{x})$  then  $\xi = Df(\hat{x})$ , where  $Df(\hat{x})$  stands for the Gâteaux derivative of  $f$  at  $\hat{x}$ .*

**REMARK 2.2** *The form (2.5) and (2.6) of the anomalous dual principle suits well a simply supported thin beam, cf. Section 3 in Galka, Telega (submitted). For a simply supported moderately thick beam, however, more general framework is needed. Now let us take  $U = U_1 \times U_2 \times U_3$  and  $Y = Y_1 \times Y_2 \times Y_3$ . The relation (2.1) generalizes obviously to*

$$\begin{aligned} \langle (u_1^*, u_2^*, u_3^*), (u_1, u_2, u_3) \rangle_{U^* \times U} &= \\ = \langle u_1^*, u_1 \rangle_{U_1^* \times U_1} + \langle u_2^*, u_2 \rangle_{U_2^* \times U_2} + \langle u_3^*, u_3 \rangle_{U_3^* \times U_3} \end{aligned} \quad (2.19)$$

We formulate the following anomalous dual variational principle, still denoted as

**PROBLEM ( $P^\otimes$ ):**  
Find

$$\inf_{y_1^* \in Y_1} \sup_{y_2^* \in Y_2^*, y_3^* \in Y_3^*} K(y_1^*, y_2^*, y_3^*), \quad (2.20)$$

where

$$K(y_1^*, y_2^*, y_3^*) = \sup_{(u_1, u_2, u_3) \in \mathcal{K}} L(u_1, u_2, u_3, y_1^*, y_2^*, y_3^*) \quad (2.21)$$

One can also consider the following problem

$$(\mathcal{P}^\otimes) \quad \left| \begin{array}{l} \text{Find} \\ \inf_{y_1^* \in Y_1^*} \sup_{y_2 \in Y_2^*} \inf_{y_3 \in Y_3^*} K(y_1^*, y_2^*, y_3^*) \end{array} \right.$$

As previously,  $L$  is a Lagrangian of the type I. Thus for any

$$(u_1, u_2, u_3) \in U_1 \times U_2 \times U_3,$$

we have

$$J(u_1, u_2, u_3) = \sup \{ L(u_1, u_2, u_3, y_1^*, y_2^*, y_3^*) \mid (y_1^*, y_2^*, y_3^*) \in Y^* \}. \quad (2.22)$$

The operator  $\Lambda$  consists of a linear part  $\Lambda_1$  as well as a nonlinear part  $\Lambda_2$ . The following form is taken as a model for developments of Section 3:

$$\Lambda(u_1, u_2, u_3) = (\Lambda_1 u_1, \Lambda_2(u_2, u_3)), \quad (2.23)$$

where

$$\Lambda_2(u_2, u_3) = {}_2\tilde{\Lambda}_{(u_2, u_3)}(u_2, \frac{1}{2}u_3) \quad (2.24)$$

Here  ${}_2\tilde{\Lambda}_{(u_2, u_3)}$  denotes the linearized operator of  $\Lambda_2$  at a point  $(u_2, u_3) \in U_2 \times U_3$ . We also introduce the Hamiltonian

$$H(u_1, u_2, u_3, y_1^*, y_2^*, y_3^*) = \langle \Lambda(u_1, u_2, u_3), (y_1^*, y_2^*, y_3^*) \rangle - L(u, y^*), \quad (2.25)$$

where

$$u = (u_1, u_2, u_3) \in U_1 \times U_2 \times U_3 \text{ and } y^* \in Y_1 \times Y_2 \times Y_3.$$

Since the anomalous dual problem (2.20) has now the form of inf sup principle, it is natural that Lagrangians should exhibit a "mixed" behaviour. It is convenient to introduce the notion of a *saddle - ( $\partial$  - critical) - point* of a Lagrangian. A point  $(\hat{u}_1, \hat{u}_2, \hat{u}_3; \hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*)$  is said to be a *saddle - ( $\partial$  - critical) - point* of  $L$  iff

$$\begin{aligned} 0 \in \partial_{y_1^*} L(\hat{u}, \hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*), \quad 0 \in \partial_{(y_2^*, y_3^*)} (-L)(\hat{u}, \hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*) \\ \text{and } 0 \in \partial_u L(\hat{u}, \hat{y}). \end{aligned} \quad (2.26)$$

Obviously, this definition may further be generalized. If  $L$  is Gateaux differentiable, a *saddle - ( $\partial$  - critical) - point* of  $L$  is a critical point of  $L$ . Then we have

$$\begin{aligned} D_{y_1^*} L(\hat{u}, \hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*) = 0, \quad -D_{(y_2^*, y_3^*)} L(\hat{u}, \hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*) = 0, \\ D_u L(\hat{u}, \hat{y}^*) = 0, \end{aligned} \quad (2.27)$$

where  $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ ,  $\hat{y} = (\hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*)$ . The definition of a *saddle - ( $\partial$  - critical) - point* implies that:

(i) the point  $(y_1^*; y_2^*, y_3^*)$  is a saddle point of  $L(\hat{u}, \cdot, \cdot, \cdot)$

$$L(\hat{u}, \hat{y}_1^*, y_2^*, y_3^*) \leq L(\hat{u}, \hat{y}^*) \leq L(\hat{u}, y_1^*, \hat{y}_2^*, \hat{y}_3^*), \quad (2.28)$$

$$\forall y_1^* \in Y_1^*, (y_2^*, y_3^*) \in Y_2^* \times Y_3^*.$$

(ii) The point  $(\hat{u}, \hat{y}_1^*)$  is a  $\partial$ -critical point of  $L(\cdot, \cdot, \cdot, \cdot, \hat{y}_2^*, y_3^*)$

$$L(u, \hat{y}^*) \geq L(\hat{u}, \hat{y}^*) \quad \forall u \in U_1 \times U_2 \times U_3$$

and

$$L(\hat{u}, y_1^*, \hat{y}_2^*, \hat{y}_3^*) \geq L(\hat{u}, \hat{y}^*) \quad \forall y_1^* \in Y_1^*. \quad (2.29)$$

As previously, the notion of an anomalous critical point of a Lagrangian plays a crucial role. Now a point  $(\hat{u}, \hat{y}^*) = \{(\hat{u}_1, \hat{u}_2, \hat{u}_3), (\hat{y}_1^*, \hat{y}_2^*, \hat{y}_3^*)\}$  is said to be an anomalous critical point of  $L$  if

$$\Lambda(\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \partial_{y^*} H(\hat{u}, \hat{y}^*), \quad (2.30)$$

and

$$(\Lambda_1^* \hat{y}_1^*, {}_2\tilde{\Lambda}_{(\hat{u}_2, \hat{u}_3)}^*(\hat{y}_2^*, \hat{y}_3^*)) \in \partial_u H(\hat{u}, \hat{y}^*). \quad (2.31)$$

The most important conclusion of the present remark is that Lemmas 2.1, 2.2 and Corollary 2.1 can readily be extended to the more general case of the dual functional  $K(y_1^*, y_2^*, y_3^*)$ . To corroborate this statement it is sufficient to observe that  $L$  persists to be of type I and  $K$  is defined by (2.21).

