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Optimal control in coefficients for parabolic variational inequalities. Singular perturbations in optimal control problem for a parabolic variational inequality

by

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This paper concerns an optimal control problem for a parabolic variational inequality with controls appearing in coefficients, right hand sides and convex sets of states. Moreover, the paper deals with an optimal control problem of parabolic singular perturbations in variational inequality. Existence of an optimal control is verified.

Introduction

This paper is concerned with an optimal control problem for parabolic variational inequalities, where the linear symmetric operators as well as the convex sets of possible states depend on the control parameter. Moreover, we shall deal with singular perturbation of an optimal control problem for a parabolic variational inequality appearing in coefficients, right hand sides and convex sets of states. We introduce an abstract framework for the theoretical study design problem in the parabolic variational inequality context for singular perturbation of an optimal control problem. We give the first properties of the solutions of the distributed control problems governed by parabolic variational inequalities. The existence theorem of this problem will be applied to the singular perturbed optimal control. Singular perturbations in variational inequalities were considered by Huet (1960, 1973), Lions (1973), Greenlee (1969, 1970) and Eckhaus, Moet (1966, 1978), while the respective optimal control problems were considered by Lions (1973). The main concern is the existence of solutions with some weak convergence theorems, but the above authors obtained the weak convergence theorems for the singular perturbations of variational inequalities.

1. Existence and uniqueness theorem for a parabolic variational inequality

1.1. Basic assumptions

We describe some function spaces. More details can be found in the books by Barbu (1978, 1984) and Brezis (1972, 1973). If $E(\Omega)$ is a Banach space, then we shall denote by $L_p(0,T,E(\Omega))$ the space of all p-integrable E-valued functions on $[0,T]$, and by $C([0,T],E(\Omega))$ the usual Banach space of all continuous functions from $[0,T]$ to $E(\Omega)$. Further, $C^k([0,T],E(\Omega))$ denotes the space of all k-times continuously differentiable functions $(: [0,T] \to E(\Omega))$. We shall denote by $W_p^m(0,T,V(\Omega))$ the space $\{v \in L_p(0,T,E(\Omega));\ dv/dt \in L_p(0,T,E(\Omega))\ldots;\ d^m v/dt^m \in L_p(0,T,E(\Omega))\}$, where the derivatives dv/dt, $d^2v/dt^2, \ldots$ of v are taken in the sense of vectorial distributions on $(0,T)$. Equivalently, $v \in W_p^1(0,T,E(\Omega))$ means that $v : [0,T] \to E(\Omega)$ is absolutely continuous, a.e. differentiable on $(0,T)$ and

$$v(t) = v(0) + \textstyle\int_0^t dv(s)/ds \qquad \text{for } t \in [0,T],\ dv/ds \in L_p(0,T,E(\Omega)) \tag{1.1}$$

The spaces $L_p(0,T,E(\Omega))$, $1 < p < \infty$, are reflexive and the dual spaces $[L_p(0,T,E(\Omega))]^*$ can be identified with the space $L_q(0,T,E^*(\Omega))$, $1/p+1/q=1$. The space $L_\infty(0,T,E^*(\Omega))$ can be identified with the dual space $[L_1(0,T,E(\Omega))]^*$, i.e. for every $\mathcal{F} \in [L_1(0,T,E(\Omega))]^*$ there exists a unique function $\theta \in L_\infty(0,T,E^*(\Omega))$ satisfying the relations:

$$||\mathcal{F}||_{[L_1(0,T,E(\Omega))]^*} = ||\theta||_{L_\infty(0,T,E^*(\Omega))}$$

and

$$\mathcal{F}(v) = \textstyle\int_0^T \langle \theta(t), v(t)\rangle_{L_1(0,T,E(\Omega))} dt \qquad \text{for every } v \in L_1(0,T,E(\Omega))$$

On the other hand, if $E(\Omega)$ is a Hilbert space with the inner product $(.,.)_{E(\Omega)}$, then $L_2(0,T,E(\Omega))$ is a Hilbert space with the inner product

$$(v,z)_{L_2(0,T,E(\Omega))} = \textstyle\int_0^T (v(t),z(t))_{E(\Omega)} dt \qquad v,z \in L_2(0,T,E(\Omega))$$

Moreover, $W_2^m(0,T,E(\Omega))$ is a Hilbert space with the inner product

$$\begin{aligned}(v,z)_{W_2^m(0,T,E(\Omega))} &= (v,z)_{L_2(0,T,E(\Omega))} + (dv/dt, dz/dt)_{L_2(0,T,E(\Omega))} \\ &+ (d^2v/dt^2, d^2z/dt^2)_{L_2(0,T,E(\Omega))} \cdots (d^m v/dt^m, d^m z/dt^m)_{L_2(0,T,E(\Omega))}\end{aligned}$$

Given a lower semicontinuous convex functional F: $E(\Omega) \to \overline{R} = (-\infty, +\infty]$ we shall denote by $\partial F(v) \in E^*(\Omega)$ (the dual space of $E(\Omega)$) the set of all subgradients of F at v, i.e.

$$\partial F(v) = \{v^* \in E^*(\Omega) : F(v) \le F(z) + \langle v^*, v-z\rangle_{E(\Omega)} \qquad \text{for all } z \in E(\Omega)\}$$

If F is Gâteaux differentiable at v, then $\partial F(v)$ consists a single element, namely the gradient $(\operatorname{grad} F(v))$ of F at v. The mapping $\partial F : E(\Omega) \to E^*(\Omega)$ is called the subdifferential of F. We shall denote by $D(F) = \{v : F(v) < +\infty\}$ the effective domain of F and by $D(\partial F)$ the domain of ∂F; i.e., $D(\partial F) = \{v \in E(\Omega) : \partial F(v) \neq \emptyset\}$.

Let $V(\Omega)$ be a Hilbert space with an inner product $(.,.)_{V(\Omega)}$ and a norm $\|.\|_{V(\Omega)}$, $\|.\|_{V^*(\Omega)}$ its dual space with the duality pairing $\langle .,. \rangle_{V(\Omega)}$ and the norm $\|.\|_{V^*(\Omega)}$. Moreover, $L(V(\Omega), V^*(\Omega))$ is the space of all linear bounded operators from $V(\Omega)$ into $V^*(\Omega)$ with the norm $\|.\|_{L(V(\Omega),V^*(\Omega))}$. We suppose that $V(\Omega) \subset H(\Omega)$ where $H(\Omega)$ is a Hilbert space, $V(\Omega)$ is dense in $H(\Omega)$. If we identify $H(\Omega)$ with its dual we have $V(\Omega) \subset H(\Omega) \subset V^*(\Omega)$ and the notation $(.,.)_{H(\Omega)}$ to denote the scalar product in $H(\Omega)$. As a consequence of the previous identifications, the scalar product in $H(\Omega)$ of $\mathcal{F} \in H(\Omega)$ and $v \in V(\Omega)$ is the same as the scalar product of $\mathcal{F}$ and v in the duality between $V(\Omega)$ and $V^*(\Omega)$. And, we put

$$\langle \mathcal{F}, v\rangle_{V(\Omega)} = (\mathcal{F}, v)_{H(\Omega)} \qquad \text{for any } \mathcal{F} \in H(\Omega), \text{ for any } v \in V(\Omega) \tag{1.2}$$

Let constants $\alpha_*, M_* (0 < \alpha_* < M_*)$ be given. We denote by $E_{V(\Omega)}(\alpha_*, M_*)$ the class of the linear operators $\mathcal{A}(t) : V(\Omega) \to V^*(\Omega)$ (for any $t \in [0,T]$) such that

$$\left\{\begin{array}{l} 1.\ \alpha_* \|v\|^2_{V(\Omega)} \leq \langle \mathcal{A}(t)v, v\rangle_{V(\Omega)} \leq M_* \|v\|^2_{V(\Omega)} \\ \quad \text{for all } v \in V(\Omega) \\ 2.\ \mathcal{A}(.) \in C^1([0,T], L(V(\Omega), V^*(\Omega))) \\ 3.\ \langle \mathcal{A}(t)v, z\rangle_{V(\Omega)} = \langle \mathcal{A}(t)z, v\rangle_{V(\Omega)} \\ \quad \text{for all } v, z \in V(\Omega) \text{ and } t \in [0,T] \end{array}\right. \tag{M1}$$

We consider the initial value problem

$$\left\{\begin{array}{l} u(t) \in \mathcal{K}(\Omega),\ (t \text{ traversing the interval } [0,T]), \text{ such that} \\ \langle du(t)/dt, v - u(t)\rangle_{V(\Omega)} + \langle A(t)u(t), v - u(t)\rangle_{V(\Omega)} \\ +\Phi(v) - \Phi(u(t)) \geq \langle L(t), v - u(t)\rangle_{V(\Omega)} \\ \text{for all } v \in \mathcal{K}(\Omega), \text{ for a.e. } t \in [0,T] \\ u(0) = u_0 \in \mathcal{K}(\Omega) \cap D(\Phi) \end{array}\right. \tag{1.3}$$

where $\mathcal{K}(\Omega)$ is a closed convex subset of $V(\Omega)$, du/dt is the strong derivative of $u; [0,T] \to V^*(\Omega)$ and

$$\left\{\begin{array}{l} A(t) \in E_{V(\Omega)}(\alpha, M), t \in [0,T], L \in W_2^1(0,T,H(\Omega)), \\ \{A(0)u_0 + \partial I_{\mathcal{K}(\Omega)}(u_0) + \partial\Phi(u_0) - L(0)\} \cap H(\Omega) \neq \emptyset, \\ \text{the functional } v \to \Phi(v) \text{ is convex, lower semi-continuous} \\ \text{for the weak topology of } V(\Omega), \text{ with values in } (-\infty, +\infty] \end{array}\right. \tag{M2}$$

($I_{\mathcal{K}(\Omega)}$ is the indicator function of some closed convex subset $\mathcal{K}(\Omega)$ of $V(\Omega)$, i.e. $I_{\mathcal{K}(\Omega)}(v) = 0$ if $v \in \mathcal{K}(\Omega)$, $I_{\mathcal{K}(\Omega)}(v) = +\infty$ if $v \notin \mathcal{K}(\Omega)$.

1.2. The approximation result for the solutions to (1.3)

Consider the approximating equations (a penalized parabolic initial value problem) corresponding to (1.3):

$$\begin{cases} du_\epsilon(t)/dt + A(t)u_\epsilon(t) + \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(u_\epsilon(t)) \\ + \operatorname{grad}\Phi_\epsilon(u_\epsilon(t)) = L(t), \qquad \epsilon > 0 \\ u_\epsilon(0) = u_0 \end{cases} \tag{1.4}$$

We approximate

$$du(t)/dt + A(t)u(t) + \partial I_{\mathcal{K}(\Omega)}(u(t)) + \partial\Phi(u(t)) \ni L(t) \qquad \text{a.e. } t \in [0,T] \tag{1.5}$$

by replacing $\partial I_{\mathcal{K}(\Omega)}$ by its Lipschitz-continuous Yoshida approximation $\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon$, $\epsilon > 0$ where

$$(I_{\mathcal{K}(\Omega)})_\epsilon(v) = (2\epsilon)^{-1}\left\|v - P_{\mathcal{K}(\Omega)}(v)\right\|^2_{V(\Omega)}, \quad \epsilon > 0,\ v \in V(\Omega)$$
$$\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(v)(\text{the Fréchet derivative}) = \epsilon^{-1}(v - P_{\mathcal{K}(\Omega)}(v))$$

where $P_{\mathcal{K}(\Omega)}$ is the orthogonal projection onto $\mathcal{K}(\Omega)$, monotone and Lipschitz continuous. The projection operator is defined by: $(P_{\mathcal{K}(\Omega)} : V(\Omega) \to \mathcal{K}(\Omega))$

$$\left\|v - P_{\mathcal{K}(\Omega)}(v)\right\|_{V(\Omega)} = \min_{z \in \mathcal{K}(\Omega)} \|v - z\|_{V(\Omega)},\ v \in V(\Omega)$$

and $P_{\mathcal{K}(\Omega)}$ has the following properties arising directly from its definition (see Barbu, 1984, Lions, 1969)

$$\begin{cases} 1^\circ\ P_{\mathcal{K}(\Omega)}(v) = v \Leftrightarrow v \in \mathcal{K}(\Omega) \\ 2^\circ\ (P_{\mathcal{K}(\Omega)}(v) - v, z - P_{\mathcal{K}(\Omega)}(v))_{V(\Omega)} \geq 0 \\ \quad \text{for all } v \in V(\Omega),\ z \in \mathcal{K}(\Omega) \\ 3^\circ\ \left\|P_{\mathcal{K}(\Omega)}(v) - P_{\mathcal{K}(\Omega)}(z)\right\|_{V(\Omega)} \leq \|v - z\|_{V(\Omega)} \\ \quad \text{for all } v, z \in V(\Omega) \end{cases} \tag{1.6}$$

On the other hand, the operator $\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon$ fulfils then the conditions

$$\begin{cases} 1^\circ\ \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(v) = 0 \Leftrightarrow v \in \mathcal{K}(\Omega) \\ 2^\circ\ \left\langle\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(v) - \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(z), v - z\right\rangle_{V(\Omega)} \geq 0 \\ 3^\circ\ \left\|\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(v) - \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(z)\right\|_{V^*(\Omega)} \leq 2\epsilon\|v - z\|_{V(\Omega)} \\ \quad \text{for all } v, z \in V(\Omega) \end{cases} \tag{1.7}$$

This means, that $(\partial I_{\mathcal{K}(\Omega)})_\epsilon$ is monotone and Lipschitz continuous. Moreover, let

$\{\Phi_\epsilon\}$ be a family of Fréchet differentiable convex functionals on $V(\Omega)$ satisfying the following conditions:

$$\begin{cases} \lim\limits_{\epsilon\to 0} \int_0^T \Phi_\epsilon(v(t))dt = \int_0^T \Phi(v(t))dt \qquad \text{for any } v \in L_2(0,T,V(\Omega)) \\ \text{There exists a real positive constant with} \\ \|\text{grad}\,\Phi_\epsilon(w(t))\|_{V^*(\Omega)} \le constant,\ t\in[0,T] \text{ for any } w(t)\in V(\Omega) \\ \text{If } v_\epsilon \rightharpoonup v,\ dv_\epsilon/dt \rightharpoonup dv/dt \text{ (weakly) in } L_2(0,T,V(\Omega)) \text{ and} \\ \int_0^T \Phi_\epsilon(v_\epsilon)dt \le constant, \text{ then } \liminf\limits_{\epsilon\to 0} \int_0^T \Phi_\epsilon(v_\epsilon)dt \ge \int_0^T \Phi(v)dt \\ \|\text{grad}\,\Phi_\epsilon(v(t)) - \text{grad}\,\Phi_\epsilon(z(t))\|_{V^*(\Omega)} \le c_\epsilon \|v(t)-z(t)\|_{V(\Omega)} \\ \text{for all } v(t), z(t) \in V(\Omega),\ c_\epsilon > 0,\ t\in[0,T] \end{cases} \qquad (M3)$$

THEOREM 1 *Let $T > 0, \epsilon > 0$. Then there exists a unique solution $u_\epsilon \in C^1([0,T], V(\Omega))$ of the initial value problem (1.4) and the sequences $\{u_\epsilon\}$, $\{du_\epsilon/dt\}$ are contained in a bounded subset of $L_2(0,T,V(\Omega)) \cap L_\infty(0,T,H(\Omega))$.*

PROOF. The initial problem can be rewritten in the form

$$\begin{cases} du_\epsilon(t)/dt + \mathcal{Z}_\epsilon(t)u_\epsilon(t) = L(t) \\ u_\epsilon(0) = u_0 \end{cases} \tag{1.8}$$

with

$$\mathcal{Z}_\epsilon(t) : V(\Omega) \to V^*(\Omega)$$
$$\mathcal{Z}_\epsilon(t) = A(t) + \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon + \text{grad}\,\Phi_\epsilon$$

Thus the operators $\mathcal{Z}_\epsilon(t)$ are uniformly Lipschitz continuous and then due to Gajewski, Gröger, Zacharias (1974) the initial value problem (1.8) has a unique solution which is also a unique solution of the problem (1.4).

Let us denote

$$z_\epsilon = u_\epsilon - u_0 \tag{1.9}$$

The function $u_\epsilon \in C^1([O,T], V(\Omega))$ is a solution of the initial value problem

$$\begin{cases} dz_\epsilon(t)/dt + A(t)z_\epsilon(t) + \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(u_0 + z_\epsilon(t)) \\ +\,\text{grad}\,\Phi_\epsilon(u_0 + z_\epsilon(t)) = L(t) - A(t)u_0 \\ z_\epsilon(0) = 0 \end{cases} \tag{1.10}$$

For any function v in $L_2(0,T,V(\Omega))$ which satisfies $dv/dt \in L_2(0,T,V^*(\Omega))$, the equation below holds:

$$\frac{d}{dt}\|v(t)\|^2_{H(\Omega)} = 2\langle dv(t)/dt, v(t)\rangle_{V(\Omega)} \tag{1.11}$$

(see Lions and Magenes, 1968 or Brezis, 1972)
This result will be used in the following step.

