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Application of Carleson's Theorem to wavelet inversion

by

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In this paper we prove, using the Carleson-Hunt theorem on the pointwise convergence of Fourier series, that the wavelet inversion formula is valid pointwise for all L^2-functions, and also without restrictions on wavelets. Moreover, we formulate and prove the same result for all L^p-functions, $1 < p < +\infty$.

Introduction

Wavelets developed from interdisciplinary origins in the last twenty years and the literature on wavelets is growing rapidly (for detailed bibliography we refer to Daubechies, 1992 and the references therein). Their importance comes from wide applicability of the subject (data analysis, image compression and enhancement, computer vision, subband filtering scheme, characterizing function spaces, like BMO, Sobolev, and Besov spaces). One of the most important features of the continuous wavelet transform is the inversion formula. The main idea is to recover a function f from its wavelet transform, just as in Fourier inversion formula. The inversion formula is achieved *in the L^2 sense*, via, so called, the resolution of the identity formula, for all L^2-functions and without restrictions on wavelets (see, for example, Daubechies, 1992 p. 24). The problem of pointwise convergence in the inversion formula was not so successfully treated so far. The results in Daubechies (1992) and Holschneider, Tchamitchian (1990)

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give the pointwise inversion formula only for bounded L^2-functions and only at continuity points, and under numerous restrictions on wavelets.

In this article we extend this pointwise inversion result in several directions. First, we remove all restrictions on wavelets. Secondly, we prove that the inversion formula is valid for all L^2-functions almost everywhere. Thirdly, we formulate and prove a pointwise inversion formula in L^p, with the sole restriction that $1 < p < +\infty$. In proving our result we will use the celebrated result of L.Carleson about the almost everywhere convergence of Fourier series. Carleson proved this theorem for $p = 2$ Carleson (1966), and R.A.Hunt extended it to the case of $p > 1$ in his work Hunt (1968).

1. Preliminaries

In this section we will briefly recall the Carleson–Hunt theorem, and some of its easy consequences that we will apply in the proof of our main result. Consider an interval $[-A, A] \subseteq \mathbb{R}$, where $A > 0$. For every $f \in L^1([-A, A])$, the Fourier coefficients $\hat{f}_n$, $n \in \mathbb{Z}$, are all defined by

$$\hat{f}_n = \int_{-A}^{A} f(t) e^{-\frac{\pi}{A} int}\, dt \quad . \tag{1}$$

For every $n \in \mathbb{N} \cup \{0\}$, we denote by $S_n(x; f)$ the nth partial sum

$$S_n(x; f) = \sum_{k=-n}^{n} \hat{f}_k e^{\frac{\pi}{A} ikx} \;, \;\; x \in [-A, A], \tag{2}$$

of the Fourier series for f. After an easy computation we obtain the well-known formula

$$S_n(x; f) = \int_{-A}^{A} f(t) D_n(x - t) dt \;, \tag{3}$$

where D_n is a periodic function with period $2A$ defined on $[-A, A]$ by

$$D_n(t) = \begin{cases} \frac{\sin\left[\frac{\pi}{A} t (n + \frac{1}{2})\right]}{\sin \frac{\pi t}{2A}} & \text{if } t \in [-A, A] \setminus \{0\} \\ 2n + 1 & \text{if } t = 0 \end{cases} \tag{4}$$

By $\mathcal{F}_A^+$ we denote the class of measurable functions defined on $[-A, A]$ with values in $[0, +\infty]$. For every p, $1 < p < +\infty$, we define the operator M_A : $L^p([-A, A]) \longrightarrow \mathcal{F}_A^+$ by

$$(M_A f)(x) = \sup \left\{ |S_n(x; f)| \, : \, n \in \mathbb{N} \cup \{0\} \right\} \; . \tag{5}$$