### 3. Dual principle for a simply supported moderately thick nonlinear beam

The dual problems  $(\mathcal{P}_1^*)$  and  $(\mathcal{P}_1^\otimes)$  will now be derived. It will become evident that the problem  $(\mathcal{P}_1^*)$  is suitable for  $q > 0$  only;  $(\mathcal{P}_1^\otimes)$ -problem is applicable to compressed beams provided that

$$0 > q \geq \frac{-\pi^2 H D}{\pi^2 D + \ell^2 H}.$$

The perturbed functional  $\Phi$  is assumed in the following form, cf. Ekeland, Temam (1976), Galka, Telega (1994A,B)

$$\begin{aligned} \Phi(u, w, \varphi, p_1, p_2, p_3, p_4) &= \\ &= \int_0^\ell \left\{ \frac{A}{2} (u_{,x} + \frac{1}{2} w_{,x}^2 + p_1)^2 + \frac{D}{2} (\varphi_{,x} + p_2)^2 + \frac{D}{2 \cdot 84} (-\varphi_{,x} - w_{,xx} + p_3)^2 + \right. \\ &\quad \left. + \frac{H}{2} (w_{,x} + \varphi + p_4)^2 \right\} dx - F(u, w, \varphi), \end{aligned} \quad (3.1)$$

where  $(u, w, \varphi) \in \mathcal{V}_1$  and  $(p_1, p_2, p_3, p_4) \in \mathcal{H} = L^2(0, \ell) \times L^2(0, \ell) \times L^2(0, \ell) \times L^2(0, \ell)$ . Hence

$$\Phi(u, w, \varphi, 0, 0, 0, 0) = J(u, w, \varphi). \quad (3.2)$$

The Lagrangian  $L$  is calculated from

$$\begin{aligned} -L(u, w, \varphi, M, S, T, N) &= \\ &= \sup_{p \in \mathcal{H}} \left\{ \int_0^\ell [Np_1 + Mp_2 + Sp_3 + Tp_4] dx - \Phi(u, w, \varphi, p_1, p_2, p_3, p_4) \right\} \end{aligned}$$

where  $(M, S, T, N) \in \mathcal{H}$ .

Finding the supremum one has

$$\begin{aligned} L(u, w, \varphi, M, S, T, N) &= \int_0^\ell \left\{ N(u, x + \frac{1}{2}w_{,x}^2) + M\varphi_{,x} - S(\varphi_{,x} + w_{,xx}) + \right. \\ &\quad \left. + T(\varphi + w_{,x}) - W^*(M(x), S(x), T(x), N(x)) \right\} dx - F(u, w, \varphi), \quad (3.3) \end{aligned}$$

where  $(u, w, \varphi) \in \mathcal{V}_1$ ,  $(M, S, T, N) \in \mathcal{H}$  and

$$\begin{aligned} W^*(a^*, b^*, c^*, d^*) &= \\ &= \sup \left\{ a^*a + b^*b + c^*c + d^*d - \right. \\ &\quad \left. - \frac{1}{2}Aa^2 - \frac{1}{2}Db^2 - \frac{1}{2} \cdot \frac{1}{84}Dc^2 - \frac{1}{2}Hd^2 \mid a, b, c, d \in \mathbf{R} \right\} = \\ &= \frac{1}{2A}a^{*2} + \frac{1}{2D}b^{*2} + \frac{84}{2D}c^{*2} + \frac{1}{2H}d^{*2}, \quad a^*, b^*, c^*, d^* \in \mathbf{R}. \end{aligned} \quad (3.4)$$

Here  $W^*$  denotes the density of the complementary energy. Performing standard calculations, from the variational equation  $\delta L = 0$  one obtains:

(i) equilibrium equations

$$\begin{aligned} N_{,x} &= 0, \\ S_{,xx} + (Nw_{,x})_{,x} + T_{,x} + p &= 0; \\ (M - S)_{,x} - T + m &= 0 \end{aligned} \quad (3.5)$$

(ii) constitutive relationships

$$\begin{aligned} N &= A\varepsilon(u, w), \quad M = D\varphi_{,x}, \\ S &= \frac{D}{84}(-w_{,xx} - \varphi_{,x}), \\ T &= H(w_{,x} + \varphi); \end{aligned} \quad (3.6)$$

(iii) boundary conditions

$$\begin{aligned} N(\ell) &= q, \quad S(0) = s_0, \\ S(\ell) &= s_1, \quad M(0) - S(0) = r_0, \\ M(\ell) - S(\ell) &= r_1. \end{aligned} \quad (3.7)$$

The equilibrium equations (3.5) are to be understood in the sense of distributions. For  $(M, S, T, N) \in \mathcal{H}$ ,  $(\overline{M}, \overline{S}, \overline{T}, \overline{N}) \in \mathcal{H}$ , and  $\overline{M} = D\rho(\varphi)$ ,  $\overline{S} = \frac{1}{84}D\eta(w, \varphi)$ ,  $\overline{T} = Hd(w, \varphi)$ ,  $\overline{N} = A\varepsilon(u, w)$ , simple calculations yield

$$\begin{aligned} L(u, w, \varphi, M, S, T, N) - L(u, w, \varphi, \overline{M}, \overline{S}, \overline{T}, \overline{N}) &= \\ &= - \int_0^\ell \left[ \frac{1}{2A}(N - \overline{N})^2 + \frac{1}{2D}(M - \overline{M})^2 + \right. \\ &\quad \left. + \frac{84}{2D}(S - \overline{S})^2 + \frac{1}{2H}(T - \overline{T})^2 \right] dx \leq 0. \end{aligned}$$

Hence

$$\begin{aligned} \sup_{(M,S,T,N) \in \mathcal{H}} L(u, w, \varphi, M, S, T, N) = \\ = L(u, w, \varphi, D\rho(\varphi), \frac{1}{84}D\eta(w, \varphi), Hd(w, \varphi), A\varepsilon(u, w)). \end{aligned}$$