After duality pairing in (1.10) we obtain

$$\begin{aligned} & d(\langle z_\epsilon(t), z_\epsilon(t)\rangle_{H(\Omega)})/dt + 2\langle A(t)z_\epsilon(t), z_\epsilon(t)\rangle_{V(\Omega)} \\ & \quad +2\langle \operatorname{grad}\Phi_\epsilon(u_0 + z_\epsilon(t)), z_\epsilon(t)\rangle_{V(\Omega)} \\ & \quad +2\big\langle \big(\partial I_{\mathcal{K}(\Omega)}\big)_\epsilon (u_0 + z_\epsilon(t)), z_\epsilon(t)\big\rangle_{V(\Omega)} \\ & = 2\langle L(t) - A(t)u_0, z_\epsilon(t)\rangle_{V(\Omega)} \end{aligned} \tag{1.12}$$

By (1.3) one has $u_0 \in \mathcal{K}(\Omega)$ and hence $\big(\partial I_{\mathcal{K}(\Omega)}\big)_\epsilon(u_0) = 0$. This means that

$$\big\langle \big(\partial I_{\mathcal{K}(\Omega)}\big)_\epsilon (u_0 + z_\epsilon(t)), z_\epsilon(t)\big\rangle_{V(\Omega)} \geq 0 \tag{1.13}$$

due to the monotonicity of $\big(\partial I_{\mathcal{K}(\Omega)}\big)_\epsilon$. Moreover, we have (due to (M3))

$$\langle \operatorname{grad}\Phi_\epsilon(u_0 + z_\epsilon(t)) - \operatorname{grad}\Phi_\epsilon(u_0), z_\epsilon(t)\rangle_{V(\Omega)} \geq 0 \tag{1.14}$$

(the operator $\operatorname{grad}\Phi_\epsilon(.) : V(\Omega) \to V^*(\Omega)$ is monotone), $\|grad\Phi_\epsilon(u_0)\|_{V^*(\Omega)} \leq M_\Phi$. Thus, by virtue of (1.13), (1.14) and (1.11) we get the inequality

$$\begin{aligned} & \frac{d}{dt}\|z_\epsilon(t)\|^2_{H(\Omega)} + 2\alpha\|z_\epsilon(t)\|^2_{V(\Omega)} \leq \\ & \leq 2\langle L(t) - A(t)u_0, z_\epsilon(t)\rangle_{V(\Omega)} - 2\langle \operatorname{grad}\Phi_\epsilon(u_0), z_\epsilon(t)\rangle_{V(\Omega)} \end{aligned} \tag{1.15}$$

The r.h.s. of (1.15) is majorized by

$$\begin{aligned} & 2\|L(t) - A(t)u_0\|_{V^*(\Omega)}\|z_\epsilon(t)\|_{V(\Omega)} + 2\|\operatorname{grad}\Phi_\epsilon(u_0)\|_{V^*(\Omega)}\|z_\epsilon(t)\|_{V(\Omega)} \\ & \qquad \leq \alpha\|z_\epsilon(t)\|^2_{V(\Omega)} + 2\alpha^{-1}(\|L(t) - A(t)u_0\|^2_{V^*(\Omega)} + M^2_\Phi) \end{aligned}$$

Therefore

$$\tfrac{d}{dt}\|z_\epsilon(t)\|^2_{H(\Omega)} + \alpha\|z_\epsilon(t)\|^2_{V(\Omega)} \leq 2\alpha^{-1}(\|L(t) - A(t)u_0\|^2_{V^*(\Omega)} + M^2_\Phi) \tag{1.16}$$

Integrating (1.16) from 0 to s, $0 < s < T$, we obtain in particular

$$\begin{aligned} & \|z_\epsilon(s)\|^2_{H(\Omega)} \leq \|u_0\|^2_{V(\Omega)} \\ & \qquad +2\alpha^{-1}\textstyle\int_0^s(\|L(t) - A(t)u_0\|^2_{V^*(\Omega)} + M^2_\Phi)dt \\ & \quad \leq \|u_0\|^2_{V(\Omega)} + 2\alpha^{-1}\textstyle\int_0^T(\|L(t) - A(t)u_0\|^2_{V^*(\Omega)} + M^2_\Phi)dt \end{aligned} \tag{1.17}$$

Hence:

$$\begin{aligned} & \sup_{s\in[0,T]}\|z_\epsilon(s)\|^2_{H(\Omega)} \leq \\ & \leq \|u_0\|^2_{V(\Omega)} + 2\alpha^{-1}\int_0^T(\|L(t) - A(t)u_0\|^2_{V^*(\Omega)} + M^2_\Phi)dt \end{aligned} \tag{1.18}$$

The r.h.s. of (1.18) is finite and independent of ϵ, therefore

The sequence z_ϵ remains in a bounded set of $L_\infty(0, T, H(\Omega))$ $\qquad (1.18)_1$

We then integrate (1.16) from 0 to T and get

$$\begin{aligned} &\|z_\epsilon(T)\|^2_{H(\Omega)} + \alpha \int_0^T \|z_\epsilon(t)\|^2_{V(\Omega)} dt \\ &\leq \|u_0\|^2_{V(\Omega)} + 2\alpha^{-1} \int_0^T (\|L(t) - A(t)u_0\|^2_{V^*(\Omega)} + M_\Phi^2) dt \end{aligned} \tag{1.18$_2$}$$

This shows that the sequence $\{z_\epsilon\}$ remains in a bounded set of $L_2(0,T,V(\Omega))$. This means that the sequence $\{u_\epsilon\}$ is bounded in

$$L_2(0,T,V(\Omega)) \cap L_\infty(0,T,H(\Omega)) \qquad \text{as } \epsilon \to 0 \tag{1.18$_3$}$$

On the other hand, in order to obtain the estimate for the sequence $\{du_\epsilon(t)/dt\}$ we formally differentiate the equation (1.10) and arrive at

$$\begin{aligned} &d[dz_\epsilon(t)/dt]/dt + d[A(t)z_\epsilon(t)]/dt + d[(\partial I_{\mathcal{K}(\Omega)})_\epsilon (u_0 + z_\epsilon(t))]/dt \\ &\quad + d[\operatorname{grad}\Phi_\epsilon(u_0 + z_\epsilon(t))]/dt = d[L(t)]/dt - (dA(t)/dt)u_0 \end{aligned} \tag{1.19}$$

Next, we observe that the functions, $(\partial I_{\mathcal{K}(\Omega)}(u_0 + z(.)))_\epsilon$, $\operatorname{grad}\Phi_\epsilon(u_0 + z(.))$: $[0,T] \to V^*(\Omega)$ are Lipschitz continuous (by virtue of (1.7,3°), (M3)). As the space $V^*(\Omega)$ is reflexive the functions $(\partial I_{\mathcal{K}(\Omega)})_\epsilon(u_0+z_\epsilon(.))$, $\operatorname{grad}\Phi_\epsilon(u_0+z(.))$ belongs to the space $W^1_\infty(0,T,V^*(\Omega))$, (see Brezis, 1973). Moreover, the functions $\mathcal{Z}_\epsilon(.)u_\epsilon(.)$, $L(.)$ from the equation (1.8) belong to the spaces $W^1_\infty(0,T,V(\Omega))$ and $W^1_2(0,T,V(\Omega))$, respectively. This means that $u_\epsilon \in W^2_2(0,T,V(\Omega))$ and by virtue of (1.9), (1.19) we can write

$$\begin{aligned} &\langle d^2u_\epsilon(t)/dt^2, du_\epsilon(t)/dt\rangle_{V(\Omega)} + \langle A(t)du_\epsilon(t)/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \\ &\quad + \langle d[(\partial I_{\mathcal{K}(\Omega)})_\epsilon (u_\epsilon(t))]/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \\ &\quad + \langle d[\operatorname{grad}\Phi_\epsilon(u_\epsilon(t))]/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \\ &\quad = \langle dL(t)/dt - (dA(t)/dt)u_\epsilon(t), du_\epsilon(t)/dt\rangle_{V(\Omega)} \quad \text{for a.e. } t \in [0,T] \end{aligned} \tag{1.20}$$

Further, due to (1.11) we have

$$\begin{aligned} &\frac{d}{dt}\|du_\epsilon(t)/dt\|^2_{H(\Omega)} + 2\langle A(t)du_\epsilon(t)/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \\ &\quad + 2\langle d[\operatorname{grad}\Phi_\epsilon(u_\epsilon(t))]/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \\ &\quad + 2\langle d[(\partial I_{\mathcal{K}(\Omega)})_\epsilon (u_\epsilon(t))]/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \\ &\quad = 2\langle dL(t)/dt - (dA(t)/dt)u_\epsilon(t), du_\epsilon(t)/dt\rangle_{V(\Omega)} \end{aligned} \tag{1.21}$$

But (due to the monotonicity of $\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon$ and grad Φ_ϵ), we can write

$$\left\{\begin{array}{ll} \left\langle d[\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(u_\epsilon(t))]/dt, du_\epsilon(t)/dt\right\rangle_{V(\Omega)} \geq 0 & \text{for a.e. } t \in [0,T] \\ \langle d[\operatorname{grad}\Phi_\epsilon(u_\epsilon(t))]/dt, du_\epsilon(t)/dt\rangle_{V(\Omega)} \geq 0 & \end{array}\right. \tag{1.22}$$

On the basis of (M1), (M2) and $(1.18)_3$, (1.21), (1.22) we obtain the inequality

$$\begin{aligned} &\frac{d}{dt}\|du_\epsilon(t)/dt\|^2_{H(\Omega)} + \alpha\|du_\epsilon(t)/dt\|^2_{V(\Omega)} \\ &\leq 2\alpha^{-1}(\|dL(t)/dt\|_{V^*(\Omega)} + \|(dA(t)/dt)u_\epsilon(t)\|_{V^*(\Omega)})^2 \end{aligned} \tag{1.23}$$

Putting $t = 0$ in the equality (1.4) we get (due to the previous estimates: $\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(u_0) = 0$)

$$\langle du_\epsilon(0)/dt, v\rangle_{V(\Omega)} = \langle L(0), v\rangle_{V(\Omega)} - \langle A(0)u_0, v\rangle_{V(\Omega)} - \langle \operatorname{grad}\Phi_\epsilon(u_0), v\rangle_{V(\Omega)}$$

and this gives

$$du_\epsilon(0)/dt = L(0) - A(0)u_0 - \operatorname{grad}\Phi_\epsilon(u_0) \in H(\Omega) \quad \text{(by (M2))} \tag{1.24}$$

which implies (due to (M2), (1.23) and (1.24), $(1.18)_2$)

$$du_\epsilon(t)/dt \in L_2(0,T,V(\Omega)) \cap L_\infty(0,T,H(\Omega)) \tag{1.25}$$

1.3. Solution of a parabolic variational inequality

Due to the a priori estimates obtained above, we obtain existence, uniqueness for a solution of the unilateral problem (1.3). We have:

THEOREM 2 *There exists a unique solution* $u \in W^1_\infty(0,T,H(\Omega)) \cap W^1_2(0,T,V(\Omega))$ *of the initial value problem (1.3)*

PROOF. Let $\epsilon \to 0$, $\epsilon > 0$. Then, due to the a priori estimates $(1.18)_2$ and (1.23) the sequence $\{u_\epsilon\}_\epsilon$ is bounded in all spaces $W^1_p(0,T,V(\Omega))$, $1 \leq p < \infty$. Hence there exists a sequence $\{\epsilon_n\}$, $\epsilon_n > 0$ and a function $u_* \in W^1_2(0,T,V(\Omega))$ such that

$$\lim_{n\to\infty} \epsilon_n = 0 \tag{1.26}$$

$$u_{\epsilon_n} \rightharpoonup u_* \qquad \text{(weakly) in } W^1_2(0,T,V(\Omega)) \tag{1.27}$$

Further, due to (1.4) we have the relation

$$(u_{\epsilon_n}(t), v)_{V(\Omega)} = \left(\int_0^t (du_{\epsilon_n}(\theta)/d\theta)d\theta, v\right)_{V(\Omega)} + (u_0, v)_{V(\Omega)}$$

for each $n \in N$ and $v \in V(\Omega)$. The expression $\left(\int_0^t (dz(\theta)/d\theta)d\theta, v\right)_{V(\Omega)}$,

$z \in W_2^1(0,T,V(\Omega))$ represents (for each fixed $t \in [0,T]$, and $v \in V(\Omega)$) a linear continuous functional over $W_2^1(0,T,V(\Omega))$. This shows that the sequence $\{(u_{\epsilon_n}(t),v)_{V(\Omega)}\}_n$ is (due to (1.27)) convergent for every $t \in [0,T]$ and $v \in V(\Omega)$. Consequently, there exists a function $u : [0,T] \to V(\Omega)$ such that

$$u_{\epsilon_n}(t) \rightharpoonup u(t) \qquad \text{(weakly) in } V(\Omega) \text{ for each } t \in [0,T] \tag{1.28}$$

According to the Fatou lemma and the Lebesgue theorem (Brezis, 1973, App. 1) we see that

$$\begin{cases} u \in L_1(0,T,V(\Omega)) \\ u_{\epsilon_n} \rightharpoonup u \qquad \text{(weakly) in } L_1(0,T,V(\Omega)) \end{cases} \tag{1.29}$$

By comparison of (1.27) with (1.29) we conclude that $u(t) = u_*(t)$ for a.e. $t \in [0,T]$ and

$$u_{\epsilon_n} \rightharpoonup u \qquad \text{(weakly) in } W_2^1(0,T,V(\Omega)) \tag{1.30}$$

On the other hand, the a priori estimates $(1.18)_1$, (1.25) imply that the sequences $\{u_{\epsilon_n}\}_n$, $\{du_{\epsilon_n}/dt\}_n$ are bounded in the space $L_\infty(0,T,H(\Omega))$ which is the adjoint space to $L_1(0,T,H(\Omega))$. Hence, by virtue of (1.30) and due to the theorem of Banach-Alaoglu-Bourbaki (Brezis, 1982, Th.III 15) we have

$$\begin{cases} u_{\epsilon_n} \overset{*}{\rightharpoonup} u \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \\ du_{\epsilon_n}/dt \rightharpoonup du/dt \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \end{cases} \tag{1.31}$$

Then by virtue of Proposition III.12 from Brezis (1982) and using (1.31) we get the inequalities

$$\|u - u_0\|_{L_\infty(0,T,H(\Omega))} \le \liminf_{n\to\infty} \|u_{\epsilon_n} - u_0\|_{L_\infty(0,T,H(\Omega))}$$

and

$$\|du/dt\|_{L_\infty(0,T,H(\Omega))} \le \liminf_{n\to\infty} \|du_{\epsilon_n}/dt\|_{L_\infty(0,T,H(\Omega))}$$

which imply the estimates

$$\|u - u_0\|_{L_\infty(0,T,H(\Omega))} \le c, \qquad \|du/dt\|_{L_\infty(0,T,H(\Omega))} \le c \tag{1.32}$$

In virtue of the equality (1.4) we can write

$$\begin{aligned} \epsilon_n \big(\partial I_{\mathcal{K}(\Omega)}\big)_{\epsilon_n}(u_{\epsilon_n}(t)) = \epsilon_n [L(t) - du_{\epsilon_n}(t)/dt - A(t)u_{\epsilon_n}(t) \\ - \operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t))] \text{for every } t \in [0,T] \end{aligned}$$

Moreover, the sequences $\{u_{\epsilon_n}(t)\}_n$, $\{du_{\epsilon_n}(t)/dt\}_n$ and $\{\operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t))\}_n$ are (due to $(1.18)_3$, (1.25), (M3)) bounded for every $t \in [0,T]$. Then one has

$$\lim_{\epsilon_n \to 0} \epsilon_n \big(\partial I_{\mathcal{K}(\Omega)}\big)_{\epsilon_n}(u_{\epsilon_n}(t)) = 0 \quad \text{(strongly) in } V^*(\Omega) \text{ for every } t \in [0,T]$$

Using then the monotonocity of $\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon$ and the relation (1.28) we obtain

$$\left\langle \left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(v), u(t)-v\right\rangle_{V(\Omega)} \leq 0 \qquad \text{for every } t \in [0,T], v \in V(\Omega) \tag{1.33}$$

Then, inserting $v = u(t) + \theta z$, $\theta > 0$, $z \in V(\Omega)$, into (1.33) we obtain

$$\left\langle \left(\partial I_{\mathcal{K}(\Omega)}\right)_{\epsilon_n}(u(t)+\theta z), z\right\rangle_{V(\Omega)} \geq 0 \qquad \text{for all } z \in V(\Omega)$$

whence (due to the Lipschitz continuity of $\partial I_{\mathcal{K}(\Omega)})_\epsilon$) the limiting process $\theta \to 0$ yields

$$\left\langle \left(\partial I_{\mathcal{K}(\Omega)}\right)_{\epsilon_n}(u(t)), z\right\rangle_{V(\Omega)} \geq 0 \qquad \text{for all } z \in V(\Omega) \quad .$$

This means that

$$\left(\partial I_{\mathcal{K}(\Omega)}\right)_\epsilon(u(t)) = 0 \qquad \text{for all } t \in [0,T] \tag{1.34}$$

which due to ((1.7),1°) gives the relation: $u(t) \in \mathcal{K}(\Omega)$. We have (after changing u on the set of zero measure)

$$u \in W^1_\infty(0,T,H(\Omega)) \cap C([0,T],H(\Omega)) \tag{1.35}$$

and thus

$$u(t) = u(0) + \textstyle\int_0^t (du(\xi)/d\xi)d\xi \qquad \text{for every } t \in [0,T] \tag{1.36}$$