The crucial step in proving the Carleson–Hunt theorem is a result that the operator M_A is of type p, for every $1 < p < +\infty$ (see, for example, Mozzochi, 1971, p.8), i.e., there exists a positive constant $C_{p,A}$, which depends only on p and A, such that, for every $f \in L^p([-A, A])$,

$$||M_A f||_p \leq C_{p,A} ||f||_p \quad . \tag{6}$$

In particular, (3), (5), and (6) show that, for every $f \in L^p([-A, A])$, there exists a subset $H_A(f) \subseteq [-A, A]$ of Lebesgue measure zero, such that, for every $x \in [-A, A] \setminus H_A(f)$,

$$\sup_{n \in \mathbb{N} \cup \{0\}} \left| \int_{-A}^{A} f(t) D_n(x-t) dt \right| < +\infty \quad . \tag{7}$$

Consider the family of functions $F_\lambda : \mathbb{R} \longrightarrow \mathbb{R}$, $\lambda \in [0, +\infty)$, defined by

$$F_\lambda = \begin{cases} \frac{\sin\left[\frac{\pi}{A}\lambda t\right]}{\frac{\pi t}{2A}} & \text{if } t \neq 0 \\ 2\lambda & \text{if } t = 0 \end{cases} \tag{8}$$

Using the same type of estimates as in Lemma 3.3 on p.12 in Mozzochi (1971) we can prove easily that there exists a constant $K > 0$, such that, for every $n \in \mathbb{N} \cup \{0\}$ and for every $t \in [-(3/2)A, (3/2)A]$,

$$|\, F_n(t) - D_n(t) \,| \leq K \ . \tag{9}$$

We apply (9) and Hölder's inequality on (7), to obtain that, for every $f \in L^p([-A, A])$, there exists a subset $H_A(f) \subseteq [-A, A]$ of Lebesgue measure zero, such that, for every $x \in [-A/2, A/2] \setminus H_A(f)$,

$$\sup_{n \in \mathbb{N} \cup \{0\}} \left| \int_{-A}^{A} f(t) F_n(x-t) dt \right| < +\infty \quad . \tag{10}$$

Notice that in (10) we had to restrict x to $[-A/2, A/2]$, because of (9), which is not valid on $[-2A, 2A]$, but only on $[-(3/2)A, (3/2)A]$.

For every $\lambda \in [0, +\infty)$, there exist $n \in \mathbb{N} \cup \{0\}$ and $\theta \in [0, 1)$, such that $\lambda = n + \theta$. It follows that

$$\begin{aligned} &|F_\lambda(t) - F_n(t)| \\ &\leq |\sin\left(\frac{\pi}{A} nt\right)| \cdot \left| \frac{1 - \cos\left(\frac{\pi}{A}\theta t\right)}{\frac{\pi \theta t}{A}} \right| \cdot |2\theta| + |\cos\left(\frac{\pi}{A} nt\right)| \cdot \left| \frac{\sin\left(\frac{\pi}{A}\theta t\right)}{\frac{\pi \theta t}{A}} \right| \cdot |2\theta| \\ &\leq 2 \left[\left| \frac{1 - \cos\left(\frac{\pi}{A}\theta t\right)}{\frac{\pi \theta t}{A}} \right| + \left| \frac{\sin\left(\frac{\pi}{A}\theta t\right)}{\frac{\pi \theta t}{A}} \right| \right] . \end{aligned}$$

Since the functions (defined to be continuous at zero)

$$\alpha \mapsto \left| \frac{1 - \cos \alpha}{\alpha} \right| \quad \text{and} \quad \alpha \mapsto \left| \frac{\sin \alpha}{\alpha} \right|$$

are bounded on the entire real line, we obtain, using (10), that for every $f \in L^p([-A,A])$, there exists a subset $H_A(f) \subseteq [-A,A]$ of Lebesgue measure zero, such that, for every $x \in [-A/2, A/2] \setminus H_A(f)$,

$$\sup_{\lambda \geq 0} \left| \int_{-A}^{A} f(t) F_\lambda(x-t)\,dt \right| < +\infty \ . \tag{11}$$

The following lemma is now an easy consequence of (11). We prove it here for the sake of completeness.