Thus we have

$$\begin{aligned} J(u, w, \varphi) = L(u, w, \varphi, D\rho(\varphi), \frac{1}{84}D\eta(w, \varphi), Hd(w, \varphi), A\varepsilon(u, w)) \\ \forall (u, w, \varphi) \in \mathcal{V}_1. \end{aligned}$$

Consequently, the problem  $(\mathcal{P}_1)$  may be written in the following form

PROBLEM  $\mathcal{P}_1$

Find

$$\inf_{(u,w,\varphi) \in \mathcal{V}_1} \sup_{(M,S,T,N) \in \mathcal{H}} L(u, w, \varphi, M, S, T, N).$$

Performing integration by parts we obtain

$$\begin{aligned} L(u, w, \varphi, M, S, T, N) = \\ = \int_0^\ell [-N_{,x}u - (M - S)_{,x}\varphi - (S_{,xx} + T_{,x})w + T\varphi + \frac{1}{2}Nw_{,x}^2]dx - \\ - W^*(M, S, T, N) - F(u, w, \varphi) + N(\ell)u(\ell) + (M(\ell) - S(\ell))\varphi(\ell) - \\ - (M(0) - S(0))\varphi(0) + S(\ell)w_{,x}(\ell) - S(0)w_{,x}(0), \end{aligned}$$

where

$$W^*(M, S, T, N) = \int_0^\ell [\frac{1}{2A}N^2 + \frac{1}{2D}M^2 + \frac{84}{2D}S^2 + \frac{1}{2H}T^2]dx.$$

For  $(u, w, \varphi) \in \mathcal{V}_1$ ,  $(\bar{u}, \bar{w}, \bar{\varphi}) \in \mathcal{V}_1$  and

$$\begin{aligned} N_{,x} &= 0, \\ (M - S)_{,x} - T + m &= 0, \\ S_{,xx} + (N\bar{w}_{,x})_{,x} + T_{,x} + p &= 0, \end{aligned} \tag{3.8}$$

$N(\ell) = q$ ,  $S(0) = s_0$ ,  $S(\ell) = s_1$ ,  $M(0) - S(0) = r_0$ ,  $M(\ell) - S(\ell) = r_1$ ,  
one has

$$\begin{aligned} L(u, w, \varphi, M, S, T, N) - L(\bar{u}, \bar{w}, \bar{\varphi}, M, S, T, N) = \\ = \frac{1}{2} \int_0^\ell N(w_{,x} - \bar{w}_{,x})^2 dx. \end{aligned} \tag{3.9}$$

Thus for  $N > 0$ , Eq.(3.9) gives

$$L(u, w, \varphi, M, S, T, N) - L(\bar{u}, \bar{w}, \bar{\varphi}, M, S, T, N) \geq 0$$

while for  $N < 0$  we have

$$L(u, w, \varphi, M, S, T, N) - L(\bar{u}, \bar{w}, \bar{\varphi}, M, S, T, N) \leq 0.$$

Integrating Eq.(3.8)<sub>2</sub> one finds

$$\bar{w}_{,x} = -\frac{1}{q}(M_{,x} + Z(x)),$$

where

$$Z(x) = m(x) + R(x) - \frac{1}{\ell} \int_0^\ell (m + R) dx - \frac{1}{\ell} (M(\ell) - M(0)), \quad (3.10)$$

provided that  $q \neq 0$ ; here  $R(x) = \int_0^x p(t) dt$ .

Let us pass now to the formulation of the dual problems for  $N > 0$ .

For Lagrangian (3.3) we have

$$\inf_{(u, w, \varphi) \in \mathcal{V}_1} L(u, w, \varphi, M, S, T, N) = \begin{cases} G_1(S, M) & \text{if } N = q > 0, \\ & T = (M - S)_{,x} + m(x), \\ & S(0) = s_0, S(\ell) = s_1, \\ & M(0) - S(0) = r_0, \\ & M(\ell) - S(\ell) = r_1, \\ -\infty, & \text{otherwise;} \end{cases}$$

where

$$\begin{aligned} G_1(S, M) &= \\ &= - \int_0^\ell \left[ \left( \frac{1}{2} q^2 \bar{w}_{,x}^2 + \frac{1}{2A} q^2 + \frac{1}{2D} M^2 \right) + \frac{84}{2D} S^2 + \right. \\ &\quad \left. + \frac{1}{H} [(M - S)_{,x} + m]^2 + \frac{1}{2q} (M_{,x} + Z(x))^2 \right] dx. \end{aligned} \quad (3.11)$$

To assess the nature of  $G_1$  we find

$$\begin{aligned} G_1(M, S) - G_1(\bar{M}, \bar{S}) &= \\ &= -\frac{1}{2} \int_0^\ell \left\{ \frac{84}{D} [S - \bar{S}]^2 + \frac{1}{H} [(M - S)_{,x} - (\bar{M} - \bar{S})_{,x}]^2 + \right. \\ &\quad \left. + \frac{1}{q} (M_{,x} - \bar{M}_{,x})^2 + \frac{1}{D} (M - \bar{M})^2 \right\} dx \leq 0. \end{aligned} \quad (3.12)$$

provided that

$$\begin{aligned} \frac{84}{D} \bar{S} + \frac{1}{H} [(\bar{M} - \bar{S})_{,xx} + m_{,x}] &= 0, \\ \bar{M}_{,xx} + m_{,x} + p - \frac{q}{D} \bar{M} + \frac{q}{H} [(\bar{M} - S)_{,xx} + m_{,x}] &= 0, \end{aligned}$$



and

$$S(\ell) = \bar{S}(\ell) = s_1, \quad S(0) = \bar{S}(0) = s_0, \\ M(0) - S(0) = \bar{M}(0) - \bar{S}(0) = r_0, \quad M(\ell) - S(\ell) = \bar{M}(\ell) - \bar{S}(\ell) = r_1.$$

It follows that

$$\sup_{S, M} G_1(M, S) = G_1(\bar{M}, \bar{S}).$$

Consequently, the dual problem  $(\mathcal{P}_1^*)$  means evaluating

$$\sup_{(M, S, T, N) \in \mathcal{H}} \inf_{(u, w, \varphi) \in \mathcal{V}_1} L(u, w, \varphi, M, S, T, N),$$

where  $N = q > 0$ .

Consider now the case  $q < 0$ .