Simultaneously, we have the relation

$$u_{\epsilon_n}(t) = u_0 + \textstyle\int_0^t (du_{\epsilon_n}(\xi)/d\xi)d\xi \qquad \text{for every } t \in [0,T],\ n \in N \tag{1.37}$$

Using then the convergences (1.28) and (1.30) we obtain the initial condition: $u(0) = u_0$. Let us suppose again that z is given in $L_1(0,T,V(\Omega))$ (be an arbitrary function) where

$$z(t) \in \mathcal{K}(\Omega) \qquad \text{for a.e. } t \in [0,T] \tag{1.38}$$

We then have the inequalities

$$\begin{gathered}\left\langle \left(\partial I_{\mathcal{K}(\Omega)}\right)_{\epsilon_n}(u_{\epsilon_n}(t)), z(t)-u_{\epsilon_n}(t)\right\rangle_{V(\Omega)} \leq 0 \\ \text{for a.e. } t \in [0,T], \text{ and every } n \in N\end{gathered} \tag{1.39}$$

We then come back to the equalities

$$du_{\epsilon_n}(t)/dt + A(t)u_{\epsilon_n}(t) + \left(\partial I_{\mathcal{K}(\Omega)}\right)_{\epsilon_n}(u_{\epsilon_n}(t)) + \operatorname{grad}\Phi_{\epsilon_n}(u_{\epsilon_n}(t)) = L(t) \tag{1.40}$$

and forming the $V^*(\Omega) - V(\Omega)$ scalar product betwen (1.40) and $[z(t) - u_{\epsilon_n}(t)]$ and integrating from 0 to T we arrive at the inequalities

$$
\begin{aligned}
&\|u_{\epsilon_n}(T)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t)u_{\epsilon_n}(t), u_{\epsilon_n}(t)\rangle_{V(\Omega)} dt \\
&\leq \|u_{\epsilon_n}(0)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t)u_{\epsilon_n}(t), z(t)\rangle_{V(\Omega)} dt \\
&\quad + 2\int_0^T \langle du_{\epsilon_n}(t)/dt, z(t)\rangle_{V(\Omega)} dt \\
&\quad - 2\int_0^T \langle \operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t)), u_{\epsilon_n}(t) - z(t)\rangle_{V(\Omega)} dt \\
&\quad + 2\int_0^T \langle L(t), u_{\epsilon_n}(t) - z(t)\rangle_{V(\Omega)} dt \\
&\qquad \text{for all } n \in N
\end{aligned}
\tag{1.41}
$$

But, using then the assumptions (M2) we easily see that the functionals on the left-hand side of (1.41) are weakly semicontinuous on the spaces $V(\Omega)$ and $L_2(0,T,V(\Omega))$, respectively. The passage to the limit for $n \to \infty$ in the integrals of the inequalities (1.41) is easy, using the relations (1.28), (1.30), (M3) and the initial conditions in (1.3) and in (1.4). Hence we find in the limit

$$
\begin{aligned}
&\|u(T)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t)u(t), u(t)\rangle_{V(\Omega)} dt \\
&\leq \liminf_{n\to\infty} \left[\|u_{\epsilon_n}(T)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t)u_{\epsilon_n}(t), u_{\epsilon_n}(t)\rangle_{V(\Omega)} dt \right] \\
&\leq \|u(0)\|^2_{H(\Omega)} + 2\int_0^T \langle du(t)/dt, z(t)\rangle_{V(\Omega)} dt + 2\int_0^T \langle A(t)u(t), z(t)\rangle_{V(\Omega)} dt \\
&\quad + 2\int_0^T \Phi(z(t))dt - 2\int_0^T \Phi(u(t))dt + 2\int \langle L(t), u(t) - z(t)\rangle_{V(\Omega)} dt
\end{aligned}
$$

and this gives

$$
\begin{aligned}
&\int_0^T \langle du(t)/dt + A(t)u(t) - L(t), z(t) - u(t)\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \Phi(z(t))dt - \int_0^T \Phi(u(t))dt \geq 0 \\
&\text{for all } z \in L_1(0,T,V(\Omega)) \text{ such that } z(t) \in \mathcal{K}(\Omega) \text{ for a.e. } t \in [0,T]
\end{aligned}
\tag{1.42}
$$

Consequently, it follows from Proposition 3 (Brezis, 1972), that

$$
\begin{aligned}
&\langle du(t)/dt, v - u(t)\rangle_{V(\Omega)} + \langle A(t)u(t), v - u(t)\rangle_{V(\Omega)} \\
&\quad + \Phi(v) - \Phi(u(t)) \geq \langle L(t), v - u(t)\rangle_{V(\Omega)} \\
&\qquad \text{for a.e. } t \in [0,T] \text{ and for all } v \in \mathcal{K}(\Omega)
\end{aligned}
$$

This inequality implies that u is a solution of the problem (1.3). On the other hand, let u_* and u_{**} be two solutions of the problem (1.3). We take successively:

$$u(t) = u_*(\xi),\ v = u_{**}(\xi)$$
$$u(t) = u_{**}(\xi),\ v = u_*(\xi)$$

in (1.3). Then adding these inequalities, we get (integrating from 0 to t)

$$\begin{aligned}\int_0^t \langle (du_*(\xi)/d\xi - du_{**}(\xi)/d\xi) + A(\xi)(u_*(\xi) - u_{**}(\xi)),\\ u_*(\xi) - u_{**}(\xi)\rangle_{V(\Omega)} d\xi \le 0\\ \text{for every } t \in [0,T]\end{aligned} \tag{1.43}$$

Let us denote $z = u_* - u_{**}$. The function z fulfils the initial condition

$$z(0) = 0 \tag{1.44}$$

The inequality (1.43) then implies (by the relation (1.11))

$$\|z(t)\|^2_{H(\Omega)} + 2\int_0^t \langle A(\xi)z(\xi), z(\xi)\rangle_{V(\Omega)} d\xi \le 0 \qquad \text{for all } t \in [0,T] \tag{1.45}$$

This estimation, together with (M2) gives

$$z(t) = u_*(t) - u_{**}(t) = 0$$

and

$$u_*(t) = u_{**}(t) \quad \text{for all } t \in [0,T]$$

which proves uniqueness of the solution of the initial value problem (1.3)

REMARK 1 The a priori estimate $(1.18)_1$ shows the existence of an elemnt u in $L_\infty(0,T,H(\Omega))$ and a subsequence $\epsilon_n \to 0$ (or $n \to +\infty$) such that

$$u_{\epsilon_n} \text{ converges to } u, \text{ for the weak-star topology of } L_\infty(0,T,H(\Omega)) \tag{1.46}$$

Then (1.46) means that for each $v \in L_1(0,T,H(\Omega))$

$$\int_0^T (u_{\epsilon_n}(t) - u(t), v(t))_{H(\Omega)} dt \to 0 \qquad \epsilon_n \to 0 \tag{1.47}$$

By (1.18) the subsequence u_{ϵ_n} belongs to a bounded set of $L_2(0,T,V(\Omega))$, therefore another passage to a subsequence shows the existence of some $u_* \in L_2(0,T,V(\Omega))$ and some subsequence $\{u_{\epsilon_k}\}_k$ such that

$$u_{\epsilon_k} \text{ converges to } u_*, \text{ for the weak topology of } L_2(0,T,V(\Omega)) \tag{1.48}$$

The convergence (1.48) means

$$\int_0^T \langle u_{\epsilon_k}(t) - u_*(t), v(t)\rangle_{V(\Omega)} \to 0 \qquad \text{for any } v \in L_2(0,T,V^*(\Omega)),\ \epsilon_k \to 0$$

In particular by (1.2) one has

$$\int_0^T (u_{\epsilon_k}(t), v(t))_{H(\Omega)} dt \to \int_0^T (u_*(t), v(t))_{H(\Omega)} dt \tag{1.49}$$

for each v in $L_2(0,T,H(\Omega))$, $\epsilon_k \to 0$. Thus, comparing (1.49) with (1.47) we see that

$$\int_0^T (u(t) - u_*(t), v(t))_{H(\Omega)} dt = 0$$

for each v in $L_2(0,T,H(\Omega))$, hence

$$u = u_* \in L_2(0,T,V(\Omega)) \cap L_\infty(0,T,H(\Omega)).$$

2. Optimal control problem

2.1. Formulation of the problem

We assume that the data in the problem (1.3) depend on a control parameter e. Control problems for parabolic (pseudoparabolic) equations were studied in Barbu (1984), Lions (1973) and Bock, Lovišek (1986, 1992). We assume that the convex set of admissible states depends also on a control parameter e. Such problem in the pseudoparabolic case was investigated in Bock, Lovišek (1992).

We consider the following state problem:

$$\begin{cases} u(t,e) \in \mathcal{K}(e,\Omega) \qquad \text{for a.e. } t \in [0,T] \\ \text{and for a.e. } t \in [0,T] \\ \langle du(t,e)/dt, v - u(t,e)\rangle_{V(\Omega)} + \langle A(t,e)u(t,e), v - u(t,e)\rangle_{V(\Omega)} \\ +\Phi(e,v) - \Phi(e,u(t,e)) \geq \langle L(t,e), v - u(t,e)\rangle_{V(\Omega)} \ \text{ for all } v \in \mathcal{K}(e,\Omega) \\ u(0,e) = u_0(e) \in \mathcal{K}(e,\Omega) \end{cases} \tag{2.1}$$

where $\mathcal{K}(e,\Omega)$ is a closed convex subset of a Hilbert space $V(\Omega)$.

The optimal control problem we consider here is: (according to the state problem (2.1))

$$\begin{cases} \text{Minimize} \\ \mathcal{L}(e, u(e)) \qquad \text{in } u \in W_2^1(0,T,V(\Omega)) \text{ and } e \in U_{ad}(\Omega) \\ \text{subject to state inequality in (2.1)} \end{cases} \tag{$\mathcal{P}$}$$

where $U_{ad}(\Omega)$ is a compact subset of a Banach space $U(\Omega)$ and the cost functional $\mathcal{L} : U(\Omega) \times W_2^1(0,T,V(\Omega)) \to R$ is lower bounded and fulfils the assumption:

$$\begin{cases} \text{If } v_n \rightharpoonup v \text{ (weakly) in } W_2^1(0,T,V(\Omega)) \\ \text{and } e_n \to e \text{ (strongly) in } U(\Omega), \text{ then one has} \\ \mathcal{L}(e,v) \leqq \liminf_{n\to\infty} \mathcal{L}(e_n, v_n) \end{cases} \tag{EO}$$

In order to characterize the dependence $e \to \mathcal{K}(e,\Omega)$ we recall a special type of convergence of set sequences introduced in Mosco (1969)

DEFINITION 1. A sequence $\{K_n(\Omega)\}$ of subset of a normed space $V(\Omega)$ convergs to a set $K(\Omega) \subset V(\Omega)$ if

$$\begin{cases} 1^\circ\ K(\Omega) \text{ contains all weak limits of sequences } \{v_{n_k}\}_k,\ v_{n_k} \in K_{n_k}(\Omega), \\ \text{where } \{K_{n_k}(\Omega)\}_k \text{ is an arbitrary subsequence of } \{K_n(\Omega)\}_n \\ 2^\circ \text{ Every element } v \in K(\Omega) \text{ is the strong limit of a sequence} \\ \{v_n\},\ v_n \in K_n(\Omega),\ n \in N \end{cases}$$

NOTATION: $K(\Omega) = \mathrm{Lim}_{n\to\infty} K_n(\Omega)$

A sequence $\{\mathcal{W}_n\}$ of functionals from $V(\Omega)$ into $(-\infty, +\infty]$ converges to a functional $\mathcal{W}$ if

$$\begin{cases} 1^\circ \text{ For every } v \in V(\Omega) \text{ there exists a sequence } \{v_n\} \subset V(\Omega) \text{ such that} \\ \lim\limits_{n\to\infty} v_n = v \text{ (strongly) in } V(\Omega) \text{ and } \limsup\limits_{n\to\infty} \mathcal{W}_n(v_n) \leqq \mathcal{W}(v) \\ 2^\circ \text{ For every subsequences } \{\mathcal{W}_{n_k}\}_k \text{ of } \{\mathcal{W}_n\}_n \text{ and every sequence} \\ \{v_k\}_k \subset V(\Omega) \text{ weakly convergent to } v \in V(\Omega) \text{ holds} \\ \mathcal{W}(v) \leq \liminf\limits_{n\to\infty} \mathcal{W}_{n_k}(v_k) \\ \text{NOTATION: } \mathcal{W} = \mathop{\mathrm{Lim}}\limits_{n\to\infty} \mathcal{W}_n \end{cases}$$

We introduce the systems $\{\mathcal{K}(e,\Omega)\}$, $\{A(t,e)\}$ of convex closed subset $\mathcal{K}(e,\Omega) \subset V(\Omega)$ and linear bounded operators $A(.,e) \in C^1([0,T], L(V(\Omega), V^*(\Omega)))$, $e \in U_{ad}(\Omega)$, $t \in [0,T]$, satisfying the following assumptions:

$$\left\{\begin{array}{l}
1^\circ \displaystyle\bigcap_{e\in U_{ad}(\Omega)} \mathcal{K}(e,\Omega) \neq \emptyset \\
2^\circ\ e_n \to e \qquad \text{(strongly) in } U(\Omega) \Rightarrow \mathcal{K}(e,\Omega) = \displaystyle\operatorname*{Lim}_{n\to\infty} \mathcal{K}(e_n,\Omega) \\
3^\circ\ \langle A(t,e)v, z\rangle_{V(\Omega)} = \langle A(t,e)z, v\rangle_{V(\Omega)} \\
\quad \text{for all } v, z \in V(\Omega),\ t \in [0,T],\ e \in U_{ad}(\Omega) \\
4^\circ\ \|A(t,e)\|_{L(V(\Omega),V^*(\Omega))} \leq M \qquad \text{for all } e \in U_{ad}(\Omega) \text{ and } t \in [0,T] \\
5^\circ\ \|dA(t,e)/dt\|_{L(V(\Omega),V^*(\Omega))} \leq M_* \text{ for all } e \in U_{ad}(\Omega) \text{ and } t \in [0,T] \\
6^\circ\ \langle A(t,e)v, v\rangle_{V(\Omega)} \geq \alpha\|v\|^2_{V(\Omega)}, \qquad \alpha > 0 \\
\quad \text{for all } v \in V(\Omega),\ t \in [0,T],\ e \in U_{ad}(\Omega) \\
\quad \text{(a real number } \alpha \text{ not depending on } [e,t] \text{ and } v, \\
\quad A(t,.) \text{ is said to be uniformly coercive with respect to } U(\Omega)) \\
7^\circ\ e_n \to e \text{ (strongly) in } V(\Omega) \Rightarrow A(.,e_n) \to A(.,e) \\
\quad \text{in } C^1([0,T],\ L(V(\Omega), V^*(\Omega))) \\
8^\circ\ u_0(e_n) \to u_0(e) \text{ (strongly) in } V(\Omega) \text{ for } e_n \to e \text{ (strongly) in } U(\Omega)
\end{array}\right. \tag{H0}$$

Thus, by virtue of ((H0),3°,4°,6°), $A(.,e_n)$, $n = 1, 2, \ldots$ and $A(e)$ are elements of the class $E(\alpha, M)$ for each sequence $\{e_n\}_n$, where $e_n \to e$ (strongly) in $U(\Omega)$. Moreover we suppose

$$\left\{\begin{array}{l}
1^\circ \text{ There is a system of functionals } \{\Phi(e_n, .(t))\}_n \text{ on } V(\Omega) \\
\text{with values in } (-\infty, +\infty] \text{ (not identically equal to } +\infty) \\
\text{semicontinuous and convex on } V(\Omega),\ t \in [0,T] \\
\{v(t) \in V(\Omega) : \Phi(e_n, v(t)) < \infty\} \subset \mathcal{K}(e_n, \Omega), \qquad t \in [0,T] \\
\Phi(e, .(t)) = \displaystyle\operatorname*{Lim}_{n\to\infty} \Phi(e_n, .(t)) \text{ as } e_n \to e \text{ (strongly) in } U(\Omega), \\
e, e_n \in U_{ad}(\Omega),\ t \in [0,T] \\
2^\circ\ \|L(.,e)\|_{W_2^1(0,T,V(\Omega))} \leq M_L^* \\
3^\circ\ \{L(.,e_n)\}_n \text{ is a sequence in } V^*(\Omega) \text{ such that} \\
L(.,e_n) \to L(.,e) \text{ in } C^1([0,T], V^*(\Omega)) \text{ as } e_n \to e \text{ (strongly) in } U(\Omega).
\end{array}\right. \tag{H1}$$

Further we assume that for each sequence $\{e_n\}_n$, $e_n \to e$ (strongly) in $U(\Omega)$ there is a bounded sequence $\{a_n(t)\}_n$ with $a_n(t) \in \mathcal{K}(e_n, \Omega)$ and $\Phi(e_n, a_n(t)) < \infty$ for all $n, e, e_n \in U_{ad}(\Omega)$ such that

$$\limsup_{n\to\infty} \Phi(e_n, a_n(t)) < \infty \qquad \text{for all } t \in [0,T] \tag{2.2}$$

Moreover, there exist two constants c_1, c_2 such that for each sequence $\{e_n\}$, $e_n \to e$ (strongly) in $U(\Omega)$ one has:

$$\Phi(e_n, v_n(t)) \geq -c_1\|v_n(t)\|_{V(\Omega)} - c_2 \;\text{ for } n = 1, 2, \ldots,\ t \in [0,T] \tag{2.3}$$

(see Mosco, 1971)
Then, since $A(t, e_n) \in E(\alpha, M)$ (for $t \in [0,T]$) for any sequence of pairs $\{[e_n, v_n]\}_n$, $e, e_n \in U_{ad}(\Omega)$ $n = 1, 2, \ldots$ with $\|v_n\|_{V(\Omega)} \to \infty$ and $e_n \to e$ (strongly) in $U(\Omega)$ we have

$$\left[\langle A(t,e_n)v_n(t), v_n(t) - a_n(t)\rangle_{V(\Omega)} + \Phi(e_n, v_n(t))\right] / \|v_n(t)\|_{V(\Omega)} \to \infty \tag{2.4}$$

Moreover, for each $n \in N$ ($t \in [0,T]$)

$$\left[\langle A(t,e_0)v(t), v(t) - a_n(t)\rangle_{V(\Omega)} + \Phi(e_n, v_n(t))\right] / \|v(t)\|_{V(\Omega)} \to \infty \tag{2.5}$$

as $\|v(t)\|_{V(\Omega)} \to \infty$, $v(t) \in \mathcal{K}(e_n, \Omega)$ where $e_0 \in U_{ad}(\Omega)$ is arbitrary but fixed in $U_{ad}(\Omega)$, $e_n \in U_{ad}(\Omega)$, $n = 1, 2, \ldots$ $A(t, e_0) \in E(\alpha, M)$
REMARK 2 By virtue of ((H0),4°,6°) and (2.2) we can write

$$\begin{aligned}&\left[\langle A(t,e_n)v_n(t), v_n(t) - a_n(t)\rangle_{V(\Omega)} + \Phi(e_n, v_n(t))\right] \\ &\quad \geq \alpha\|v_n(t) - a_n(t)\|^2_{V(\Omega)} - c_3\|v_n(t) - a_n(t)\|_{V(\Omega)} - c_4\end{aligned}$$

where $a_n(t)$ is bounded in $\mathcal{K}(e_n, \Omega)$ $(n = 1, 2, \ldots)$ $t \in [0,T]$ and when $\|v_n(t)\|_{V(\Omega)} \to \infty$ then also $\|v_n(t) - a_n(t)\|_{V(\Omega)} \to \infty$. In a similar way (for each $n \in N$ and $t \in [0,T]$) we obtain relation (2.5).