LEMMA 1 *Let $p \in (1, +\infty)$. For every $f \in L^p(\mathbb{R})$, there exists a subset $H(f) \subseteq \mathbb{R}$ of Lebesgue measure zero, such that, for every $x \in \mathbb{R} \setminus H(f)$,*

$$\sup_{\lambda \geq 0} \left| \int_{-\infty}^{+\infty} f(t)\, \frac{\sin[2\pi\lambda(x-t)]}{(x-t)}\,dt \right| < +\infty \ , \tag{12}$$

where the function $u \mapsto [\sin(2\pi\lambda u)]/u$ is defined to be continuous at zero.

PROOF. Since the countable union of sets of Lebesgue measure zero is again of Lebesgue measure zero, it is enough to prove (12) for every x in an arbitrary interval $[-B, B]$.

Consider $[-B, B] \subseteq \mathbb{R}$, $B > 0$. For every $x \in [-B, B]$ we have

$$\begin{aligned} &\left| \int_{-\infty}^{+\infty} f(t)\, \frac{\sin[2\pi\lambda(x-t)]}{(x-t)}\,dt \right| \\ &\leq \left| \int_{-2B}^{2B} f(t)\, \frac{\sin[2\pi\lambda(x-t)]}{(x-t)}\,dt \right| + \int_{|t|>2B} \frac{|f(t)|}{|x-t|}\,dt \ . \end{aligned}$$

Since $|x| \leq B$ and $|t| > 2B$, we have that $|x-t| > B$. Using Hölder's inequality we obtain that the second integral above is smaller than

$$\left(\int_{\mathbb{R}} |f(t)|^p\,dt \right)^{1/p} \left(\int_{|u|>B} \frac{du}{|u|^q} \right)^{1/q} \ , \quad \frac{1}{p} + \frac{1}{q} = 1 \ . \tag{13}$$

Since $f \in L^p(\mathbb{R})$, the expression in (13) is finite, and does not depend on λ.

The first integral is estimated by (11) already; taking $A = 2B$, and realizing that f restricted to $[-A, A]$ is in $L^p([-A,A])$ and that

$$\begin{aligned} &\sup_{\lambda \geq 0} \left| \int_{-A}^{A} f(t)\, \frac{\sin[2\pi\lambda(x-t)]}{(x-t)}\,dt \right| \\ &\quad = \frac{2A}{\pi} \cdot \sup_{\lambda_1 \geq 0} \left| \int_{-A}^{A} f(t)\, \frac{\sin[\frac{\pi}{A}\lambda_1(x-t)]}{\frac{\pi(x-t)}{2A}}\,dt \right| \ , \end{aligned}$$

where $\lambda_1 = 2A\lambda$. ■

2. Main result

For every $f \in L^2(\mathbb{R})$ we denote by $\hat{f}$ its Fourier transform, i.e., for every $\omega \in \mathbb{R}$,

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi i\omega t}\, dt \quad . \tag{14}$$

We will say that $\psi \in L^2(\mathbb{R})$ is *a mother wavelet* if $\psi \in L^1(\mathbb{R})$, $||\psi||_2 = 1$, and $\hat{\psi}(0) = 0$. To simplify the notation, we assign to each mother wavelet ψ two families of functions $\{\, \psi_a^0 \;:\; a \in \mathbb{R}, a \neq 0 \,\}$ and $\{\, \psi_a^{00} \;:\; a \in \mathbb{R}, a \neq 0 \,\}$ defined by

$$\psi_a^0(x) = |a|^{-1/2}\bar{\psi}\left(\frac{-x}{a}\right) \quad \text{and} \quad \psi_a^{00}(x) = |a|^{-1/2}\psi\left(\frac{x}{a}\right) \quad , \tag{15}$$

where $\bar{z}$ denotes the complex conjugate of the complex number z. Notice that ψ_a^0, $\psi_a^{00} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $||\psi_a^0||_2 = ||\psi_a^{00}||_2 = 1$, and

$$\hat{\psi}_a^0(\omega) = |a|^{1/2}\operatorname{sign}(a)\bar{\hat{\psi}}(a\omega) \quad , \quad \hat{\psi}_a^{00}(\omega) = |a|^{1/2}\operatorname{sign}(a)\hat{\psi}(a\omega) \quad . \tag{16}$$