Towards this end we find

$$\sup_{(u, w, \varphi) \in \mathcal{V}_1} L(u, w, \varphi, M, S, T, N) = \begin{cases} G_1(S, M) & \text{for } N = q > 0, \\ & T = (M - S)_{,x} + m(x), \\ & S(0) = s_0, S(\ell) = s_1, \\ & M(0) - S(0) = r_0, \\ & M(\ell) - S(\ell) = r_1, \\ +\infty, & \text{otherwise.} \end{cases}$$

To assess the nature of the functional  $G_1$  for  $q < 0$  we calculate

$$G_1(M, S) - G_1(M, \bar{S}) = \\ = -\frac{1}{2} \int_0^\ell \left\{ \frac{84}{D} [S - \bar{S}]^2 + \frac{1}{H} [(S - \bar{S})_{,x}]^2 \right\} dx \leq 0.$$

provided that

$$S(0) = \bar{S}(0) = s_0, \quad S(\ell) = \bar{S}(\ell) = s_1.$$

Now we have to calculate

$$G_1(M, \bar{S}(M)) - G_1(\bar{M}, \bar{S}(\bar{M})) \geq \\ \geq -\frac{1}{2} \int_0^\ell \left\{ \frac{1}{D} [M - \bar{M}]^2 + \left( \frac{1}{H} + \frac{1}{q} \right) (M - \bar{M})_{,x}^2 \right\} dx.$$

We can use now the Friedrichs' inequality, cf. Oden, Reddy (1976), Rektorys (1980)

$$\forall v \in H_0^1(0, \ell), \quad \|v\|_{L^2(0, \ell)}^2 \leq \frac{\ell}{\pi^2} \|\nabla v\|_{L^2(0, \ell)}^2,$$

where  $\nabla v = \frac{dv}{dx} = v_{,x}$  to  $v = M - \bar{M} \in H_0^1(0, \ell)$ . Then one has

$$G_1(M, \bar{S}(M)) - G_1(\bar{M}, \bar{S}(\bar{M})) \geq 0$$

provided that

$$0 > q \geq \frac{-\pi^2 HD}{\pi^2 D + \ell^2 H}. \quad (3.13)$$

The anomalous dual variational principle  $(\mathcal{P}_1^\otimes)$  is to find

$$(\mathcal{P}_1^\otimes) \quad \inf_{M \in L^2(0, \ell)} \sup_{S \in L^2(0, \ell)} \inf_{T \in L^2(0, \ell)} \sup_{(u, w, \varphi) \in \mathcal{V}} L(u, w, \varphi, M, S, T, N).$$

Suppose that  $(\tilde{u}, \tilde{w}, \tilde{\varphi})$  solves the primal problem  $(P_1)$ , whereas  $(\tilde{M}, \tilde{S}, \tilde{T}, \tilde{N} = q)$  is a solution to  $(\mathcal{P}_1^\otimes)$ . From the considerations of the present Section we conclude that  $\{(\tilde{u}, \tilde{w}, \tilde{\varphi}), (\tilde{M}, \tilde{S}, \tilde{T}, \tilde{N} = q)\}$  is the critical point of  $L$ , which is also an anomalous critical point. In this case Remark 2.2 and Corollary 2.1 imply

$$J(\tilde{u}, \tilde{w}, \tilde{\varphi}) = L(\tilde{u}, \tilde{w}, \tilde{\varphi}, \tilde{M}, \tilde{S}, \tilde{T}, q) = K(\tilde{M}, \tilde{S}, \tilde{T}, q), \quad (3.14)$$

where

$$K(M, S, T, N) = \sup_{(u, w, \varphi) \in \mathcal{V}_1} L(u, w, \varphi, M, S, T, N). \quad (3.15)$$

Let us pass to a closer inspection of this anomalous critical point. The operator  $\Lambda$  is assumed in the form

$$\begin{aligned} \Lambda(u, w, \varphi) &= (\Lambda_1 \varphi, \Lambda_2(w, \varphi), \Lambda_3(w, \varphi), \Lambda_4(u, w)) = \\ &= (\rho(\varphi), \eta(w, \varphi), d(w, \varphi), \varepsilon(u, w)), \quad (u, w, \varphi) \in \mathcal{V}_1, \end{aligned}$$

$\Lambda : \mathcal{V}_1 \rightarrow Y = Y_1 \times Y_2 \times Y_3 \times Y_4 = Y^*$ , where  $Y_i = L^2(0, \ell)$ ,  $i = 1, 2, 3, 4$ . The Lagrangian  $L$ , earlier given by (3.3), is still written in the following way

$$\begin{aligned} L(u, w, \varphi, M, S, T, N) &= \langle \Lambda(u, w, \varphi), (M, S, T, N) \rangle - \\ &- \int_0^\ell W^*[M(x), S(x), T(x), N(x)] dx - F(u, w, \varphi). \end{aligned} \quad (3.16)$$

Next, the Hamiltonian assumes the form

$$\begin{aligned} H(u, w, \varphi, M, S, T, N) &= \\ &= \langle \Lambda(u, w, \varphi), (M, S, T, N) \rangle - L(u, w, \varphi, M, S, T, N) = \\ &= \int_0^\ell W^*[M(x), S(x), T(x), N(x)] dx + F(u, w, \varphi). \end{aligned} \quad (3.17)$$

Let  ${}_4\tilde{\Lambda}_w$  denote the linearized operator of  $\Lambda_4$  at  $(0, w)$ , where  $w \in H^2(0, \ell)$  and  $w(0) = w(\ell) = 0$ . To obtain the explicit form of the dual operator

$$(\Lambda_w^0)^* = (\Lambda_1^*, \Lambda_2^*, \Lambda_3^*, {}_4\tilde{\Lambda}_w^*) \quad \text{of} \quad \Lambda_w^0 := (\Lambda_1, \Lambda_2, \Lambda_3, {}_4\tilde{\Lambda}_w),$$

we calculate

$$\begin{aligned}
 & \langle (\Lambda_1 \varphi, \Lambda_2(z, \varphi), \Lambda_3(z, \varphi), {}_4\tilde{\Lambda}_w(u, z)), (M, S, T, N) \rangle = \\
 & = \int_0^\ell \{M(x) \nabla \varphi(x) - S(x) [\nabla^2 z(x) + \nabla \varphi(x)] + T(x) [\nabla z(x) + \varphi(x)] + \\
 & \quad + N(x) [\nabla u(x) + \nabla w(x) \nabla z(x)]\} dx = - \int_0^\ell N_{,x} u dx - \\
 & \quad - \int_0^\ell [S_{,xx} + T_{,x} + (Nw_{,x})_{,x}] z dx + \int_0^\ell [(S - M)_{,x} + T] \varphi dx + \\
 & \quad + N(\ell) u(\ell) + [M(\ell) - S(\ell)] \varphi(\ell) - [M(0) - S(0)] \varphi(0) \\
 & \quad - S(\ell) z_{,x}(\ell) + S(0) z_{,x}(0) = \langle (\Lambda_1^* M, \Lambda_2^* S, \Lambda_3^* T, {}_4\tilde{\Lambda}_w^* N), (u, z, \varphi) \rangle = \\
 & = \langle (\Lambda_w^0)^* (M, S, T, N), (u, z, \varphi) \rangle, \tag{3.18}
 \end{aligned}$$

where  $\nabla \varphi = \varphi_{,x}$ ,  $\nabla^2 z = z_{,xx}$ , etc.