Let $\{\Phi_\epsilon(e, .(t))\} : V(\Omega) \to R$, $t \in [0,T]$ be a family of Fréchet differentiable convex functionals on $V(\Omega)$ with the following properties:

$$\left\{\begin{array}{l} 1^\circ\ \Phi_\epsilon(e, v(t)) \geq -c(\|v(t)\|_{V(\Omega)} + 1) \\ \text{for all } \epsilon > 0,\ t \in [0,T],\ e \in U_{ad}(\Omega),\ v(t) \in V(\Omega) \\ 2^\circ\ \lim\limits_{\epsilon \to 0} \Phi_\epsilon(e, v(t)) = \Phi(e, v(t)) \\ \text{for all } e \in U_{ad}(\Omega),\ v(t) \in V(\Omega),\ t \in [0,T] \\ 3^\circ \text{ If } e_n \to e \text{ (strongly) in } U(\Omega), \\ \text{then } \Phi_\epsilon(e, .(t)) = \operatorname*{Lim}\limits_{n \to \infty} \Phi_\epsilon(e_n, .(t)),\ t \in [0,T] \\ 4^\circ \text{ If } e_n \to e \text{ (strongly) in } U(\Omega), \\ \epsilon_n \to 0 \Rightarrow \Phi(e, .(t)) = \operatorname*{Lim}\limits_{n \to \infty} \Phi_{\epsilon_n}(e_n, .(t)),\ t \in [0,T] \\ 5^\circ\ \|\operatorname{grad}\Phi_\epsilon(e, v(t)) - \operatorname{grad}\Phi_\epsilon(e, z(t))\|_{V^*(\Omega)} \\ \leq M(\epsilon)\|v(t) - z(t)\|_{V(\Omega)} \\ \text{for all } \epsilon > 0,\ e \in U_{ad}(\Omega),\ v(t), z(t) \in V(\Omega),\ t \in [0,T] \\ 6^\circ\ \|\operatorname{grad}\Phi_\epsilon(e, w(t))\|_{V^*(\Omega)} \leq M_{**} \\ \text{for any } w(t) \in V(\Omega) \text{ and all } e \in U_{ad}(\Omega),\ \epsilon > 0,\ t \in [0,T] \end{array}\right. \tag{H1$_\epsilon$}$$

We shall formulate the optimal control problem in the following way:

PROBLEM($\mathcal{P}_*$): *Find a control* $e_* \in U_{ad}(\Omega)$ *such that*

$$\begin{cases} u(t,e_*) \in \mathcal{K}(e_*\Omega) \qquad \text{for a.e. } t \in [0,T] \\ \langle du(t,e_*)/dt, v - u(t,e_*)\rangle_{V(\Omega)} + \langle A(t,e_*)u(t,e_*), v - u(t,e_*)\rangle_{V(\Omega)} \\ +\Phi(e_*,v) - \Phi(e_*,u(t,e_*)) \geq \langle L(t,e_*), v - u(t,e_*)\rangle_{V(\Omega)} \\ \text{for all } v \in \mathcal{K}(e_*,\Omega) \\ u(0,e_*) = u_0(e_*) \in \mathcal{K}(e_*,\Omega) \\ \mathcal{L}(e_*,u(e_*)) = \min\limits_{e\in U_{ad}(\Omega)} \mathcal{L}(e,u(e)) \end{cases} \tag{2.6}$$

Concerning the existence of solution of these problems, we will prove the following result.

THEOREM 3 *Let the assumptions (H0), (E0), (H1) and (2.2), (2.3) be satisfied. Then there exists at least one solution* e_* *of the optimal control problem* $(\mathcal{P}_*)$.

PROOF. As the solution (the state function) $u(e)$ of the variational inequality in (2.1) is uniquely determined for every $e \in U_{ad}(\Omega)$, we can introduce the functional $J(e)$ as

$$J(e) = \mathcal{L}(e,u(e)), \qquad e \in U_{ad}(\Omega) \tag{2.7}$$

(By virtue of Theorem 2 for any $e \in U_{ad}(\Omega)$ there exists a unique solution $u(e) \in W^1_\infty(0,T,H(\Omega)) \cap W^1_2(0,T,V(\Omega))$ of the state initial value problem (2.6))

Let $\{e_n\}_n \subset U_{ad}(\Omega)$ be a minimizing sequence for $J(e)$:

$$\lim_{n\to\infty} J(e_n) = \inf_{e\in U_{ad}(\Omega)} J(e) \tag{2.8}$$

On the other hand, the set $U_{ad}(\Omega)$ is compact in $U(\Omega)$, so that there exists an element $e_* \in U_{ad}(\Omega)$ and a subsequence of $\{e_n\}_n$, $\{e_{n_k}\}_k$ such that:

$$\lim_{k\to\infty} e_{n_k} = e_* \qquad \text{in } U(\Omega) \tag{2.9}$$

Denoting $u(t,e_n) := u_n(t) \in \mathcal{K}(e_n,\Omega)$ we rewrite the state problem (2.1) in the form

$$\begin{cases} u_n(t) \in \mathcal{K}(e_n,\Omega) \text{ for a.e. } t \in [0,T] \\ \text{and for a.e. } t \in [0,T],\ n = 1,2,3,\ldots \\ \langle du_n(t)/dt, v - u_n(t)\rangle_{V(\Omega)} + \langle A(t,e_n)u_n(t), v - u_n(t)\rangle_{V(\Omega)} \\ +\Phi(e_n,v) - \Phi(e_n,u_n(t)) \geq \langle L(t,e_n), v - u_n(t)\rangle_{V(\Omega)} \\ \text{for all } v \in \mathcal{K}(e_n,\Omega) \\ u_n(0) = u_0(e_n) \in \mathcal{K}(e_n,\Omega) \end{cases} \tag{2.10}$$

We apply Theorem 2 (to take account of (H0), $(H1)_\epsilon$ and (2.2), (2.4)), and we obtain the estimates:

$$\begin{cases} \|u_n\|_{W_2^1(0,T,V(\Omega))} \leq c_* \\ \|u_n\|_{W_\infty^1(0,T,H(\Omega))} \leq c_* \qquad \text{for all } n \in N \end{cases} \tag{2.11}$$

where the constant c_* involves only the constants $[M, M_*, \alpha, M_L^*]$ and the upper bound for the sequence $u_0(e_n)$. On the other hand if we compare the estimates $(1.18)_2$, (1.23) and (1.32) we can see that c_* does not depend on the sequence $\{\mathcal{K}(e_n, \Omega)\}_n$.

It follows by estimate (2.11) that there exist a function $u_* \in W_\infty^1(0,T,H(\Omega)) \cap W_2^1(0,T,V(\Omega))$ and a subsequence of $\{u_{n_k}\}_k$ such that:

$$u_{n_k} \rightharpoonup u_* \qquad \text{(weakly) in } W_2^1(0,T,V(\Omega)) \tag{2.12}$$

$$u_{n_k}(t) \rightharpoonup u_*(t) \qquad \text{(weakly) in } V(\Omega) \text{ for a.e. } t \in [0,T] \tag{2.13}$$

$$\begin{cases} u_{n_k} \overset{*}{\rightharpoonup} u_* \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \\ du_{n_k}/dt \overset{*}{\rightharpoonup} du_*/dt \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \end{cases} \tag{2.14}$$

Then we also have (by the relations (2.10), (2.13) and the assumption (H0,2°))

$$u_*(t) \in \mathcal{K}(e_*, \Omega) \qquad \text{for a.e. } t \in [0,T] \tag{2.15}$$

We infer from (2.1), (2.2), (H1,1°) and Definition 1, that

$$\begin{aligned} \Phi(e_*, u_*(t)) &\leq \liminf_{k\to\infty} \Phi(e_{n_k}, u_{n_k}(t)) \\ &\leq \limsup_{k\to\infty} \{\Phi(e_{n_k}, a_{n_k}(t)) \\ &\quad - \langle A(t, e_{n_k})u_{n_k}(t) - L(t, e_{n_k}), u_{n_k}(t) - a_{n_k}(t)\rangle_{V(\Omega)}\} < \infty \\ &\qquad \text{for a.e. } t \in [0,T] \end{aligned} \tag{2.16}$$

since by virtue of the monotonicity of $A(t, e_{n_k})$ one has

$$\begin{aligned} &\left|\langle A(t, e_{n_k})u_{n_k}(t), a_{n_k}(t) - u_{n_k}(t)\rangle_{V(\Omega)}\right| \\ &\leq \langle A(t, e_{n_k})a_{n_k}(t), a_{n_k}(t)\rangle_{V(\Omega)} + \left|\langle A(e_{n_k})a_{n_k}(t), u_{n_k}(t)\rangle_{V(\Omega)}\right| \leq 2Mc^2 \end{aligned}$$

where

$$(\|u_{n_k}(t)\|_{V(\Omega)}, \|a_{n_k}(t)\|_{V(\Omega)}) \leq c \quad \text{for } t \in [0,T]$$

Next according to the relations:

$$\begin{aligned} u_{n_k}(t) &= u_0(e_{n_k}) + \int_0^t (du_n(\xi)/d\xi)d\xi \\ u_*(t) &= u_*(0) + \int_0^t (du_*(\xi)/d\xi)d\xi \qquad t \in [0,T] \end{aligned} \tag{2.17}$$

we obtain, due to (2.12), (2.13) and (H0,7°), the initial condition

$$u_*(0) = u_0(e_*) \in \mathcal{K}(e_*, \Omega) \tag{2.18}$$

Let z (arbitrary function) be given in $L_1(0, T, V(\Omega))$ such that

$$z(t) \in \mathcal{K}(e_*, \Omega) \qquad \text{for a.e. } t \in [0, T] \tag{2.19}$$

Now, since the set $\mathcal{K}(e_*, \Omega)$ is closed in the space $V(\Omega)$, we can use (Brezis, 1973, Lemma A.0, App.) according to which for every $\epsilon > 0$ there exists a measurable function $z : [0, T] \to \mathcal{K}(e_*, \Omega)$ with only a finite number of values and such that:

$$\int_0^T \|z(t) - v_{n_k}(t)\|_{V(\Omega)} dt < \epsilon$$

On the other hand, by Definition 1 and assumption (H0,2°) there are a subsequence $\{e_{n_k}\}_k$ ($\subset \{e_n\}_n$) and a sequence $\{v_{n_k}\}_k \subset L_1(0, T, V(\Omega))$ such that

$$\begin{cases} v_{n_k} \in \mathcal{K}(e_{n_k}, \Omega) \qquad \text{for all } t \in [0, T],\ k \in N \\ and \\ \lim\limits_{k\to\infty} \|v_{n_k} - z\|_{L_1(0,T,V(\Omega))} = \lim\limits_{k\to\infty} \int_0^T \|v_{n_k}(t) - z(t)\|_{V(\Omega)} dt = 0 \\ \lim\limits_{k\to\infty} \Phi(e_{n_k}, v_{n_k}(t)) = \Phi(e_*, z(t)) \end{cases} \tag{2.20}$$

On the other hand, we can easily show that

$$A(t, e_{n_k}) v_{n_k}(t) \rightharpoonup A(t, e_*) v(t) \qquad \text{(weakly) in } V(\Omega) \tag{2.21}$$

$$\langle A(t, e_*) v(t), v(t) \rangle_{V(\Omega)} \le \liminf_{k\to\infty} \langle A(t, e_{n_k}) v_{n_k}(t), v_{n_k}(t) \rangle_{V(\Omega)}, \tag{2.22}$$

if $v_{n_k}(t) \rightharpoonup v(t)$ (weakly) in $V(\Omega)$ and $A(t, e_{n_k}) v_{n_k}(t) \rightharpoonup A(t, e_*) v(t)$ (weakly) in $V^*(\Omega)$ for $e_{n_k} \to e_*$ (strongly) in $U(\Omega)$, for $t \in [0, T]$.

Indeed, for any $w(t) \in V(\Omega)$ we can write

$$\begin{aligned} \lim_{k\to\infty} \langle A(t, e_{n_k}) v_{n_k}(t), w(t) \rangle_{V(\Omega)} &= \lim_{k\to\infty} \langle A(t, e_{n_k}) w(t), v_{n_k}(t) \rangle_{V(\Omega)} \\ &= \langle A(t, e_*) w(t), v(t) \rangle_{V(\Omega)} = \langle A(t, e_*) v(t), w(t) \rangle_{V(\Omega)} \end{aligned}$$

Moreover one has

$$\langle A(t, e_{n_k})(v_{n_k}(t) - v(t)), v_{n_k}(t) - v(t) \rangle_{V(\Omega)} \ge 0$$

Hence we may write

$$\begin{aligned} &\lim_{k\to\infty} 2\langle A(t, e_{n_k}) v(t), v_{n_k}(t) \rangle_{V(\Omega)} \\ &\quad \ge \liminf_{k\to\infty} \langle A(t, e_{n_k}) v_{n_k}(t), v_{n_k}(t) \rangle_{V(\Omega)} + \lim_{k\to\infty} \langle A(t, e_{n_k}) v(t), v(t) \rangle_{V(\Omega)} \end{aligned}$$

This yields (2.22).