Let $p \in (1, +\infty)$, and consider $L^p(\mathbb{R})$. We define the continuous wavelet transform with respect to ψ by

$$(T^{wav}f)(a,b) = \left(f * \psi_a^0\right)(b), \quad f \in L^p(\mathbb{R}), \tag{17}$$

where $a, b \in \mathbb{R}$, $a \neq 0$, and $*$ is the convolution operator. Recall that Young's theorem guarantees that

$$b \mapsto (T^{wav}f)(a,b)$$

is in $L^p(\mathbb{R})$, for every $a \in \mathbb{R}$, $a \neq 0$. In the case when $p = 2$, i.e., $f \in L^2(\mathbb{R})$, formula (16) implies that

$$(\hat{T}^{wav}f)(a,\cdot)(\omega) = \hat{f}(\omega)|a|^{1/2}\operatorname{sign}(a)\bar{\hat{\psi}}(a\omega) \quad . \tag{18}$$

We can state our main result now.

THEOREM 1 *Let $1 < p, q < +\infty$, $1/p + 1/q = 1$. Suppose that ψ and φ are mother wavelets such that,*

$$\int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)| \cdot |\hat{\varphi}(\omega)|}{|\omega|}\, d\omega < +\infty \quad . \tag{19}$$

Then, for every $f \in L^p(\mathbb{R})$,

$$\lim_{\lambda \to 0+} \int_{|a| \geq \lambda} \frac{G(a,x)}{a^2}\, da = C_{\psi,\varphi} \cdot f(x) \quad (a.e.), \tag{20}$$

where $G(a,x)$ is defined by

$$G(a,x) = |a|^{-1/2} \int_{-\infty}^{+\infty} (T^{wav} f)(a,b) \cdot \varphi\left(\frac{x-b}{a}\right) db \quad , \tag{21}$$

where T^{wav} denotes the continuous wavelet transform with respect to ψ, and the constant $C_{\psi,\varphi}$ is given by

$$C_{\psi,\varphi} = \int_{-\infty}^{+\infty} \frac{\bar{\hat{\psi}}(\omega) \cdot \hat{\varphi}(\omega)}{|\omega|} \, d\omega < +\infty \quad . \tag{22}$$

Notice that, as indicated at the beginning of this paper, the inverse formula (20) is given for very general class of functions f and wavelets ψ and φ. The condition (19) is the only assumption on the pair of mother wavelets ψ and φ, and (20) is valid for every $f \in L^p(\mathbb{R})$.

The rest of the paper is devoted to the proof of Theorem 1. Consider first $G(a,x)$ in the case when $f \in L^2(\mathbb{R})$. Then, $G(a,x)$ can be expressed by

$$G(a,x) = \{ [(T^{wav} f)(a,\cdot)] * \varphi_a^{00} \} (x) \quad . \tag{23}$$

Since $\varphi_a^{00}(x) = \bar{\varphi}_a^0(-x)$, it follows that $G(a,x)$ is equal to the scalar product (in $L^2(\mathbb{R})$) of $b \mapsto (T^{wav} f)(a,b)$ and $b \mapsto \varphi_a^0(b-x)$. Now, applying Plancherel's theorem and (16) and (18) we obtain

$$\begin{aligned} G(a,x) &= \int_{-\infty}^{+\infty} [(\hat{T}^{wav} f)(a,\cdot)](\omega) \cdot [\bar{\varphi}_a^0(\cdot - x)](\omega) \, d\omega \\ &= \int_{-\infty}^{+\infty} \hat{f}(\omega) |a|^{1/2} \mathrm{sign}(a) \bar{\hat{\psi}}(a\omega) \cdot |a|^{1/2} \mathrm{sign}(a) e^{2\pi i \omega x} \hat{\varphi}(a\omega) \, d\omega \\ &= |a| \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{2\pi i \omega x} \bar{\hat{\psi}}(a\omega) \hat{\varphi}(a\omega) \, d\omega \quad . \end{aligned}$$