Hence

$$(\Lambda_w^0)^* (M, S, T, N) = \begin{cases} -N_{,x}, & \text{in } (0, \ell) \\ -[S_{,xx} + (Nw_{,x})_{,x} + T_{,x}], & \text{in } (0, \ell) \\ -[(M - S)_{,x} - T], & \text{in } (0, \ell) \\ N(\ell), & \text{if } x = \ell \\ S(0), & \text{if } x = 0 \\ -S(\ell), & \text{if } x = \ell \\ -[M(0) - S(0)], & \text{if } x = 0 \\ M(\ell) - S(\ell), & \text{if } x = \ell. \end{cases} \tag{3.19}$$

The inclusion (2.30) is written as follows

$$\Lambda(\tilde{u}, \tilde{w}, \tilde{\varphi}) \in \partial_{(M, S, T, N)} H(\tilde{u}, \tilde{w}, \tilde{\varphi}, \tilde{M}, \tilde{S}, \tilde{T}, \tilde{N}), \text{ where } \tilde{N} = q.$$

Hence the constitutive relationships, inverse to the these given by (1.2), are readily recovered

$$\begin{aligned}
 \rho(\tilde{\varphi}(x)) &= \frac{\partial W^*[\tilde{M}(x), \tilde{S}(x), \tilde{T}(x), \tilde{N}]}{\partial M} = \frac{S}{D}, \\
 \eta(\tilde{w}(x), \tilde{\varphi}(x)) &= \frac{\partial W^*[\tilde{M}(x), \tilde{S}(x), \tilde{T}(x), \tilde{N}]}{\partial S} = \frac{84}{D} S, \\
 d(\tilde{w}(x), \tilde{\varphi}(x)) &= \frac{\partial W^*[\tilde{M}(x), \tilde{S}(x), \tilde{T}(x), \tilde{N}]}{\partial T} = \frac{T}{H}, \\
 \varepsilon(\tilde{u}(x), \tilde{w}(x)) &= \frac{\partial W^*[\tilde{M}(x), \tilde{S}(x), \tilde{T}(x), \tilde{N}]}{\partial N} = \frac{q}{A}.
 \end{aligned}$$

The differential inclusion (2.31) takes now the form

$$(\Lambda_w^0)^* (\tilde{M}, \tilde{S}, \tilde{T}, \tilde{N}) \in \partial_{(u, w, \varphi)} H(\tilde{u}, \tilde{w}, \tilde{\varphi}, \tilde{M}, \tilde{S}, \tilde{T}, \tilde{N}), \quad \tilde{N} = q \tag{3.20}$$

The local form of the r.h.s. is

$$\begin{aligned} & \partial_{(u(x), w(x), \varphi(x))} H[\tilde{u}(x), \tilde{w}(x), \tilde{\varphi}(x), \tilde{M}(x), \tilde{S}(x), \tilde{T}(x), \tilde{N})] = \\ & = DF(\tilde{u}, \tilde{w}, \tilde{\varphi})(x) = \begin{cases} p(x), & \text{in } (0, \ell) \\ m(x), & \text{in } (0, \ell) \\ q, & \text{if } x = \ell \\ s_0, & \text{if } x = 0 \\ -s_1, & \text{if } x = \ell \\ -r_0, & \text{if } x = 0 \\ r_1, & \text{if } x = \ell. \end{cases} \end{aligned} \quad (3.21)$$

Consequently, (3.20) and (3.21) yield:

$$\begin{aligned} & -\tilde{N}_{,x} = 0, \\ & \tilde{S}_{,xx} + (\tilde{N}\tilde{w}_{,x})_{,x} + \tilde{T}_{,x} + p = 0, \text{ in } (0, \ell), \\ & (\tilde{M} - \tilde{S})_{,x} - \tilde{T} + m = 0, \text{ in } (0, \ell), \\ & \tilde{N}(\ell) = q, \tilde{S}(0) = s_0, \tilde{S}(\ell) = s_1, \\ & \tilde{M}(0) - \tilde{S}(0) = r_0, \tilde{M}(\ell) - \tilde{S}(\ell) = r_1. \end{aligned}$$

#### 4. Problem ( $P_2$ ) and duality

If one of the ends of the beam is fixed, the dual formulation for  $q < 0$  significantly complicates. It will be shown, however, that the dual principle in the case of the boundary conditions (1.10) and  $q < 0$  assumes the form of sup inf or inf sup principles, depending upon the choice of Lagrangians.

##### 4.1. Case 1 : sup inf principle

First, we will formulate the dual problem as an sup inf principle.

For  $(u, w) \in \mathcal{V}_2$  we define the operator  $\Lambda$

$$\begin{aligned} \Lambda(u, w) &= (\Lambda_1 u, \Lambda_2 w, \Lambda_3 w) = (\nabla u, \nabla w, \nabla^2 w), \\ \Lambda : \mathcal{V}_2 &\rightarrow Y = Y^* = L^2(0, \ell) \times L^2(0, \ell) \times L^2(0, \ell). \end{aligned} \quad (4.1)$$