According to inequality in (2.10) we have

$$\begin{aligned}\int_0^T &\langle du_{n_k}(t)/dt + A(t,e_{n_k})u_{n_k}(t)\\ &-L(t,e_{n_k}), v_{n_k}(t)-u_{n_k}(t)\rangle_{V(\Omega)}\\ &\geq \int_0^T \Phi(e_{n_k},u_{n_k}(t))dt - \int_0^T \Phi(e_{n_k},v_{n_k}(t))dt\end{aligned} \tag{2.23}$$

Then we write the last inequality in the form

$$\begin{aligned}&\|u_{n_k}(T)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t,e_{n_k})u_{n_k}(t), u_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+\int_0^T \Phi(e_{n_k},u_{n_k}(t))dt\\ &\leq \|u_{n_k}(0)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t,e_{n_k})u_{n_k}(t), v_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+\int_0^T \langle du_{n_k}(t)/dt, v_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+2\int_0^T \langle L(t,e_{n_k}), u_{n_k}(t)-v_{n_k}(t)\rangle_{V(\Omega)}dt + \int_0^T \Phi(e_{n_k},v_{n_k}(t))dt\end{aligned} \tag{2.24}$$

Thus, by passing to the limit in (2.24) we have

$$\begin{aligned}&\liminf_{k\to\infty}\|u_{n_k}(T)\|^2_{H(\Omega)} + 2\liminf_{k\to\infty}\int_0^T \langle A(t,e_{n_k})u_{n_k}(t), u_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+\liminf_{k\to\infty}\int_0^T \Phi(e_{n_k},u_{n_k}(t))dt \leq \lim_{k\to\infty}\|u_0(e_{n_k})\|^2_{H(\Omega)}\\ &\quad+2\lim_{k\to\infty}\int_0^T \langle A(t,e_{n_k})u_{n_k}(t), v_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+\lim_{k\to\infty}\int_0^T \langle du_{n_k}(t)/dt, v_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+2\lim_{k\to\infty}\int_0^T \langle L(t,e_{n_k}), u_{n_k}(t)-v_{n_k}(t)\rangle_{V(\Omega)}dt\\ &\quad+\lim_{k\to\infty}\int_0^T \Phi(e_{n_k},v_{n_k}(t))dt\end{aligned}$$

and hence (using Definition 1, (H0,8°) and the relations, (2.12), (2.13), (2.14), (2.15) and (2.16), (2.20), (2.21), (2.22))

$$
\begin{aligned}
&\|u_*(T)\|^2_{H(\Omega)} + 2\int_0^T \langle A(t,e_*)u_*(t), u_*(t)\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \Phi(e_*, u_*(t))dt \le \|u_0(e_*)\|^2_{H(\Omega)} \\
&\quad + \int_0^T \langle A(t,e_*)u_*(t), z(t)\rangle_{V(\Omega)} dt + \int_0^T \langle du_*(t)/dt, z(t)\rangle_{V(\Omega)} dt \\
&\quad + 2\int_0^T \langle L(t,e_*), u_*(t) - z(t)\rangle_{V(\Omega)} dt + \int_0^T \Phi(e_*, z(t))dt \\
&\qquad\qquad \text{for all } z \in L_1(0,T,V(\Omega))
\end{aligned}
\tag{2.25}
$$

such that $z(t) \in \mathcal{K}(e_*, \Omega)$ for a.e. $t \in [0,T]$. On the other hand, we infer from (2.25) that (using the initial condition in (2.6), the differentiability and symmetry of the operator function $A(.,e_*)$)

$$
\begin{aligned}
&\int_0^T \langle du_*(t)/dt + A(t,e_*)u_*(t) - L(t,e_*), z(t) - u_*(t)\rangle_{V(\Omega)} \\
&\quad \ge \int_0^T \Phi(e_*, u_*(t))dt - \int_0^T \Phi(e_*, z(t))dt
\end{aligned}
\tag{2.26}
$$

for all $z \in L_1(0,T,V(\Omega))$ such that $z(t) \in \mathcal{K}(e_*, \Omega)$ for a.e. $t \in [0,T]$.

Then by the Proposition 3 (Brezis, 1972) we deduce from (2.26) that

$$
\begin{aligned}
&\langle du_*(t)/dt, v - u_*(t)\rangle_{V(\Omega)} + \langle A(t,e_*)u_*(t), v - u_*(t)\rangle_{V(\Omega)} + \\
&\quad + \Phi(e_*, v) - \Phi(e_*, u_*(t)) \ge \langle L(t,e_*), v - u_*(t)\rangle_{V(\Omega)} \\
&\qquad\qquad \text{for a.e. } t \in [0,T], \text{ for all } v \in \mathcal{K}(e_*, \Omega)
\end{aligned}
\tag{2.27}
$$

This inequality (together with (2.12), (2.15) and (2.18)) implies that (by the uniqueness of a solution of (2.1))

$$
u_* = u(e_*) \tag{2.28}
$$

$$
u(e_{n_k}) \rightharpoonup u(e_*) \quad \text{(weakly) in } W_2^1(0,T,V(\Omega)) \tag{2.29}
$$

Finally we come back to (E0) and (2.8). We write the inequalities

$$
\begin{aligned}
&\mathcal{L}(e_*, u(e_*)) \le \liminf_{k\to\infty} \mathcal{L}(e_{n_k}, u(e_{n_k})) = \\
&\quad = \lim_{k\to\infty} J(e_{n_k}) = \inf_{e\in U_{ad}(\Omega)} J(e) = \inf_{e\in U_{ad}(\Omega)} \mathcal{L}(e, u(e))
\end{aligned}
$$

and hence $(\mathcal{P}_*)$ follows. This means that the proof of the theorem is finished.

3. Singular perturbations in optimal control problem with parabolic variational inequality.

Let $V(\Omega)$ and $W(\Omega)$ be two real Hilbert spaces. Assume that

$$\begin{aligned} &V(\Omega) \subset W(\Omega) \subset H(\Omega), \\ &V(\Omega) \text{ dense in } W(\Omega) \text{ and } W(\Omega) \text{ dense in } H(\Omega) \end{aligned} \tag{3.1}$$

where $H(\Omega)$ is a Hilbert space (in (3.1) each space is dense in the following one). Moreover, if we identify $H(\Omega)$ with its dual, we have:

$$V(\Omega) \subset W(\Omega) \subset H(\Omega) \subset W^*(\Omega) \subset V^*(\Omega)$$

Next we assume that $U_{ad}(\Omega) \subset U(\Omega)$ is compact in $U(\Omega)$.

We introduce the systems $\{\mathcal{K}(e,\Omega)\}_{e\in U_{ad}(\Omega)}$, $\{\hat{\mathcal{K}}(e,\Omega)\}_{e\in U_{ad}(\Omega)}$ of convex closed subset $\mathcal{K}(e,\Omega) \subset V(\Omega)$, $\hat{\mathcal{K}}(e,\Omega) \subset W(\Omega)$, $e \in U_{ad}(\Omega)$ and a system operators $\{\mathcal{A}(t,e)\}_{e\in U_{ad}(\Omega)}$, $\{\mathcal{B}(t,e)\}_{e\in U_{ad}(\Omega)}$ and

$$\mathcal{A}(t,e) : V(\Omega) \to V^*(\Omega), \qquad \mathcal{B}(t,e) : W(\Omega) \to W^*(\Omega)$$

for any $t \in [0,T]$, $e \in U_{ad}(\Omega)$ satisfying the following assumptions:

$$\left\{ \begin{array}{l} 1^\circ \bigcap\limits_{e\in U_{ad}(\Omega)} \mathcal{K}(e,\Omega) \neq \emptyset \qquad (0 \in \bigcap\limits_{e\in U_{ad}(\Omega)} \mathcal{K}(e,\Omega)) \\ 2^\circ\ e_n \to e_0 \qquad \text{(strongly) in } U(\Omega) \Rightarrow \mathcal{K}(e_0,\Omega) = \operatorname*{Lim}\limits_{n\to\infty} \mathcal{K}(e_n,\Omega) \\ 3^\circ\ \mathcal{A}(.,e) \in C^1([0,T], L(V(\Omega), V^*(\Omega))) \\ 4^\circ\ \{\mathcal{A}(t,e)v\}_{e\in U_{ad}(\Omega)} \subset E_{V(\Omega)}(0,c_{\mathcal{A}}) \\ 5^\circ\ \langle \mathcal{A}(t,e)v, z\rangle_{V(\Omega)} = \langle \mathcal{A}(t,e)z, v\rangle_{V(\Omega)} \\ \text{for any } v,z \in V(\Omega), \text{ for any } t \in [0,T] \\ 6^\circ\ e_n \to e_0 \qquad \text{(strongly) in } U(\Omega) \Rightarrow \mathcal{A}(.,e_n) \to \mathcal{A}(.,e_0) \\ \text{in } C^1([0,T], L(V(\Omega), V^*(\Omega))) \\ 7^\circ \text{ There exists } \alpha_{\mathcal{A}} > 0 \text{ such that for all } e \in U_{ad}(\Omega) \text{ and } v \in V(\Omega) \\ \langle \mathcal{A}(t,e)v, v\rangle_{V(\Omega)} + \|v\|^2_{W(\Omega)} \geq \alpha_{\mathcal{A}} \|v\|^2_{V(\Omega)} \qquad \text{for any } t \in [0,T] \end{array} \right. \tag{H0$_\mathcal{A}$}$$

and

$$\left\{\begin{array}{l} 1^\circ \displaystyle\bigcap_{e\in U_{ad}(\Omega)} \hat{\mathcal{K}}(e,\Omega) \neq \emptyset \\ 2^\circ\ e_n \to e_0 \qquad \text{(strongly) in } U(\Omega) \Rightarrow \hat{\mathcal{K}}(e_0,\Omega) = \displaystyle\operatorname*{Lim}_{n\to\infty} \hat{\mathcal{K}}(e_n,\Omega) \\ 3^\circ\ \mathcal{B}(.,e) \in C^1([0,T], L(W(\Omega), W^*(\Omega))) \\ 4^\circ\ \{\mathcal{B}(t,e)v\}_{e\in U_{ad}(\Omega)} \subset E_{W(\Omega)}(\alpha_{\mathcal{B}}, c_{\mathcal{B}}) \\ \qquad \text{with } \alpha_{\mathcal{B}} > 0, \text{ for any } t \in [0,T] \\ 5^\circ\ \langle \mathcal{B}(t,e)v, z\rangle_{W(\Omega)} = \langle \mathcal{B}(t,e)z, v\rangle_{W(\Omega)} \\ \qquad \text{for any } v,z \in W(\Omega), \text{ for any } t \in [0,T] \\ 6^\circ\ e_n \to e_0 \qquad \text{(strongly) in } U(\Omega) \Rightarrow \mathcal{B}(.,e_n) \to \mathcal{B}(.,e_0) \\ \text{in } C^1([0,T], L(W(\Omega), W^*(\Omega))) \end{array}\right. \tag{H0$_{\mathcal{B}}$}$$

$$\left\{\begin{array}{l} \text{Let us consider a continuous functional } v \to \Phi(v), \\ \text{on } W(\Omega) \text{ which is convex and nonnegative,} \\ \text{with } \Phi(0) = 0, \text{ and with the properties (M3).} \end{array}\right. \tag{M4}$$

Note that $W^*(\Omega) \circlearrowleft V^*(\Omega)$ continuously, and one has the transposition formula

$$\begin{aligned} &\langle F(t), v\rangle_{V(\Omega)} = \langle F(t), v\rangle_{W(\Omega)} \\ &\qquad\qquad \text{for any } v \in V(\Omega),\ t \in [0,T] \text{ and for any } F(t) \in W^*(\Omega). \end{aligned} \tag{M5}$$

REMARK 3 Instead of $((HO)_{\mathcal{A}}), 4^\circ, 7^\circ)$ and $((HO)_{\mathcal{B}}), 4^\circ)$ we may assume that $\mathcal{A}(t,e)$, $\mathcal{B}(t,e)$ be two operators (i.e. there are constants $M_{\mathcal{A}}$, $M_{\mathcal{B}} > 0$) such that

$$\begin{aligned} &||\mathcal{A}(t,e)v - \mathcal{A}(t,e)z||_{V^*(\Omega)} \leq M_{\mathcal{A}}||v - z||_{V(\Omega)} \\ &||\mathcal{B}(t,e)v - \mathcal{B}(t,e)z||_{W^*(\Omega)} \leq M_{\mathcal{B}}||v - z||_{W(\Omega)} \\ &\qquad\qquad \text{for any } v,z \in V(\Omega), \text{ for any } t \in [0,T] \end{aligned}$$

and assume $\mathcal{B}(t,e)$ is strongly coercive in the usual sense:
there is $(\alpha_{\mathcal{B}} > 0)$ such that

$$\begin{aligned} &\langle \mathcal{B}(t,e)v - \mathcal{B}(t,e)z, v - z\rangle_{W(\Omega)} \geq \alpha_{\mathcal{B}}||v - z||^2_{W(\Omega)} \\ &\qquad\qquad v,z \in W(\Omega), \text{ for each } t \in [0,T],\ e \in U_{ad}(\Omega) \end{aligned}$$

and $\mathcal{A}(t,e)$ is such that for some constants $\alpha^*_{\mathcal{A}} > 0$, $\beta_{\mathcal{B}} \geq 0$:

$$\begin{aligned} &\langle \mathcal{A}(t,e)v - \mathcal{A}(t,e)z, v - z\rangle_{V(\Omega)} \geq \alpha^*_{\mathcal{A}}||v - z||^2_{V(\Omega)} - \beta_{\mathcal{B}}||v - z||^2_{W(\Omega)} \\ &\qquad\qquad \text{for any } v,z \in V(\Omega),\ t \in [0,T],\ e \in U_{ad}(\Omega) \end{aligned}$$

We consider now inequalities of evolution. Let t denote the time variable $t \in [0,T]$, $T \leq \infty$. Let $f = f(t)$ be given such that

$$f + Be \in W_2^1(0,T,H(\Omega)),\ f(0) \in H(\Omega),\ B : U(\Omega) \to H(\Omega) \tag{M5}$$

be a linear continuous operator. Then for every $\epsilon > 0$ and for every $e \in U_{ad}(\Omega)$ there exists a unique function $u_\epsilon(e) = u_\epsilon(t,e)$ such that

$$\begin{cases} u_\epsilon(e) = W_\infty^1(0,T,H(\Omega)) \cap W_2^1(0,T,V(\Omega)) \\ u_\epsilon(t,e) \in \mathcal{K}(e,\Omega),\ u_\epsilon(0,e) = 0 \qquad (\in \mathcal{K}(e,\Omega)) \\ \langle du_\epsilon(t,e)/dt, v - u_\epsilon(t,e)\rangle_{V(\Omega)} \\ +\langle \epsilon\mathcal{A}(t,e)u_\epsilon(t,e) + \mathcal{B}(t,e)u_\epsilon(t,e), v - u_\epsilon(t,e)\rangle_{V(\Omega)} \\ +\Phi(v) - \Phi(u_\epsilon(t,e)) \geq \langle f(t) + Be, v - u_\epsilon(t,e)\rangle_{W(\Omega)} \\ \text{for any } v \in \mathcal{K}(e,\Omega) \text{ and } e \in U_{ad}(\Omega) \\ u_\epsilon(0,e) = 0 \in \mathcal{K}(e,\Omega) \qquad \text{for any } e \in U_{ad}(\Omega) \end{cases} \tag{3.3}$$

Indeed, thanks to Theorem 2 it is enough to prove that there is $c_\epsilon > 0$ such that

$$\langle \epsilon\mathcal{A}(t,e)v, v\rangle_{V(\Omega)} + \langle \mathcal{B}(t,e)v, v\rangle_{W(\Omega)} \geq c_\epsilon \|v\|^2_{V(\Omega)}$$
$$\text{for } v \in V(\Omega) \text{ and for each } t \in [0,T]$$

and this immediately follows from $((H0)_{\mathcal{A}}, 4°, 7°)$, $((H0)_{\mathcal{B}}, 4°)$.

Similarly there exists a unique function (thanks to $((H0)_{\mathcal{B}}, 4°)$ for any $e \in U_{ad}(\Omega)$) $u_0(e_0) = u_0(t,e_0)$ such that

$$\begin{cases} u_0(e_0) \in W_\infty^1(0,T,H(\Omega)) \cap W_2^1(0,T,W(\Omega)) \\ u_0(t,e_0) \in \hat{\mathcal{K}}(e_0,\Omega),\ u_0(0,e_0) = 0 \qquad (\in \hat{\mathcal{K}}(e_0,\Omega)) \\ \langle du_0(t,e_0)/dt, v - u_0(t,e_0)\rangle_{V(\Omega)} \\ +\langle \mathcal{B}(t,e_0)u_0(t,e_0), v - u_0(t,e_0)\rangle_{W(\Omega)} \\ +\Phi(v) - \Phi(u_0(t,e_0)) \geq \langle f(t) + Be_0, v - u_0(t,e_0)\rangle_{W(\Omega)} \\ \text{for any } v \in \hat{\mathcal{K}}(e_0,\Omega) \end{cases} \tag{3.4}$$

Let us consider a functional $\mathcal{L} : U(\Omega) \times W_2^1(0,T,W(\Omega)) \to R^+ \equiv \{a \in R, a \geq 0\}$ for which the following condition holds:

$$\left\{\begin{array}{l} 1^\circ\ \{v_n\}_n \subset W_2^1(0,T,V(\Omega)),\ v \in W_2^1(0,T,W(\Omega)),\ v_n \to v \\ \text{(strongly) in } W_2^1(0,T,W(\Omega)) \Rightarrow \mathcal{L}(e,v) = \lim_{n\to\infty} \mathcal{L}(e,v_n) \\ 2^\circ\ \{v_n\}_n \subset W_2^1(0,T,V(\Omega)),\ v \in W_2^1(0,T,W(\Omega)),\ \{e_n\}_n \subset U_{ad}(\Omega) \\ e \in U_{ad}(\Omega),\ e_n \to e \qquad \text{(strongly) in } U(\Omega),\ v_n \rightharpoonup v \text{ (weakly) in} \\ W_2^1(0,T,W(\Omega)) \Rightarrow \mathcal{L}(e,v) \le \liminf_{n\to\infty} \mathcal{L}(e_n,v_n) \end{array}\right. \tag{E1}$$

Let us set now $\overset{\wedge}{\mathcal{K}}(e,\Omega) = cl\mathcal{K}(e,\Omega)$ in $W(\Omega)$ for all $e \in U_{ad}(\Omega)$ where cl denotes closure.

We shall solve the following optimization problem $(\mathcal{P}_\epsilon)$:

$$\left\{\begin{array}{l} \text{Find a control } e_\epsilon \in U_{ad}(\Omega) \text{ such that} \\ J_\epsilon(e_\epsilon) = \inf_{e\in U_{ad}(\Omega)} J_\epsilon(e) \end{array}\right. \tag{$\mathcal{P}_\epsilon$}$$

where $J_\epsilon(e) = \mathcal{L}(e, u_\epsilon(e))$, $e \in U_{ad}(\Omega)$
$u_\epsilon(e)$ is the uniquely determined solution of (3.3), $e \in U_{ad}(\Omega)$.