Let us define the function ϱ by

$$\varrho = \psi_1^0 * \varphi_1^{00} \quad . \tag{24}$$

Then, we obtain that, for every $f \in L^2(\mathbb{R})$,

$$G(a,x) = |a| \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{2\pi i \omega x} \hat{\varrho}(a\omega) \, d\omega \quad . \tag{25}$$

Notice that (24) implies that $\hat{\varrho} \in L^1(\mathbb{R})$, and that condition (19) is equivalent to the condition

$$\frac{|\hat{\varrho}(\omega)|}{|\omega|} \in L^1(\mathbb{R}) \quad . \tag{26}$$

We claim that (25) implies that

$$\int_{|a| \geq \lambda} \frac{G(a,x)}{a^2} da = \int_{\mathbb{R}} \frac{\hat{\varrho}(a)}{|a|} da \int_{|\omega| \leq |a|/\lambda} \hat{f}(\omega) e^{2\pi i \omega x} \, d\omega \quad . \tag{27}$$

Notice that (27) follows by application of Fubini's theorem. Hence, to prove (27), it is enough to justify Fubini's theorem. Using (25) we get

$$\begin{aligned}
&\int_{|a|\geq\lambda} \frac{|a|}{a^2}da \int_{\mathbb{R}} |\hat{f}(\omega)| \cdot |\hat{\varrho}(a\omega)|\, d\omega \\
&= \int_{\mathbb{R}} |\hat{f}(\omega)|\, d\omega \int_{|a|\geq|\omega|\lambda} \frac{|\hat{\varrho}(a)|}{|a|} da \\
&= \int_{\mathbb{R}} \frac{|\hat{\varrho}(a)|}{|a|} da \int_{|\omega|\leq|a|/\lambda} |\hat{f}(\omega)|\, d\omega \\
&\leq \int_{|a|\leq 1} \frac{|\hat{\varrho}(a)|}{|a|} da \int_{|\omega|\leq 1/\lambda} |\hat{f}(\omega)| d\omega + \int_{|a|>1} \frac{|\hat{\varrho}(a)|}{|a|} da \int_{|\omega|\leq|a|/\lambda} |\hat{f}(\omega)| d\omega.
\end{aligned}$$

The first integral in the last line above is finite, since (26) is valid and

$$\int_{|\omega|\leq 1/\lambda} |\hat{f}(\omega)|\, d\omega \leq \|\hat{f}\|_2 \cdot \sqrt{\frac{2}{\lambda}} < +\infty \ .$$

That the second integral is finite follows from the fact that $\hat{\varrho} \in L^1(\mathbb{R})$, and from the estimate

$$\frac{1}{|a|} \int_{|\omega|\leq|a|/\lambda} |\hat{f}(\omega)|\, d\omega \leq \frac{1}{|a|} \|\hat{f}\|_2 \left[\frac{2|a|}{\lambda}\right]^{1/2} \leq \|\hat{f}\|_2 \cdot \sqrt{\frac{2}{\lambda}} \, ,$$

where the last inequality follows since $|a| > 1$. Therefore, we just proved (27).

Recall a well-known formula, which is an easy consequence of (14), and says that, for every $f \in L^2(\mathbb{R})$,

$$\int_{|u|\leq\xi} \hat{f}(u) e^{2\pi i u x} du = \frac{1}{\pi} \int_{\mathbb{R}} f(t)\, \frac{\sin[2\pi\xi(x-t)]}{(x-t)}\, dt \ . \tag{28}$$