We observe that now  $\Lambda$  is a linear operator. The functional  $J_2$  of the total potential energy (1.13) may be written as follows

$$J_2(u, w) = \tilde{W}_1(\Lambda_1 u, \Lambda_2 w) + \tilde{W}_2(\Lambda_3 w) + F(u, w), (u, w) \in \mathcal{V}_2, \quad (4.2)$$

where

$$\tilde{W}_1(r, s) = \int_0^\ell \frac{A}{2} (r + \frac{1}{2} s^2)^2 dx, \quad (r, s) \in \mathcal{H}, \quad (4.3)$$

$$\tilde{W}_2(t) = \int_0^\ell \frac{D}{2} t^2 dx, \quad t \in L^2(0, \ell), \quad (4.4)$$

provided that the stored energy function is given by (1.13). The perturbed functional  $\Phi$  is now assumed in the following form

$$\Phi(u, w, p_1, p_2, p_3) = \tilde{W}_1(\Lambda_1 u + p_1, \Lambda_2 w + p_2) + \tilde{W}_2(\Lambda_3 w + p_3) + F(u, w), \\ (u, w) \in \mathcal{V}_2, \quad (p_1, p_2, p_3) \in Y. \quad (4.5)$$

According to the "standard" theory of duality (Ekeland, Temam, 1976) we determine the first Lagrangian

$$L_1(u, w, N, Q, M) = \\ = \sup \left\{ \int_0^\ell (Np_1 + Qp_2 + Mp_3)dx - \Phi(u, w, p_1, p_2, p_3) \mid (p_1, p_2, p_3) \in Y \right\}. \quad (4.6)$$

After some calculations one obtains

$$L_1(u, w, N, Q, M) = \quad (4.7) \\ = \begin{cases} F(u, w) + \int_0^\ell [Nu_{,x} + Qw_{,x} + Mw_{,xx} - \\ - (\frac{1}{2D}M^2 + \frac{1}{2A}N^2 + \frac{1}{2N}Q^2)]dx, & \text{if } N > 0 \\ F(u, w) + \int_0^\ell (Mw_{,xx} - \frac{1}{2D}M^2)dx & \text{if } N = 0 \text{ and } Q = 0 \\ -\infty, & \text{if } N < 0 \text{ or } N = 0 \text{ and } Q \neq 0. \end{cases}$$

Moreover

$$J(u, w) = \sup \{ L_1(u, w, N, Q, M) \mid (N, Q, M) \in Y^* \}. \quad (4.8)$$

From (4.7) and (4.8) we conclude that the Lagrangian  $L_1$  is appropriate for  $N \geq 0$ , i.e. for  $q \geq 0$ . Then the dual problem  $(\mathcal{P}_2^*)$  is to find

$$\sup_{(N, Q, M) \in Y^*} \inf_{(u, w) \in \mathcal{V}_2} L(u, w, N, Q, M).$$

In the sequel we will focus on the case  $q < 0$ . Let us introduce the following Lagrangian

$$-L_2(u, w, N, Q, M) = \\ = \inf_{p_2 \in L^2(0, \ell)} \sup_{(p_1, p_3) \in \mathcal{L}} \int_0^\ell (Np_1 + Qp_2 + Mp_3)dx - \Phi(u, w, p_1, p_2, p_3), \quad (4.9)$$

where  $\mathcal{L} = L^2(0, \ell) \times L^2(0, \ell)$  and  $\Phi$  is still given by (4.5). Hence

$$L_2(u, w, N, Q, M) = F(u, w) - \tilde{W}_2^*(M) + \int_0^\ell (Nu_{,x} + Qw_{,x})dx + \\ - \inf_{p_2 \in L^2(0, \ell)} \sup_{p_1 \in L^2(0, \ell)} \left[ \int_0^\ell (Np_1 - \frac{A}{2}p_1^2)dx + \int_0^\ell (Qp_2 - \frac{1}{2}Np_2^2)dx \right],$$

and finally

$$L_2(u, w, N, Q, M) = \quad (4.10)$$

$$= \begin{cases} F(u, w) + \int_0^\ell [Nu_{,x} + Qw_{,x} + Mw_{,xx} - \\ - \frac{1}{2D}M^2 + \frac{1}{2A}N^2 + \frac{1}{2N}Q^2]dx, & \text{if } N < 0; \\ F(u, w) + \int_0^\ell (Mw_{,xx} - \frac{1}{2D}M^2)dx, & \text{if } N = 0 \text{ and } Q = 0; \\ +\infty, & \text{if } N > 0 \text{ or } N = 0 \text{ and } Q \neq 0. \end{cases}$$

Here

$$\tilde{W}_2^*(M) = \sup \left\{ \int_0^\ell (Mt - \frac{D}{2}t^2)dx \mid t \in L^2(0, \ell) \right\} = \int_0^\ell \frac{1}{2D}M^2(x)dx.$$

Let

$$G_2(N, Q, M) := \sup_{(u, w) \in \mathcal{V}_2} L(u, w, N, Q, M). \quad (4.11)$$

Taking account of (4.10) we obtain

$$G_2(N, Q, M) = \quad (4.12)$$

$$= \begin{cases} - \int_0^\ell (\frac{1}{2D}M^2 + \frac{1}{2A}N^2 + \frac{1}{2N}Q^2)dx & \text{if } N < 0 \text{ and } N_{,x} = 0, N(\ell) = q, \\ & M_{,xx} - Q_{,x} - p = 0, M(\ell) = -m; \\ - \int_0^\ell \frac{1}{2D}M^2dx, & \text{if } N = 0, Q = 0 \\ & M_{,xx} - p = 0, M(\ell) = -m; \\ +\infty, & \text{otherwise.} \end{cases}$$

The equilibrium equation  $M_{,xx} - Q_{,x} - p = 0$  is formally equivalent to

$$Q(x) = \nabla M(x) - \nabla M(0) + Q(0) - R(x), \quad (4.13)$$

where  $R$  has been defined in the previous section. It will now be shown that the dual problem

$$(\mathcal{P}_2^\oplus) \quad \sup_{M \in \mathcal{M}} \inf_{(Q, N) \in \mathcal{L}} G_2(N, Q, M)$$

is well defined provided that

$$\frac{q}{D} + \frac{2}{\ell^2} > 0, \quad (4.14)$$

where

$$\mathcal{M} = \left\{ M \mid M_{,xx} - \frac{q}{D}M - p = 0, \text{ in } (0, \ell); M(\ell) = m \right\}. \quad (4.15)$$