We say that e_ϵ is an optimal control of the problem $(\mathcal{P}_\epsilon)$.

THEOREM 4 *There exists at least one solution to* $(\mathcal{P}_\epsilon)$.

PROOF. Due to the compactness of $U_{ad}(\Omega)$ in $U(\Omega)$, there exists a sequence $\{e_\epsilon^n\} \subset U_{ad}(\Omega)$ such that

$$\lim_{n\to\infty} e_\epsilon^n = e_\epsilon^0 \qquad \text{in } U(\Omega),\ e_\epsilon^0 \in U_{ad}(\Omega) \tag{3.5}$$

and

$$\lim_{n\to\infty} J_\epsilon(e_\epsilon^n) = \inf_{e\in U_{ad}(\Omega)} J_\epsilon(e) \tag{3.6}$$

Denoting $u_\epsilon(t, e_\epsilon^n) := u_\epsilon^n(t)$ we rewrite the state problem (3.3) in the form

$$\left\{\begin{array}{l} u_\epsilon^n(t) \in \mathcal{K}(e_\epsilon^n,\Omega) \qquad \text{for a.e. } t \in [0,T] \\ \langle du_\epsilon^n(t)/dt, v - u_\epsilon^n(t)\rangle_{V(\Omega)} + \\ +\langle \epsilon\mathcal{A}(t,e_\epsilon^n)u_\epsilon^n(t) + \mathcal{B}(t,e_\epsilon^n)u_\epsilon^n(t), v - u_\epsilon^n(t)\rangle_{V(\Omega)} + \Phi(v) - \Phi(u_\epsilon^n(t)) \\ \ge \langle f(t) + Be_\epsilon^n, v - u_\epsilon^n(t)\rangle_{W(\Omega)} \qquad \text{for all } v \in \mathcal{K}(e_\epsilon^n,\Omega) \\ u_\epsilon^n(0) = 0 \in \mathcal{K}(e_\epsilon^n,\Omega) \end{array}\right. \tag{3.7}$$

Applying Theorem 2 and the assumptions $((H0)_\mathcal{A}, (H0)_\mathcal{B})$ we obtain the estimates

$$\left\{\begin{array}{ll} \|u_\epsilon^n\|_{W_\infty^1(0,T,H(\Omega))} \le c(\epsilon), & n = 1,2,\ldots \text{ for fixed } \epsilon > 0 \\ \|u_\epsilon^n\|_{W_2^1(0,T,V(\Omega))} \le c(\epsilon) & \end{array}\right. \tag{3.8}$$

Comparing the estimates $(1.18)_2$, (1.23) and (1.32), we can see that $c(\epsilon)$ does not depend on the sequence $\{\mathcal{K}(e_\epsilon^n, \Omega)\}$. It results from (3.8) that there exists a function

$$u_\epsilon^0 \in W_\infty^1(0,T,H(\Omega)) \cap W_2^1(0,T,V(\Omega))$$

and a subsequence of $\{u_\epsilon^{n_k}\}$ such that

$$\begin{cases} u_\epsilon^{n_k} \rightharpoonup u_\epsilon^0 & \text{(weakly) in } W_2^1(0,T,V(\Omega)) \\ u_\epsilon^{n_k}(t) \rightharpoonup u_\epsilon^0(t) & \text{(weakly) in } V(\Omega) \text{ for a.e. } t \in [0,T] \end{cases} \tag{3.9}$$

$$\begin{cases} u_{\epsilon_{n_k}} \overset{*}{\rightharpoonup} u_\epsilon^0 & \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \\ du_{\epsilon_{n_k}}/dt \overset{*}{\rightharpoonup} du_\epsilon^0/dt & \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \end{cases} \tag{3.10}$$

Moreover, the relations (3.5), (3.7) and the assumption $((H0)_\mathcal{A}, 2°)$ imply

$$u_\epsilon^0(t) \in \mathcal{K}(e_\epsilon^0, \Omega) \qquad \text{for a.e. } t \in [0,T] \tag{3.11}$$

Next by virtue of the relations

$$u_\epsilon^n(t) = u_\epsilon^n(0) + \int_0^t (du_\epsilon^n(s)/ds)ds$$

$$u_\epsilon^0(t) = u_\epsilon^0(0) + \int_0^t (du_\epsilon^0(s)/ds)ds, \qquad t \in [0,T]$$

we obtain, due to (3.9), the initial condition

$$u_\epsilon^0(0) = 0 \in \mathcal{K}(e_\epsilon^0, \Omega) \tag{3.12}$$

We observe that $(H0)_\mathcal{A}$, $(H0)_\mathcal{B}$ and (3.9) imply

$$\begin{cases} \mathcal{A}(e_\epsilon^n)u_\epsilon^n \rightharpoonup \mathcal{A}(e_\epsilon^0)u_\epsilon^0 & \text{(weakly) in } L_2(0,T,V^*(\Omega)) \\ \text{and} & \\ \mathcal{B}(e_\epsilon^n)u_\epsilon^n \rightharpoonup \mathcal{B}(u_\epsilon^0)u_\epsilon^0 & \text{(weakly) in } L_2(0,T,W^*(\Omega)) \end{cases} \tag{3.13}$$

We can write (by the inequality in (3.3))

$$\int_0^T \langle du_\epsilon^n(t)/dt + \epsilon\mathcal{A}(t, e_\epsilon^n)u_\epsilon^n(t) + \mathcal{B}(t, e_\epsilon^n)u_\epsilon^n(t), v_n(t) - u_\epsilon^n(t)\rangle_{V(\Omega)}dt$$
$$+ \int_0^T \Phi(v_n(t))dt - \int_0^T \Phi(u_\epsilon^n(t))dt \geq \int_0^T \langle f(t) - Be_\epsilon^n, v_n(t) - u_\epsilon^n(t)\rangle_{W(\Omega)}dt$$

The last inequality can be rewritten in the form

$$
\begin{aligned}
&\|u_\epsilon^n(T)\|^2_{H(\Omega)} + 2\epsilon\int_0^T \langle \mathcal{A}(t,e_\epsilon^n)u_\epsilon^n(t), u_\epsilon^n(t)\rangle_{V(\Omega)}dt \\
&\quad + 2\int_0^T \langle \mathcal{B}(t,e_\epsilon^n)u_\epsilon^n(t), u_\epsilon^n(t)\rangle_{W(\Omega)}dt + \int_0^T \Phi(u_\epsilon^n(t))dt \\
&\quad \le 2\epsilon\int_0^T \langle \mathcal{A}(t,e_\epsilon^n)u_\epsilon^n(t), v_n(t)\rangle_{V(\Omega)}dt + 2\int_0^T \langle \mathcal{B}(t,e_\epsilon^n)u_\epsilon^n(t), v_n(t)\rangle_{W(\Omega)}dt \\
&\quad + \int_0^T \langle du_\epsilon^n(t)/dt, v_n(t)\rangle_{V(\Omega)}dt \\
&\quad + 2\int_0^T \langle f(t) + Be_\epsilon^n, u_\epsilon^n(t) - v_n(t)\rangle_{W(\Omega)}dt + \int_0^T \Phi(v_n(t))dt
\end{aligned}
$$

where

$$
\{v_n\} \subset L_1(0,T,V(\Omega)), \qquad v_n : [0,T] \to \mathcal{K}(e_\epsilon^n,\Omega)
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^T \|v_n(t) - z(t)\|_{V(\Omega)}dt = 0, \\
&z(t) \in \mathcal{K}(e_\epsilon^0,\Omega), \text{ if } e_\epsilon^n \to e_\epsilon^0, \text{ (strongly) in } U(\Omega)
\end{aligned} \tag{3.14}
$$

The functionals on the l.h.s. of this inequality are weakly lower semicontinuous on the spaces $V(\Omega)$ and $L_2(0,T,V(\Omega))$, respectively, which follows from the assumptions $(H0)_\mathcal{A}$, $(H0)_\mathcal{B}$ and (M3). Letting $n \to \infty$, we obtain (using $(H0)_\mathcal{A}$, (M5), $(H0)_\mathcal{B}$) and relations (3.5), (3.9) and (3.10), (3.14)) the inequality

$$
\begin{aligned}
&\left\|u_\epsilon^0(T)\right\|^2_{H(\Omega)} + 2\int_0^T \left\langle [\epsilon\mathcal{A}(t,e_\epsilon^0)u_\epsilon^0(t) + \mathcal{B}(t,e_\epsilon^0)u_\epsilon^0(t)], u_\epsilon^0(t)\right\rangle_{V(\Omega)}dt \\
&\quad + \int_0^T \Phi(u_\epsilon^0(t)) \le 2\int_0^T \left\langle [\epsilon\mathcal{A}(t,e_\epsilon^0)u_\epsilon^0(t) + \mathcal{B}(t,e_\epsilon^0)u_\epsilon^0(t)], z(t)\right\rangle_{V(\Omega)}dt \\
&\quad + \int_0^T \left\langle du_\epsilon^0(t)/dt, z(t)\right\rangle_{V(\Omega)}dt \\
&\quad + 2\int_0^T \left\langle f(t) + Be_\epsilon^0, u_\epsilon^0(t) - z(t)\right\rangle_{W(\Omega)}dt + \int_0^T \Phi(z(t))dt
\end{aligned} \tag{3.15}
$$

for all $z \in L_1(0,T,V(\Omega))$ such that $z(t) \in \mathcal{K}(e_\epsilon^0,\Omega)$ for a.e. $t \in [0,T]$.

Using the initial condition (3.12) we deduce from (3.15) that

$$\int_0^T \left\langle du_\epsilon^0(t)/dt + \epsilon\mathcal{A}(t,e_\epsilon^0)u_\epsilon^0(t) + \mathcal{B}(t,e_\epsilon^0)u_\epsilon^0(t), z(t) - u_\epsilon^0(t)\right\rangle_{V(\Omega)} dt$$
$$- \int_0^T \left\langle f(t) + Be_\epsilon^0, z(t) - u_\epsilon^0(t)\right\rangle_{W(\Omega)} dt \tag{3.16}$$
$$\geq \int_0^T \Phi(u_\epsilon^0(t))dt - \int_0^T \Phi(z(t))dt$$

for all $z \in L_1(0,T,V(\Omega))$ such that $z(t) \in \mathcal{K}(e_\epsilon^0,\Omega)$ for a.e. $t \in [0,T]$.
Again, using Proposition 3 from Brezis (1972) (App.I) we arrive at

$$\begin{aligned}&\left\langle du_\epsilon^0(t)/dt, z - u_\epsilon^0(t)\right\rangle_{V(\Omega)}\\ &\quad + \left\langle \epsilon\mathcal{A}(t,e_\epsilon^0)u_\epsilon^0(t) + \mathcal{B}(t,e_\epsilon^0)u_\epsilon^0(t), z - u_\epsilon^0(t)\right\rangle_{V(\Omega)}\\ &\quad + \Phi(z) - \Phi(u_\epsilon^0(t)) \geq \left\langle f(t) + Be_\epsilon^0, z - u_\epsilon^0(t)\right\rangle_{W(\Omega)}\end{aligned} \tag{3.17}$$

for a.e. $t \in [0,T]$, for all $z \in \mathcal{K}(e_\epsilon^0,\Omega)$.
The last inequality, together with (3.9), (3.11), (3.12) and the uniqueness of a solution of (3.3), imply the relations

$$\begin{cases} u_\epsilon^0 = u_\epsilon(e_\epsilon^0) \\ u_\epsilon(e_\epsilon^n) \rightharpoonup u_\epsilon(e_\epsilon^0) \qquad \text{(weakly) in } W_2^1(0,T,V(\Omega)) \end{cases} \tag{3.18}$$

Then (E1) and (3.6) yield

$$\begin{aligned}&\mathcal{L}(e_\epsilon^0, u_\epsilon(e_\epsilon^0)) \leq \liminf_{n\to\infty} \mathcal{L}(e_\epsilon^n, u_\epsilon(e_\epsilon^n)) =\\ &\quad = \lim_{n\to\infty} J_\epsilon(e_\epsilon^n) = \inf_{e\in U_{ad}(\Omega)} J_\epsilon(e) = \inf_{e\in U_{ad}(\Omega)} \mathcal{L}(e, u_\epsilon(e))\end{aligned}$$

Hence $\mathcal{L}(e_\epsilon^0, u_\epsilon(e_\epsilon^0)) = \inf\{\mathcal{L}(e,u_\epsilon(e)), e \in U_{ad}(\Omega)\}$ which completes the proof. ■

Limit state function and limit cost function

We define the limit state function for any $e \in U_{ad}(\Omega)$, by the following variational inequality

$$\begin{cases} \text{Find } u_0(e) = u_0(t,e) \in \hat{\mathcal{K}}(e,\Omega) \text{ such that} \\ \langle du_0(t,e)/dt, v - u_0(t,e)\rangle_{W(\Omega)} + \langle \mathcal{B}(t,e)u_0(t,e), v - u_0(t,e)\rangle_{W(\Omega)} \\ +\Phi(v) - \Phi(u_0(t,e)) \geq \langle f(t) + Be, v - u_0(t,e)\rangle_{W(\Omega)} \\ \text{for any } v \in \hat{\mathcal{K}}(e,\Omega) \end{cases} \tag{3.19}$$

and the limit cost function:

$$J_0(e) = \mathcal{L}(e_0, u_0(e))$$

In this case one has the limit control problem $(\mathcal{P}_0)$ defined as follows:

$$\text{Find: } e_0 \in \operatorname{Arginf}\{J_0(e), e \in U_{ad}(\Omega)\} \qquad (\mathcal{P}_0)$$

THEOREM 5 *There exists at least one solution to* $(\mathcal{P}_0)$.

PROOF. Analogous to that of Theorem 4.

There arises a natural question concerning the type of relation between solution to $(\mathcal{P}_0)$ and $(\mathcal{P}_\epsilon)$ if $\epsilon \to 0_+$. We prove the following theorem:

THEOREM 6 *Let the assumptions* $((H0)_{\mathcal{A}})$, $((H0)_{\mathcal{B}})$ *and (E1), (M5) be satisfied. Let* e_{ϵ_n}, e_0 *be the solutions of the problems* $(\mathcal{P}_{\epsilon_n})$, $(\mathcal{P}_0)$, *respectively. Then there exists a subsequence* $\{\epsilon_{n_k}\}$ *of* $\{\epsilon_n\}$ *such that*

$$\begin{cases} e_{\epsilon_{n_k}} \to e_0 \qquad \text{(strongly) in } U(\Omega) \\ du_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})/dt \rightharpoonup du_0(e_0)/dt \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \\ u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup u_0(e_0) \qquad \text{(weakly) in } W_2^1(0,T,W(\Omega)) \\ u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \to u_0(e_0) \qquad \text{(strongly) in } L_2(0,T,W(\Omega)) \\ J_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) = \inf\limits_{e\in U_{ad}(\Omega)} J_{\epsilon_{n_k}}(e) \to J_0(e_0) = \inf\limits_{e\in U_{ad}(\Omega)} J_0(e) \end{cases} \tag{3.20}$$

PROOF. Due to the compactness of $U_{ad}(\Omega)$ there exists a sequence $\{e_{\epsilon_n}\} \subset U_{ad}(\Omega)$ such that

$$e_n \to e_0^* \qquad \text{(strongly) in } U(\Omega) \tag{3.21}$$

Then the state function $u_\epsilon(t, e_{\epsilon_n}) \in \mathcal{K}(e_{\epsilon_n}, \Omega)$ for a.e. $t \in [0,T]$ of the state variational inequality

$$\begin{aligned} &\langle du_{\epsilon_n}(t,e_{\epsilon_n})/dt, v - u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} \\ &\quad + \langle \epsilon_n \mathcal{A}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}) + \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), v - u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} \\ &\quad + \Phi(v) - \Phi(u_{\epsilon_n}(t,e_{\epsilon_n})) \geq \langle f(t) + Be_{\epsilon_n}, v - u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} \\ &\qquad\qquad \text{for any } v \in \mathcal{K}(e_{\epsilon_n},\Omega) \end{aligned} \tag{3.22}$$

for a.e. $t \in [0,T]$ and for given $e_{\epsilon_n} \in U_{ad}(\Omega)$, $\epsilon_n > 0$, $n = 1, 2, \dots$

Taking $v = 0(\in \bigcap\limits_{e\in U_{ad}(\Omega)} \mathcal{K}(e,\Omega))$ we obtain

$$\begin{aligned} &\langle du_{\epsilon_n}(t,e_{\epsilon_n})/dt, u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} \\ &\langle \epsilon_n \mathcal{A}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}) + \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} \\ &\quad \leq \langle f(t) + Be_{\epsilon_n}, u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} \end{aligned}$$

It follows that

$$
\begin{aligned}
&d(\|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{H(\Omega)})/dt + \epsilon_n(\langle \mathcal{A}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} \\
&\qquad + \|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{W(\Omega)}) + (\alpha_{\mathcal{B}} - \epsilon_n)\|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{W(\Omega)} \\
&\quad \le \langle f(t) + Be_{\epsilon_n}, u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)}
\end{aligned} \tag{3.23}
$$