Apply (28) on (27), and we get that for every $f \in L^2(\mathbb{R})$ and for every $x \in \mathbb{R}$ and $\lambda > 0$

$$\int_{|a|\geq\lambda} \frac{G(a,x)}{a^2} da = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\hat{\varrho}(a)}{|a|} da \int_{\mathbb{R}} f(t)\, \frac{\sin[2\pi(|a|/\lambda)(x-t)]}{(x-t)}\, dt \ . \tag{29}$$

Notice that on both sides of (29) there is no more $\hat{f}$, but only f, and that, in particular, (29) is valid for every f continuous with compact support, i.e., (29) is valid on a dense subspace of $L^p(\mathbb{R})$. Therefore, (29) is valid for every $f \in L^p(\mathbb{R})$, if we can prove that both sides of (29) define continuous linear functionals on $L^p(\mathbb{R})$. Indeed, this is the case.

Consider the integral on the right hand side first. Since the function

$$t \mapsto \frac{\sin[2\pi(|a|/\lambda)(x-t)]}{(x-t)}$$

is in $L^q(\mathbb{R})$, we obtain, using Hölder's inequality, that the integral on the right–hand side is bounded above by

$$\left(\frac{2}{\lambda}\right)^{1/p} \cdot \frac{1}{\pi^{1/q}} \cdot \|f\|_p \cdot \|\frac{\sin t}{t}\|_q \cdot \int_{\mathbb{R}} \frac{|\hat{\varrho}(a)|}{|a|^{1/q}} da \, .$$

Recall (26) and $\hat{\varrho} \in L^1(\mathbb{R})$ to prove that

$$\int_{\mathbb{R}} \frac{|\hat{\varrho}(a)|}{|a|^{1/q}} da \leq \int_{|a|\leq 1} \frac{|\hat{\varrho}(a)|}{|a|} da + \int_{|a|>1} |\hat{\varrho}(a)| da < +\infty \quad .$$

Therefore, the right–hand side is a continuous linear functional on $L^p(\mathbb{R})$.

Consider the integral on the left–hand side of (29). Using (17) and (23) we obtain that

$$\left| \int_{|a|\geq\lambda} \frac{G(a,x)}{a^2} da \right| = \left| \int_{|a|\geq\lambda} [f * (\psi_a^0 * \varphi_a^{00})](x) \frac{da}{a^2} \right| \, .$$

Let us denote by μ the measure $1_{\{a:|a|\geq\lambda\}} da/a^2$. We claim that the function

$$b \mapsto \int (\psi_a^0 * \varphi_a^{00})(b) d\mu(a) \tag{30}$$

is in $L^q(\mathbb{R})$. By Schwarz inequality and the fact that $\|\psi_a^0\|_2 = \|\varphi_a^{00}\|_2 = 1$, we obtain that $|(\psi_a^0 * \varphi_a^{00})(b)| \leq 1$, for every $b \in \mathbb{R}$. Therefore, we obtain

$$\begin{aligned}
\left\| \int (\psi_a^0 * \varphi_a^{00})(\cdot) d\mu(a) \right\|_q &\leq \int \|(\psi_a^0 * \varphi_a^{00})(\cdot)\|_q d\mu(a) \\
&\leq \int \|(\psi_a^0 * \varphi_a^{00})(\cdot)\|_1^{1/q} d\mu(a) \\
&\leq \int \|\psi_a^0\|_1^{1/q} \cdot \|\varphi_a^{00}\|_1^{1/q} d\mu(a) \\
&= \|\psi\|_1^{1/q} \cdot \|\varphi\|_1^{1/q} \cdot \int |a|^{1/q} d\mu(a) \\
&= \|\psi\|_1^{1/q} \cdot \|\varphi\|_1^{1/q} \cdot \int_{|a|\geq\lambda} \frac{da}{|a|^{2-1/q}} \\
&= \|\psi\|_1^{1/q} \cdot \|\varphi\|_1^{1/q} \cdot \frac{1}{p\lambda^{1/p}} \\
&< +\infty \, ,
\end{aligned}$$

where in the third line above we used Young's inequality. Therefore, we just proved that the function defined by (30) is in $L^q(\mathbb{R})$. Applying this result we prove that the left-hand side of (29) is also a continuous linear functional, since