To corroborate this statement, by accounting for (4.13), we find

$$\begin{aligned} \inf_{Q \in L^2(0, \ell)} \inf_{N \in L^2(0, \ell)} G_2(N, Q, M) &= \\ &= \inf_{Q(0) \in \mathbf{R}} \left\{ - \int_0^\ell \left[ \frac{1}{2D} M^2 + \frac{1}{2A} q^2 + \right. \right. \\ &\quad \left. \left. + \frac{1}{2q} (M_{,x} - M_{,x}(0) + Q(0) - R)^2 \right] dx \right\} = \\ &= - \int_0^\ell \left[ \frac{1}{2D} M^2 + \frac{1}{2A} q^2 + \frac{1}{2q} \left( M_{,x} + \frac{M(0)}{\ell} + Z(x) \right)^2 \right] dx, \end{aligned} \quad (4.16)$$

since the infimum over  $Q(0) \in \mathbf{R}$  is attained for

$$Q(0) = \frac{m}{\ell} + \frac{M(0)}{\ell} + M_{,x}(0) + \frac{1}{\ell} \int_0^\ell R(x) dx.$$

We recall that  $Z(x)$  is given by (3.10). Let us denote by  $g$  the functional on the r.h.s. of (4.16), that is

$$g(M) := - \int_0^\ell \left[ \frac{1}{2D} M^2 + \frac{1}{2A} q^2 + \frac{1}{2q} \left( M_{,x} + \frac{M(0)}{\ell} + Z(x) \right)^2 \right] dx. \quad (4.17)$$

By  $\overline{M}$  we denote the bending moments field rendering the functional  $g$  stationary; hence

$$\begin{aligned} \overline{M}_{,xx} - \frac{q}{D} \overline{M} - p &\doteq 0, \text{ in } (0, \ell), \\ \overline{M}_{,x}(0) + \overline{M}(0) + \int_0^\ell R(x) dx + m &= 0, \quad \overline{M}(\ell) = -m. \end{aligned} \quad (4.18)$$

For  $M \in H^1(0, \ell)$ ,  $M(\ell) = -m$ , after lengthy calculations one obtains

$$\begin{aligned} g(M) - g(\overline{M}) &= - \int_0^\ell \left[ \frac{1}{2D} (M - \overline{M})^2 + \frac{1}{2q} (M_{,x} - \overline{M}_{,x})^2 \right] dx + \\ &\quad + \frac{1}{2q\ell} (M(0) - \overline{M}(0))^2. \end{aligned} \quad (4.19)$$

The last step consists in proving that for bending moments satisfying

$$M_{,xx} - \frac{q}{D} M - p = 0, \text{ for } x \in (0, \ell); \quad M(\ell) = -m, \quad (4.20)$$

one has

$$g(M) \leq g(\overline{M}). \quad (4.21)$$

Towards this end the Poincaré's inequality in the following form is employed

$$\int_0^\ell v^2(x) dx \leq \frac{\ell^2}{2} \int_0^\ell v_{,x}^2(x) dx + \frac{1}{\ell} \left\{ \int_0^\ell v(x) dx \right\}^2. \quad (4.22)$$

It holds for any  $v \in H^1(0, \ell)$ , thus particularly for  $v$  such that  $v(\ell) = 0$ . By applying the Poincaré's inequality to  $v(x) = \nabla M(x) - \nabla \overline{M}(x)$ ,  $M(\ell) = m$ ,  $\overline{M}(\ell) = -m$  or  $v(\ell) = M(\ell) - \overline{M}(\ell) = 0$ , we readily obtain

$$\int_0^\ell (M_{,x} - \overline{M}_{,x})^2 dx \leq \frac{\ell^2}{2} \int_0^\ell \frac{q^2}{D} (M - \overline{M})^2 dx + \frac{1}{\ell} [M(0) - \overline{M}(0)]^2,$$

because

$$M_{,xx} - \overline{M}_{,xx} = \frac{q}{D} (M - \overline{M}).$$

Thus we finally have

$$g(M) - g(\overline{M}) \leq - \int_0^\ell \left[ \frac{1}{2D} (M - \overline{M})^2 + \frac{\ell^2 q}{4D^2} (M - \overline{M})^2 \right] dx \leq 0,$$

provided that

$$0 > q \geq -\frac{2D}{\ell^2}. \quad (4.23)$$

#### 4.2. Case 2 : inf sup principle

In the present subsection the dual problem will be formulated in the form of inf sup principle. The perturbed functional is now given by

$$\begin{aligned} \Phi(u, w, p_1, p_2, p_3) = \int_0^\ell \left\{ \frac{A}{2} [u_{,x} + p_1 + \frac{1}{2} w_{,x} (w_{,x} + p_3)]^2 + \right. \\ \left. + \frac{D}{2} (w_{,xx} + p_2)^2 \right\} dx + F(u, w), \end{aligned} \quad (4.24)$$

where  $(u, w) \in \mathcal{V}_2$  and  $(p_1, p_2, p_3) \in Y = [L^2(0, \ell)]^3$ . One sees that

$$\Phi(u, w, 0, 0, 0) = J(u, w), \quad (u, w) \in \mathcal{V}_2.$$

The Lagrangian is defined by

$$\begin{aligned} L_3(u, w, N, M, Q) = \\ = \inf_{p_3 \in L^2(0, \ell)} \sup_{(p_1, p_2) \in \mathcal{L}} \left\{ \int_0^\ell (N p_1 + M p_2 + Q p_3) dx - \right. \\ \left. - \int_0^\ell \frac{A}{2} (u_{,x} + p_1 + \frac{1}{2} w_{,x} (w_{,x} + p_3))^2 + \frac{D}{2} (w_{,xx} + p_2)^2 dx - F(u, w) \right\}. \end{aligned}$$

Hence

$$-L_3(u, w, N, M, Q) = \quad (4.25)$$

$$= \begin{cases} F(u, w) + \int_0^\ell \left( N u_{,x} + Q w_{,x} + M w_{,xx} \right. \\ \left. - \left( \frac{1}{2D} M^2 + \frac{1}{2A} N^2 \right) \right) dx, & \text{if } Q = \frac{1}{2} N w_{,x}; \\ +\infty, & \text{if } Q \neq \frac{1}{2} N w_{,x}. \end{cases}$$



We observe that by introducing the multiplier  $\lambda$  the Lagrangian  $L_3$  may be written in the form

$$L_3(u, w, N, M, Q) = \sup_{\lambda \in R} L(u, w, N, M, Q, \lambda) \quad (4.26)$$

where

$$\begin{aligned} L(u, w, N, M, Q, \lambda) = \\ = F(u, w) + \int_0^\ell [Nu_{,x} + Qw_{,x} + Mw_{,xx} - \\ - (\frac{1}{2D}M^2 + \frac{1}{2A}N^2) + \lambda(Q - \frac{1}{2}Nw_{,x})]dx \end{aligned} \quad (4.27)$$