Then by setting $\epsilon_n \le \alpha_{\mathcal{B}}/2$ and applying $((H0)_{\mathcal{A}}, (H0)_{\mathcal{B}})$ (integrating (3.23) from 0 to s) we obtain

$$
\begin{aligned}
&\|u_{\epsilon_n}(s,e_{\epsilon_n})\|^2_{H(\Omega)} + 2\epsilon_n \int_0^s \|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{V(\Omega)}dt \\
&\qquad + 2\int_0^s \|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{W(\Omega)}dt \le \int_0^s \|f(t) + Be_{\epsilon_n}\|^2_{W^*(\Omega)}dt \\
&\qquad + \int_0^s \|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{W(\Omega)}dt
\end{aligned} \tag{3.24}
$$

Hence

$$
\sup_{s\in[0,T]} \|u_{\epsilon_n}(s,e_{\epsilon_n})\|^2_{H(\Omega)} \le \textstyle\int_0^T \|f(t) + Be_{\epsilon_n}\|^2_{W^*(\Omega)}dt \tag{3.25}
$$

which implies that:
the sequence $\{u_{\epsilon_n}(e_{\epsilon_n})\}$ remains in a bounded set of $L_\infty(0,T,H(\Omega))$. We then integrate (3.24) from 0 to T and get

$$
\begin{aligned}
&\|u_{\epsilon_n}(T,e_{\epsilon_n})\|^2_{H(\Omega)} + 2\epsilon_n \int_0^T \|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{V(\Omega)}dt \\
&\qquad + \int_0^T \|u_{\epsilon_n}(t,e_{\epsilon_n})\|^2_{W(\Omega)}dt \le \int_0^T \|f(t) + Be_{\epsilon_n}\|^2_{W^*(\Omega)}dt
\end{aligned}
$$

From this we conclude that

$$
\begin{cases}
\|u_{\epsilon_n}(e_{\epsilon_n})\|_{L_2(0,T,W(\Omega))} \le c \\
\sqrt{\epsilon_n}\|u_{\epsilon_n}(e_{\epsilon_n})\|_{L_2(0,T,V(\Omega))} \le c
\end{cases} \tag{3.26}
$$

(The sequence $\{u_{\epsilon_n}(e_{\epsilon_n})\}_n$ remains in a bounded set of $L_2(0,T,W(\Omega))\cap L_\infty(0,T,H(\Omega))$ and the sequence $\{\sqrt{\epsilon_n}u_{\epsilon_n}(e_{\epsilon_n})\}_n$ remains in a bounded set of $L_2(0,T,V(\Omega)))$

We can therefore extract a subsequence $\{u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})\}_k$ such that

$$
\begin{cases}
u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup w \\
\text{(weakly) in } L_2(0,T,W(\Omega)) \text{ for } \epsilon_{n_k} \to 0 \ (k \to +\infty) \\
\sqrt{\epsilon_{n_k}}u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup 0 \\
\text{(weakly) in } L_2(0,T,V(\Omega)) \text{ for } \epsilon_{n_k} \to 0 \ (k \to +\infty)
\end{cases} \tag{3.27}
$$

On the other hand, in order to obtain the estimate for sequence $\{du_{\epsilon_n}(e_{\epsilon_n})/dt\}_n$ we formally differentiate the equation:

$$\begin{aligned} &du_{\epsilon_n}(t, e_{\epsilon_n})/dt + \epsilon_n \mathcal{A}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}) + \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}) \\ &\quad + \left(\partial I_{\mathcal{K}(e_{\epsilon_n}, \Omega)}\right)_{\epsilon_n} (u_{\epsilon_n}(t, e_{\epsilon_n})) + \operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t, e_{\epsilon_n})) = f(t) + Be_{\epsilon_n} \end{aligned} \tag{3.28}$$

and arrive at

$$\begin{aligned} &d[du_{\epsilon_n}(t, e_{\epsilon_n})/dt]/dt + \epsilon_n d[\mathcal{A}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n})]/dt \\ &\quad + d[\mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n})]/dt + d\left[\left(\partial I_{\mathcal{K}(e_{\epsilon_n}, \Omega)}\right)_{\epsilon_n} (u_{\epsilon_n}(t, e_{\epsilon_n}))\right]/dt \\ &\quad + d[\operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t, e_{\epsilon_n}))]/dt = d[f(t) + Be_{\epsilon_n}]/dt \end{aligned}$$

We recall that $u_{\epsilon_n}(e_{\epsilon_n}) \in W_2^2(0, T, V(\Omega))$ (see the proof of Theorem1). This means that we can write

$$\begin{aligned} &\left\langle d^2 u_{\epsilon_n}(t, e_{\epsilon_n})/dt^2, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \right\rangle_{V(\Omega)} \\ &\quad + \epsilon_n \langle \mathcal{A}(t, e_{\epsilon_n}) du_{\epsilon_n}(t, e_{\epsilon_n})/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{V(\Omega)} \\ &\quad + \langle \mathcal{B}(t, e_{\epsilon_n}) du_{\epsilon_n}(t, e_{\epsilon_n})/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{W(\Omega)} \\ &\quad + \left\langle d\left[\left(\partial I_{\mathcal{K}(e_{\epsilon_n}, \Omega)}\right)_{\epsilon_n} (u_{\epsilon_n}(t, e_{\epsilon_n}))\right]/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \right\rangle_{V(\Omega)} \\ &\quad + \langle d[\operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t, e_{\epsilon_n}))]/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{V(\Omega)} \\ &= \langle d[f(t) + Be_{\epsilon_n}]/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{W(\Omega)} \\ &\quad - \epsilon_n \langle (d(\mathcal{A}(t, e_{\epsilon_n})/dt) u_{\epsilon_n}(t, e_{\epsilon_n}), du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{V(\Omega)} \\ &\quad - \langle (d(\mathcal{B}(t, e_{\epsilon_n}))/dt) u_{\epsilon_n}(t, e_{\epsilon_n}), du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{W(\Omega)} \\ &\qquad\qquad \text{for a.e. } t \in [0, T] \text{ and } \epsilon_n > 0,\ n = 1, 2, \ldots \end{aligned}$$

But, then we have

$$\begin{aligned} &d(\|du_{\epsilon_n}(t, e_{\epsilon_n})/dt\|^2_{H(\Omega)})/dt \\ &\quad + 2\epsilon_n \langle \mathcal{A}(t, e_{\epsilon_n}) du_{\epsilon_n}(t, e_{\epsilon_n})/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{V(\Omega)} \\ &\quad + 2\langle \mathcal{B}(t, e_{\epsilon_n}) du_{\epsilon_n}(t, e_{\epsilon_n})/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{W(\Omega)} \\ &\quad + 2\left\langle d\left[\left(\partial I_{\mathcal{K}(e_{\epsilon_n}, \Omega)}\right)_{\epsilon_n} (u_{\epsilon_n}(t, e_{\epsilon_n}))\right]/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \right\rangle_{V(\Omega)} \\ &\quad + 2\langle d[\operatorname{grad} \Phi_{\epsilon_n}(u_{\epsilon_n}(t, e_{\epsilon_n}))]/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{V(\Omega)} \\ &= 2\langle d[f(t) + Be_{\epsilon_n}]/dt, du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{W(\Omega)} \\ &\quad - 2\epsilon_n \langle (d\mathcal{A}(t, e_{\epsilon_n})/dt) u_{\epsilon_n}(t, e_{\epsilon_n}), du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{V(\Omega)} - \\ &\quad - \langle (d\mathcal{B}(t, e_{\epsilon_n})/dt) u_{\epsilon_n}(t, e_{\epsilon_n}), du_{\epsilon_n}(t, e_{\epsilon_n})/dt \rangle_{W(\Omega)} \\ &\qquad\qquad \text{for a.e. } t \in [0, T] \text{ and } \epsilon_n > 0,\ n = 1, 2, \ldots \end{aligned} \tag{3.29}$$

From the monotonicity of $(\partial I_{\mathcal{K}(e_{\epsilon_n}, \Omega)})_{\epsilon_n}$ (or grad Φ_ϵ) we obtain inequality

$$\left\{\begin{array}{l} \left\langle d\left[(\partial I_{K(e_{\epsilon_n},\Omega)})_{\epsilon_n}(u_{\epsilon_n}(t,e_{\epsilon_n}))\right]/dt, du_{\epsilon_n}(t,e_{\epsilon_n})/dt\right\rangle_{V(\Omega)} \geq 0 \\ \text{or} \\ \langle d[\operatorname{grad}\Phi_{\epsilon_n}(u_{\epsilon_n}(t,e_{\epsilon_n}))]/dt, du_{\epsilon_n}(t,e_{\epsilon_n})/dt\rangle_{V(\Omega)} \geq 0 \\ \text{for a.e. } t \in [0,T] \end{array}\right. \tag{3.30}$$

Then by virtue of $(H0)_{\mathcal{A}}$, $(H0)_{\mathcal{B}}$, together with (3.29), (3.30), we verify that

$$\begin{aligned} & d(\|du_{\epsilon_n}(t,e_{\epsilon_n})/dt\|^2_{H(\Omega)})/dt + c_{\mathcal{A}}\epsilon_n\|du_{\epsilon_n}(t,e_{\epsilon_n})/dt\|^2_{V(\Omega)} \\ & \qquad + c_{\mathcal{B}}\|du_{\epsilon_n}(t,e_{\epsilon_n})/dt\|^2_{W(\Omega)}/dt \leq c_{\mathcal{AB}} \end{aligned} \tag{3.31}$$

($c_{\mathcal{A}}$, $c_{\mathcal{B}}$, $c_{\mathcal{AB}}$ are positive constants, independent on ϵ_n)

Integrating (3.31) from 0 to s, $0 < s < T$, we obtain in particular

$$\|du_{\epsilon_n}(s,e_{\epsilon_n})/dt\|^2_{H(\Omega)} \leq c_{\mathcal{AB}}$$

Hence

$$\sup_{s\in[0,T]} \|du_{\epsilon_n}(s,e_{\epsilon_n})/dt\|^2_{H(\Omega)} \leq c_{\mathcal{AB}} \tag{3.32}$$

The constant $c_{\mathcal{AB}}$ in (3.32) is finite and independent of ϵ_n, therefore:
The sequence $\{du_{\epsilon_n}(e_{\epsilon_n})/dt\}_n$ remains in a bounded set of $L_\infty(0,T,H(\Omega))$.
We then integrate (3.31) from 0 to T and we get

$$\begin{aligned} & \|du_{\epsilon_n}(T,e_{\epsilon_n})/dt\|^2_{H(\Omega)} + c_{\mathcal{A}}\epsilon_n\int_0^T \|du_{\epsilon_n}(t,e_{\epsilon_n})/dt\|^2_{V(\Omega)}dt \\ & \qquad + c_{\mathcal{B}}\int_0^T \|du_{\epsilon_n}(t,e_{\epsilon_n})/dt\|^2_{W(\Omega)}dt \leq c_{\mathcal{AB}}T \end{aligned}$$

This shows that the sequence $\{du_{\epsilon_n}(e_{\epsilon_n})/dt\}_n$ remains in a bounded set of $L_2(0,T,W(\Omega)) \cap L_\infty(0,T,H(\Omega))$ and the sequence $\{\sqrt{\epsilon_n}du_{\epsilon_n}(e_{\epsilon_n})/dt\}_n$ remains in a bounded set of $L_2(0,T,V(\Omega))$.
Then, one has (for a subsequences $\{du_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})/dt\}_k$, $\{\sqrt{\epsilon_{n_k}}du_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})/dt\}_k$)

$$\left\{\begin{array}{l} du_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})/dt \rightharpoonup dw/dt \\ \text{(weakly) in } L_2(0,T,W(\Omega)) \text{ for } \epsilon_{n_k} \to 0\ (k \to +\infty) \\ \sqrt{\epsilon_{n_k}}(du_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})/dt) \rightharpoonup 0 \\ \text{(weakly) in } L_2(0,T,V(\Omega)) \text{ for } \epsilon_{n_k} \to 0\ (k \to +\infty) \end{array}\right. \tag{3.33}$$

Moreover, by virtue of (3.27) and (3.33) we conclude that

$$u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup w \qquad \text{(weakly) in } W_2^1(0,T,W(\Omega)) \tag{3.34}$$

$$u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}) \rightharpoonup w(t) \qquad \text{(weakly) in } W(\Omega) \text{ for a.e. } t \in [0,T] \tag{3.35}$$

$$u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup w \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \tag{3.36}$$

$$du_{\epsilon_{n_k}}(e_{\epsilon_{n_k}})/dt \rightharpoonup dw/dt \qquad \text{(weakly star) in } L_\infty(0,T,H(\Omega)) \tag{3.37}$$

$$\sqrt{\epsilon_{n_k}} u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup 0 \qquad \text{(weakly) in } W_2^1(0,T,V(\Omega)) \tag{3.38}$$

$$\sqrt{\epsilon_{n_k}} u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}) \rightharpoonup 0 \qquad \text{(weakly) in } V(\Omega) \text{ for a.e. } t \in [0,T] \tag{3.39}$$

Since $u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}) \in \mathcal{K}(e_{\epsilon_{n_k}}, \Omega)$ by assumption $((H0)_{\mathcal{A}}, 2°)$, (3.35) we have $w(t) \in \mathcal{K}(e_0^*, \Omega)$ as well. From this one has

$$w(t) \in \overset{\wedge}{\mathcal{K}}(e_0^*, \Omega) \qquad \text{for a.e. } t \in [0,T] \tag{a1}$$

From the relations

$$u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}) = u_{\epsilon_{n_k}}(0, e_{\epsilon_{n_k}}) + \int_0^t (du_{\epsilon_{n_k}}(s, e_{\epsilon_{n_k}})/ds)ds$$

$$w(t) = w(0) + \int_0^t (dw(s)/ds)ds \quad t \in [0,T],$$

we obtain due to (3.34), (3.35) the initial condition

$$w(0) = 0 \in \mathcal{K}(e_0^*, \Omega) \tag{a2}$$

For any $z \in L_2(0,T,V(\Omega))$ we have by assumption $((H0)_{\mathcal{A}}, 5°, 6°)$ (and by virtue of (3.39))

$$\begin{aligned}
&\lim_{\substack{k \to +\infty \\ (\epsilon_{n_k} \to 0)}} \int_0^T \left\langle \mathcal{A}(t, e_{\epsilon_{n_k}}) \sqrt{\epsilon_{n_k}} u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}), z(t) \right\rangle_{V(\Omega)} dt \\
&= \lim_{\substack{k \to +\infty \\ (\epsilon_{n_k} \to 0}} \int_0^T \left\langle \mathcal{A}(t, e_{\epsilon_{n_k}}) z(t), \sqrt{\epsilon_{n_k}} u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}) \right\rangle_{V(\Omega)} dt \\
&= \int_0^T \langle \mathcal{A}(t, e_0^*) z(t), 0 \rangle_{V(\Omega)} dt = \int_0^T \langle \mathcal{A}(t, e_0^*) 0, z(t) \rangle_{V(\Omega)} dt
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathcal{A}(e_{\epsilon_{n_k}}) \sqrt{\epsilon_{n_k}} u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup \mathcal{A}(e_0^*)0 = 0 \\
&\text{(weakly) in } L_2(0,T,V^*(\Omega)),
\end{aligned} \tag{3.40}$$

note that

$$\left| \langle \epsilon \mathcal{A}(e) v, u_\epsilon(e) \rangle_{L_2(0,T,V(\Omega))} \right| = 0(\sqrt{\epsilon})$$

On the other hand by the analogy with (3.40) we obtain

$$\begin{aligned}
&\lim_{\substack{k\to+\infty \\ (\epsilon_{n_k}\to 0)}} \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}), z(t)\right\rangle_{W(\Omega)} dt \\
&\quad = \lim_{\substack{k\to+\infty \\ (\epsilon_{n_k}\to 0)}} \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})z(t), u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}})\right\rangle_{W(\Omega)} dt \\
&\quad = \int_0^T \langle \mathcal{B}(t,e_0^*)z(t), w(t)\rangle_{W(\Omega)} dt
\end{aligned}$$

This means that

$$\begin{aligned}
&\mathcal{B}(e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(e_{\epsilon_{n_k}}) \rightharpoonup \mathcal{B}(e_0^*)w \\
&\text{(weakly) in } L_2(0,T,W^*(\Omega)) \text{ for } k\to+\infty \text{ and } \epsilon_{n_k}\to 0
\end{aligned} \tag{3.41}$$

Furthermore, in virtue of the monotonicity of $\mathcal{B}(t,e_{\epsilon_{n_k}})$ (due to the assumption $((H0)_{\mathcal{B}}, 4^\circ)$) we know that

$$\begin{aligned}
&\int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}), u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}) - w(t)\right\rangle_{W(\Omega)} \\
&\quad \geq \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})w(t), u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}) - w(t)\right\rangle_{W(\Omega)} dt, \qquad k=1,2,\ldots
\end{aligned}$$

Passing to the limit we have

$$\begin{aligned}
&2\lim_{\substack{k\to+\infty \\ (\epsilon_{n_k}\to 0)}} \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})w(t), u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}})\right\rangle_{W(\Omega)} dt \\
&\quad \leq \lim_{\substack{k\to+\infty \\ (\epsilon_{n_k}\to 0)}} \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}), u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}})\right\rangle_{W(\Omega)} dt \\
&\qquad + \lim_{\substack{k\to+\infty \\ (\epsilon_{n_k}\to 0)}} \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})w(t), w(t)\right\rangle_{W(\Omega)} dt
\end{aligned}$$