(by Hölder's inequality)

$$\begin{aligned}\left|\int_{|a|\geq\lambda}\frac{G(a,x)}{a^2}da\right| &= \left|\left\{f*\left[\int_{|a|\geq\lambda}(\psi_a^0*\varphi_a^{00})(\cdot)\frac{da}{a^2}\right]\right\}(x)\right| \\ &\leq \|f\|_p\cdot\left\|\int(\psi_a^0*\varphi_a^{00})(\cdot)d\mu(a)\right\|_q .\end{aligned}$$

Hence, (29) is valid for every $f\in L^p(\mathbb{R})$.

Consider now the family of functions $C_K^\infty(\mathbb{R})$, where $C_K^\infty(\mathbb{R})$ is the set of infinitely differentiable functions with compact support. Recall that $C_K^\infty(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$, for every $1<p<+\infty$, and that $\hat{f}\in L^1(\mathbb{R})$, for every $f\in C_K^\infty(\mathbb{R})$. Using (28), we obtain that for every $f\in C_K^\infty(\mathbb{R})$, and for every $a,x\in\mathbb{R}$, $a\neq 0$,

$$\lim_{\lambda\to 0+}\frac{1}{\pi}\int_{\mathbb{R}} f(t)\,\frac{\sin[2\pi(|a|/\lambda)(x-t)]}{(x-t)}\,dt = f(x) \quad , \tag{31}$$

and, by (27) and the dominated convergence theorem, for every $x\in\mathbb{R}$,

$$\lim_{\lambda\to 0+}\int_{|a|\geq\lambda}\frac{G(a,x)}{a^2}da = \left(\int_{\mathbb{R}}\frac{\hat{\varrho}(a)}{|a|}da\right)\cdot f(x) = C_{\psi,\varphi}\cdot f(x) . \tag{32}$$

Consider the family of operators T_λ^a defined on $L^p(\mathbb{R})$ by

$$(T_\lambda^a f)(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\,\frac{\sin[2\pi(|a|/\lambda)(x-t)]}{(x-t)}\,dt \quad , \tag{33}$$

and the operator T^* defined on $L^p(\mathbb{R})$ by

$$(T^*f)(x) = \sup_{\lambda>0}|(T_\lambda^a f)(x)| \quad . \tag{34}$$

Notice that supremum in (34) does not depend on $a\in\mathbb{R}$, $a\neq 0$, so the definition of T^* makes sense.

Consider the set D of functions $f\in L^p(\mathbb{R})$ such that, for every $a\in\mathbb{R}$, $a\neq 0$, the limit

$$\lim_{\lambda\to 0+}(T_\lambda^a f)(x) = (T^a f)(x) \tag{35}$$

exists for almost every x. By (31) D is dense in $L^p(\mathbb{R})$. However, Lemma 1 shows that for every $f\in L^p(\mathbb{R})$, $(T^*f)(x)<+\infty$, x-(a.e.). Using the Banach principle (see, for example, Garsia, 1970, pp.1–3, Theorem 1.1.1.) we conclude that D is a closed set. Therefore, $D=L^p(\mathbb{R})$. Using Lemma 1 one more time on (34) (notice that $(T^*f)(x)$ does not depend on a), we apply the dominated convergence theorem on (29), and obtain that, for every $f\in L^p(\mathbb{R})$,

$$\lim_{\lambda\to 0+}\int_{|a|\geq\lambda}\frac{G(a,x)}{a^2}da = \int_{\mathbb{R}}\frac{\hat{\varrho}(a)}{|a|}(T^a f)(x)da \quad x-(a.e.). \tag{36}$$

Now we need only to prove that, for every $a \in \mathbb{R}$, $a \neq 0$ and for every $f \in L^p(\mathbb{R})$,

$$(T^a f)(x) = f(x) \quad x - (a.e.). \tag{37}$$

It is enough to prove that the sequence of functions $\{T^a_{1/n} f\}$ converges almost everywhere to f.