It can be shown that

$$J(u, w) = \sup_{(N, M) \in \mathcal{L}} \inf_{Q \in L^2(0, \ell)} L_3(u, w, N, M, Q). \quad (4.28)$$

The second dual principle is formulated by using the functional

$$\begin{aligned} G_3(N, M, Q) = \inf_{(u, w) \in \mathcal{V}_2} L_3(u, w, N, M, Q) = \\ = \begin{cases} G(M, Q), & \text{if } N = q, Q(0) = 0, \int_0^\ell Q dx = 0, \\ +\infty, & \text{if } Q(0) \neq 0 \text{ or } \int_0^\ell \frac{Q}{N} dx \neq 0, \\ -\infty, & \text{if } N \neq 0, Q(0) = 0, \int_0^\ell \frac{Q}{N} dx = 0, \end{cases} \end{aligned}$$

where

$$G(M, Q) = \int_0^\ell (\frac{2}{q}Q^2 + \frac{2}{q}MQ_{,x} + \frac{2}{q}RQ - \frac{1}{2D}M^2)dx + \frac{2}{q}mQ(\ell) - \frac{\ell}{2A}q^2. \quad (4.29)$$

The second dual problem, denoted by  $(\mathcal{P}_2^\odot)$ , is to find

$$(\mathcal{P}_2^\odot) \quad \inf_{Q \in H^1(0, \ell)} \sup_{(N, M) \in \mathcal{L}} G_3(N, Q, M).$$

Under an additional condition (see (4.40)), the last problem is well defined. To corroborate this statement we successively find

$$\begin{aligned} \sup_{N \in L^2(0, \ell)} G_3(N, M, Q) = \\ = \begin{cases} G(M, Q), & \text{if } Q(0) = 0, \int_0^\ell Q dx = 0, \\ +\infty, & \text{if } Q(0) \neq 0 \text{ or } \int_0^\ell Q dx \neq 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \sup_{M \in L^2(0, \ell)} \sup_{N \in L^2(0, \ell)} G_3(N, Q, M) = \\ = \begin{cases} f(Q), & \text{if } Q(0) = 0, \int_0^\ell Q dx = 0, \\ +\infty, & \text{if } Q(0) \neq 0 \text{ or } \int_0^\ell Q dx \neq 0 \end{cases} \end{aligned}$$

where

$$f(Q) = \frac{2}{q} \int_0^\ell (Q^2 + RQ + \frac{D}{q} Q_{,x}^2) dx + \frac{2}{q} mQ(\ell) - \frac{\ell}{2A} q^2. \quad (4.30)$$

Further, we have

$$\inf_{Q \in H^1(0,\ell)} \sup_{(N,M) \in \mathcal{L}} G_3(N, M, Q) = \inf \{f(Q) \mid Q \in \mathcal{W}\} \quad (4.31)$$

where

$$\mathcal{W} = \left\{ Q \mid Q(0) = 0, \int_0^\ell Q dx = 0 \right\} \quad (4.32)$$

For  $Q \in \mathcal{W}$  and  $\bar{Q} \in \mathcal{W}$  one has

$$\begin{aligned} f(Q) - f(\bar{Q}) &= \frac{2}{q} \int_0^\ell [(Q - \bar{Q})^2 + \frac{D}{q} (Q_{,x} - \bar{Q}_{,x})^2] dx + \\ &\quad + \frac{4}{q} \int_0^\ell (Q - \bar{Q})(\bar{Q} - \frac{D}{q} \bar{Q}_{,xx} + \frac{R}{2}) dx + \\ &\quad + \frac{2}{q} [Q(\ell) - \bar{Q}(\ell)] [m + \frac{2D}{q} \bar{Q}_{,x}(\ell)]. \end{aligned} \quad (4.33)$$

Let  $\tilde{Q} \in \mathcal{W}$  be an element satisfying

$$\begin{aligned} \tilde{Q} - \frac{D}{q} \tilde{Q}_{,xx} + \frac{R}{2} &= C, \text{ in } (0, \ell); \\ \frac{2D}{q} \tilde{Q}_{,x}(\ell) + m &= 0. \end{aligned} \quad (4.34)$$

The constant  $C$  conforms with the fact that the second order Eq.(4.34)<sub>1</sub> has to satisfy three imposed conditions. Then (4.33) reduces to

$$f(Q) - f(\tilde{Q}) = \frac{2}{q} \int_0^\ell [(Q - \tilde{Q})^2 + \frac{D}{q} (Q_{,x} - \tilde{Q}_{,x})^2] dx, \quad (4.35)$$

for any  $Q \in \mathcal{W}$ . Hence, if

$$\tilde{\lambda}_1^2 = \inf_{v \in \mathcal{V}} \frac{\int_0^\ell v_{,x}^2 dx}{\int_0^\ell v(x) dx}, \quad (4.36)$$

where

$$\mathcal{V} = \left\{ v \in H^1(0, \ell) \mid v(0) = 0, \int_0^\ell v(x) dx = 0 \right\}, \quad (4.37)$$

then one has

$$\forall Q \in \mathcal{V}, \quad f(Q) - f(\tilde{Q}) \geq 0 \quad (4.38)$$

provided that

$$q < 0, \quad \frac{q}{D} + \tilde{\lambda}_1^2 \geq 0. \quad (4.39)$$

REMARK 4.1 For  $v \in \mathcal{V}$  the Poincaré's inequality (4.22) reduces to

$$\|v\|_{L^2(0,\ell)}^2 \leq \frac{\ell^2}{2} \|\nabla v\|_{L^2(0,\ell)}^2.$$

In this case we have  $\frac{1}{\tilde{\lambda}_1^2} \leq \frac{\ell^2}{2}$  and consequently

$$0 > q \geq -\frac{2D}{q} \geq -\tilde{\lambda}_1^2 D. \quad (4.40)$$

### Concluding remarks

The results obtained for two models of nonlinear elastic beams exhibit the role of boundary conditions on the choice of Lagrangians from which dual principles are derived. Some of the results proposed by Auchmuty (1983) have been generalized so as to include nonlinear operators  $\Lambda$ . However, the available theoretical results concerning anomalous dual variational principles are insufficient even for the compressed beam fixed at one end. Therefore new developments in the spirit of Auchmuty's paper, Auchmuty (1983) and our Section 2 are necessary in order to find, for instance, dual principles for compressed von Kármán plates. Much remains to be done in the search for dual principles in nonlinear solid and structural mechanics. Most promising seems to be an approach in which properly set up Lagrangians are assumed as a starting point.

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