This yields (together with (3.35), $((H0)_{\mathcal{B}}, 6^\circ)$ and (3.21))

$$\begin{aligned}
&\liminf_{\substack{k\to+\infty \\ (\epsilon_{n_k}\to 0)}} \int_0^T \left\langle \mathcal{B}(t,e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}}), u_{\epsilon_{n_k}}(t,e_{\epsilon_{n_k}})\right\rangle_{W(\Omega)} \\
&\quad \geq \int_0^T \langle \mathcal{B}(t,e_0^*)w(t), w(t)\rangle_{W(\Omega)} dt
\end{aligned} \tag{3.42}$$

Let $a(t) \in \mathcal{K}(e_0^*,\Omega)$ (for any $t\in[0,T]$) be an arbitrary element and $\{a_{\epsilon_{n_k}}(t)\}_k$ such a sequence ($t\in[0,T]$) that

$$a_{\epsilon_{n_k}}(t) \to a(t) \qquad \text{(strongly) in } V(\Omega) \text{ for any } t \in [0,T], \text{ where} \tag{3.43}$$
$$a_{\epsilon_{n_k}}(t) \in \mathcal{K}(e_{\epsilon_{n_k}}, \Omega), \qquad k = 1,2,\dots$$

(The existence of $\{a_{\epsilon_{n_k}}(t)\}_k$ is ensured by $((H0)_{\mathcal{A}}, 2°)$). Then we have (by (3.22))

$$\begin{aligned}
&\int_0^T \left\langle du_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}})/dt, a_{\epsilon_{n_k}}(t)\right\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \Big\langle \epsilon_{n_k}\mathcal{A}(t, e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}) \\
&\quad + \mathcal{B}(t, e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}), a_{\epsilon_{n_k}}(t)\Big\rangle_{V(\Omega)} dt + \int_0^T \Phi(a_{\epsilon_{n_k}}(t))dt \\
&\quad - \int_0^T \left\langle f(t) + Be_{\epsilon_{n_k}}, a_{\epsilon_{n_k}}(t) - u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}})\right\rangle_{W(\Omega)} dt \qquad (3.44)\\
&\geq \int_0^T \left\langle du_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}})/dt, u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}})\right\rangle_{V(\Omega)} \\
&\quad + \int_0^T \left\langle \mathcal{B}(t, e_{\epsilon_{n_k}})u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}), u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}})\right\rangle_{W(\Omega)} dt \\
&\quad + \int_0^T \Phi(u_{\epsilon_{n_k}}(t, e_{\epsilon_{n_k}}))dt
\end{aligned}$$

From this inequality (using (3.35), (3.42) and (3.43) too) we get (the functionals on the r.h.s. of (3.44) are lower semicontinuous on the spaces $V(\Omega)$ and $L_2(0,T,V(\Omega))$, respectively, which follows from $(H0)_{\mathcal{B}}$, (M4), (1.11))

$$\begin{aligned}
&\int_0^T \langle dw(t)/dt, a(t) - w(t)\rangle_{V(\Omega)} dt + \int_0^T \langle \mathcal{B}(t, e_0^*)w(t), a(t) - w(t)\rangle_{W(\Omega)} dt \\
&\quad + \int_0^T \Phi(a(t))dt - \int_0^T \Phi(w(t))dt \geq \int_0^T \langle f(t) + Be_0^*, a(t) - w(t)\rangle_{W(\Omega)} dt
\end{aligned}$$

for a.e. $t \in [0,T]$, for all $a \in L_1(0,T,V(\Omega))$ such that $a(t) \in \mathcal{K}(e_0^*, \Omega)$.
But from Proposition 3 (Brezis, 1972, App.I) we arrive at

$$\begin{aligned}
&\langle dw(t)/dt, a - w(t)\rangle_{V(\Omega)} + \langle \mathcal{B}(t, e_0^*)w(t), a - w(t)\rangle_{W(\Omega)} \\
&\quad + \Phi(a) - \Phi(w(t)) \geq \langle f(t) + Be_0^*, a - w(t)\rangle_{W(\Omega)} \qquad (3.45)\\
&\qquad \text{for a.e. } t \in [0,T], \text{ for all } a(t) \in \mathcal{K}(e_0^*, \Omega)
\end{aligned}$$

and therefore we have also (3.45) for all $a(t) \in \hat{\mathcal{K}}(e_0^*, \Omega)$ (by density). The last inequality together with (3.34), (a1), (a2) and the uniqueness of a solution of (3.4) imply the relations:

$$w = u_0(e_0^*) \tag{3.46}$$

$$u_{\epsilon_n}(e_{\epsilon_n}) \rightharpoonup u_0(e_0^*)$$
$$\text{(weakly) in } W_2^1(0,T,W(\Omega)) \text{ for } n \to +\infty \ (\epsilon_n \to 0) \tag{3.47}$$

Next, in order to prove strong convergence $u_{\epsilon_n}(e_{\epsilon_n}) \to u_0(e_0^*)$ in $L_2(0,T,W(\Omega))$ and in $C([0,T],H(\Omega))$ we consider (regarding to (3.22) and (M5))

$$\begin{aligned}
&\int_0^T \langle du_{\epsilon_n}(t,e_{\epsilon_n})/dt, u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \langle \epsilon_n \mathcal{A}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}) + \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \Phi(u_{\epsilon_n}(t,e_{\epsilon_n}))dt \le \int_0^T \langle du_{\epsilon_n}(t,e_{\epsilon_n})/dt, v_{\epsilon_n}(t)\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \langle \epsilon_n \mathcal{A}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}) + \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), v_{\epsilon_n}(t)\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \Phi(v_{\epsilon_n}(t))dt - \int_0^T \langle f(t) + Be_{\epsilon_n}, v_{\epsilon_n}(t) - u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} dt
\end{aligned} \tag{3.48}$$

where $u_{\epsilon_n}(t,e_{\epsilon_n})$, $v_{\epsilon_n}(t) \in \mathcal{K}(e_{\epsilon_n},\Omega)$ for a.e. $t \in [0,T]$, $e_{\epsilon_n} \in U_{ad}(\Omega)$.
We deduce from (3.48) that

$$\begin{aligned}
&\liminf_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} (1/2)\|u_{\epsilon_n}(T,e_{\epsilon_n})\|^2_{H(\Omega)} \\
&\qquad + \limsup_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \int_0^T \langle \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} dt \\
&\quad \le \lim_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \int_0^T \langle du_{\epsilon_n}(t,e_{\epsilon_n})/dt, v_{\epsilon_n}(t)\rangle_{V(\Omega)} dt \\
&\qquad + \lim_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \int_0^T \langle \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), v_{\epsilon_n}(t)\rangle_{W(\Omega)} dt \\
&\qquad + \lim_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \int_0^T \Phi(v_{\epsilon_n}(t))dt - \liminf_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \int_0^T \Phi(u_{\epsilon_n}(t,e_{\epsilon_n}))dt \\
&\qquad - \lim_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \int_0^T \langle f(t) + Be_{\epsilon_n}, v_{\epsilon_n}(t) - u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} dt
\end{aligned} \tag{3.49}$$

Hence by (3.41), (3.43), (M4) and (3.47) one has (we set $a_{\epsilon_n}(t) = v_{\epsilon_n}(t)$ and $v(t) = a(t)$)

$$\begin{aligned}
&\limsup_{\substack{n\to+\infty\\(\epsilon_n\to 0)}} \int_0^T \langle \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} dt \\
&\quad \le \int_0^T \langle du_0(t,e_0^*)/dt, v(t)-u_0(t,e_0^*)\rangle_{V(\Omega)} dt \\
&\quad + \int_0^T \langle \mathcal{B}(t,e_0^*)u_0(t,e_0^*), v(t)\rangle_{W(\Omega)} dt + \int_0^T \Phi(v(t))dt \\
&\quad - \int_0^T \Phi(u_0(t,e_0^*))dt - \int_0^T \langle f(t)+Be_0^*, v(t)-u_0(t,e_0^*)\rangle_{W(\Omega)} dt \\
&\quad \text{for a.e. } t\in[0,T] \text{ and for all } v(t)\in\mathcal{K}(e_0^*,\Omega)
\end{aligned} \tag{3.50}$$

(by density one concludes (3.50) also all $v \in \hat{\mathcal{K}}(e_0^*,\Omega)$ and therefore (by taking $v(t) = u_0(t,e_0^*) \in \hat{\mathcal{K}}(e_0^*,\Omega)$ in (3.50)) the inequality

$$\begin{aligned}
&\limsup_{\substack{n\to+\infty\\(\epsilon_n\to 0)}} \int_0^T \langle \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} dt \\
&\quad \le \int_0^T \langle \mathcal{B}(t,e_0^*)u_0(t,e_0^*), u_0(t,e_0^*)\rangle_{W(\Omega)} dt
\end{aligned} \tag{3.51}$$

is verified. Using it we get via (3.42), (3.47) and (3.51)

$$\begin{aligned}
&\lim_{\substack{n\to+\infty\\(\epsilon_n\to 0)}} \int_0^T \langle \mathcal{B}(t,e_{\epsilon_n})u_{\epsilon_n}(t,e_{\epsilon_n}), u_{\epsilon_n}(t,e_{\epsilon_n})\rangle_{W(\Omega)} dt \\
&\quad = \int_0^T \langle \mathcal{B}(t,e_0^*)u_0(t,e_0^*), u_0(t,e_0^*)\rangle_{W(\Omega)} dt \qquad \text{for a.e. } t\in[0,T]
\end{aligned} \tag{3.52}$$

Next we set

$$\begin{aligned}
\mathcal{V}_{\epsilon_n}^s = &\int_0^s \langle \mathcal{B}(t,e_{\epsilon_n})[u_{\epsilon_n}(t,e_{\epsilon_n}) - u_0(t,e_0^*)], \\
&u_{\epsilon_n}(t,e_{\epsilon_n}) - u_0(t,e_0^*)\rangle_{W(\Omega)}\, dt \\
&+ \int_0^s \langle du_{\epsilon_n}(t,e_{\epsilon_n})/dt - du_0(t,e_0^*)/dt, u_{\epsilon_n}(t,e_{\epsilon_n}) - u_0(t,e_0^*)\rangle_{V(\Omega)} dt
\end{aligned} \tag{3.53}$$

One has

$$\begin{aligned}
\mathcal{V}_{\epsilon_n}^s \ge &(1/2)\|u_{\epsilon_n}(s,e_{\epsilon_n}) - u_0(s,e_0^*)\|_{H(\Omega)}^2 \\
&+ \alpha_{\mathcal{B}} \int_0^s \|u_{\epsilon_n}(t,e_{\epsilon_n}) - u_0(t,e_0^*)\|_{W(\Omega)}^2 dt
\end{aligned} \tag{3.54}$$

so that it is enough to prove that $\mathcal{V}_{\epsilon_n}^s \to 0$ (for $n\to+\infty$) uniformly in s. But thanks to (3.19) where $v = u_{\epsilon_n}(t, e_{\epsilon_n})$, one has

$$\begin{aligned}
\mathcal{V}^s_{\epsilon_n} \leq & \int_0^s \langle du_{\epsilon_n}(t, e_{\epsilon_n})/dt, u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{V(\Omega)} dt \\
& + \int_0^s \langle \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{W(\Omega)} dt \\
& + \int_0^s \Phi(u_{\epsilon_n}(t, e_{\epsilon_n}))dt - \int_0^s \langle du_{\epsilon_n}(t, e_{\epsilon_n})/dt, u_0(t, e_0^*)\rangle_{V(\Omega)} dt \\
& - \int_0^s \langle \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), u_0(t, e_0^*)\rangle_{W(\Omega)} dt \\
& - \int_0^s \Phi(u_0(t, e_0^*))dt - \int_0^s \langle f(t) + Be_0^*, u_{\epsilon_n}(t, e_{\epsilon_n}) - u_0(t, e_0^*)\rangle_{W(\Omega)} dt
\end{aligned} \tag{3.55}$$

Let us consider a function $v \in L_2(0, T, V(\Omega))$, $v(t) \in \mathcal{K}(e_{\epsilon_n}, \Omega)$ (for a.e. $t \in [0, T]$). Taking $v = v(t)$ in (3.3) we obtain

$$\begin{aligned}
& \int_0^s \langle du_{\epsilon_n}(t, e_{\epsilon_n})/dt, u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{V(\Omega)} dt + \int_0^s \Phi(u_{\epsilon_n}(t, e_{\epsilon_n}))dt \\
& \quad + \int_0^s \langle \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{W(\Omega)} dt \\
& \quad \leq \int_0^s \langle du_{\epsilon_n}(t, e_{\epsilon_n})/dt, u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{V(\Omega)} \\
& \quad + \int_0^s \epsilon_n \langle \mathcal{A}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{V(\Omega)} dt \\
& \quad + \int_0^s \langle \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{W(\Omega)} dt + \int_0^s \Phi(u_{\epsilon_n}(t, e_{\epsilon_n}))dt \\
& \quad \leq \int_0^s \langle du_{\epsilon_n}(t, e_{\epsilon_n})/dt, v(t)\rangle_{V(\Omega)} dt \\
& \quad + \int_0^s \epsilon_n \langle \mathcal{A}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), v(t)\rangle_{V(\Omega)} dt \\
& \quad + \int_0^s \langle \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), v(t)\rangle_{W(\Omega)} dt \int_0^s \Phi(v(t))dt \\
& \quad - \int_0^s \langle f(t) + Be_{\epsilon_n}, v(t) - u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{W(\Omega)} dt
\end{aligned} \tag{3.56}$$

But, by virtue of (3.55), (M4) and (3.56) we can write

$$
\begin{aligned}
\mathcal{V}^s_{\epsilon_n} \leq &\int_0^s \langle du_{\epsilon_n}(t, e_{\epsilon_n})/dt, v(t) - u_0(t, e_0^*)\rangle_{V(\Omega)} dt \\
&+ \int_0^s \epsilon_n \langle \mathcal{A}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), v(t)\rangle_{V(\Omega)} dt \\
&+ \int_0^s \langle \mathcal{B}(t, e_{\epsilon_n}) u_{\epsilon_n}(t, e_{\epsilon_n}), v(t) - u_0(t, e_0^*)\rangle_{W(\Omega)} dt \\
&+ \int_0^s \Phi(v(t)) dt - \int_0^s \Phi(u_0(t, e_0^*)) dt \\
&- \int_0^s \langle f(t) + Be_{\epsilon_n}, v(t) - u_{\epsilon_n}(t, e_{\epsilon_n})\rangle_{W(\Omega)} dt
\end{aligned}
\tag{3.57}
$$

Hence by (3.47), (3.40) and (3.41) we get

$$
\begin{aligned}
\limsup_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \mathcal{V}^s_{\epsilon_n} \leq &\int_0^s \langle du_0(t, e_0^*)/dt, v(t) - u_0(e_0^*)\rangle_{V(\Omega)} dt \\
&+ \int_0^s \langle \mathcal{B}(t, e_0^*) u_0(t, e_0^*), v(t) - u_0(t, e_0^*)\rangle_{W(\Omega)} dt \\
&+ \int_0^s \Phi(v(t)) dt - \int_0^s \Phi(u_0(t, e_0^*)) dt \\
&- \int_0^s \langle f(t) + Be_0^*, v(t) - u_0(t, e_0^*)\rangle_{W(\Omega)} dt \\
&\qquad \text{for a.e. } t \in [0, T], \text{ for any } v(t) \in \mathcal{K}(e_0^*, \Omega),\ e_0^* \in U_{ad}(\Omega)
\end{aligned}
\tag{3.58}
$$

Note that (3.58) is true for any $v \in \hat{\mathcal{K}}(e_0^*, \Omega)$, hence also for $v(t) = u_0(t, e_0^*)$ and therefore $\lim\limits_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} \mathcal{V}^s_{\epsilon_n} \leq 0$, whence the strong convergence in $L_2(0, T, W(\Omega))$ follows.

Now as $u_{\epsilon_n}(e) \to u_0(e)$ (strongly) in $L_2(0, T, W(\Omega))$ for all $e \in U_{ad}(\Omega)$, we obtain $J_{\epsilon_n}(e_{\epsilon_n}) \leq J_{\epsilon_n}(e)$ for all $e \in U_{ad}(\Omega)$ and from this and ((E1),1°) we get

$$
\begin{aligned}
&\limsup_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} J_{\epsilon_n}(e_{\epsilon_n}) \leq J_0(e) \quad \text{for any } e \in U_{ad}(\Omega) \quad \Rightarrow \\
&\qquad \limsup_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} J_{\epsilon_n}(e_{\epsilon_n}) \leq \inf_{e\in U_{ad}(\Omega)} J_0(e)
\end{aligned}
\tag{3.59}
$$

Furthermore, we observe that ((E1),2°) and (3.47) imply

$$
\liminf_{\substack{n\to+\infty \\ (\epsilon_n\to 0)}} J_{\epsilon_n}(e_{\epsilon_n}) \geq \mathcal{L}(e_0^*, u_0(e_0^*)) = J_0(e_0^*)
$$

Comparing this result with (3.59) we see that necessarily

$$e_0^* = e_0, \qquad \text{thus:} \qquad u_0(t, e_0^*) = u_0(t, e_0) \qquad \text{for a.e. } t \in [0, T] \tag{3.60}$$

This means that (3.47) and (3.60) give (3.20).

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