Let us fix $a \neq 0$. By Lemma 1 we can apply the Banach principle (Garsia, 1970, p.2) one more time, to get that there exists a positive, decreasing function $C : (0, +\infty) \longrightarrow (0, +\infty)$, such that,

$$\lim_{\lambda \to +\infty} C(\lambda) = 0 \tag{38}$$

and, for every $f \in L^p(\mathbb{R})$ and $R > 0$,

$$Leb\left\{ x : \sup_{n \geq 1} |(T^a_{1/n} f)(x)| > R \right\} \leq C\left(\frac{R}{\|f\|_p} \right) \quad , \tag{39}$$

where Leb denotes the Lebesgue measure.

Let $f \in L^p(\mathbb{R})$ and $\varepsilon > 0$. There exists a sequence $\eta_k \nearrow +\infty$ such that $C(\eta_k) \leq 1/2^k$. By taking a subsequence if necessary, we can choose a sequence $\{f_k\} \subseteq C^\infty_K(\mathbb{R})$ such that

$$f_k \longrightarrow f \;\; (a.e.), \text{ as } k \to +\infty, \tag{40}$$

and, for every $k \in \mathbb{N}$,

$$\| f_k - f \|_p \leq \frac{\varepsilon}{\eta_k} \quad . \tag{41}$$

In particular $f_k \longrightarrow f$ in $L^p(\mathbb{R})$. Using (39) for $\varepsilon > 0$ and every k, we obtain, since (41) is valid, that

$$\int_{\mathbb{R}} \sum_{k=1}^{\infty} 1_{\{y : \sup_{n \geq 1} |(T^a_{1/n} f)(y) - (T^a_{1/n} f_k)(y)| > \varepsilon\}}(x) \, dx \leq 1 \quad . \tag{42}$$

Using (40) and (42) we conclude that there exists a set $H(f, \varepsilon) \subseteq \mathbb{R}$ of Lebesgue measure zero, such that, for every $x \notin H(f, \varepsilon)$, there exist $k_1(x) \in \mathbb{N}$ and $k_2(x) \in \mathbb{N}$, such that, for every $k \geq k_1(x)$,

$$| f_k(x) - f(x) | \leq \varepsilon \quad , \tag{43}$$

and, for every $k \geq k_2(x)$,

$$\sup_{n \geq 1} | (T^a_{1/n} f)(x) - (T^a_{1/n} f_k)(x) | \leq \varepsilon \quad . \tag{44}$$

Therefore, if we take $k \geq \max(k_1(x), k_2(x))$, then, since $f_k \in C^\infty_K(\mathbb{R})$, there exists $n_0 \in \mathbb{N}$, such that, for every $n \geq n_0$,

$$| (T^a_{1/n} f_k)(x) - f_k(x) | \leq \varepsilon \quad . \tag{45}$$

Using (43), (44), and (45), and the fact that

$$\begin{aligned}&|(T^a_{1/n}f)(x) - f(x)|\\ &\leq |(T^a_{1/n}f)(x) - (T^a_{1/n}f_k)(x)| + |(T^a_{1/n}f_k)(x) - f_k(x)| + |f_k(x) - f(x)|,\end{aligned}$$

we obtain that there exists a set $H(f,\varepsilon) \subseteq \mathbb{R}$ of Lebesgue measure zero, such that, for every $x \notin H(f,\varepsilon)$, there exists $n_0(x,\varepsilon) \in \mathbb{N}$, such that, for every $n \geq n_0(x,\varepsilon)$,

$$|(T^a_{1/n}f)(x) - f(x)| \leq 3\varepsilon \ . \tag{46}$$

Consider the set

$$H(f) = \bigcup_{l=1}^{\infty} H\left(f, \frac{1}{l}\right) .$$

Then $LebH(f) = 0$, and, for every $x \notin H(f)$,

$$\lim_{n\to+\infty} \left(T^a_{1/n}f\right)(x) = f(x) \ . \tag{47}$$

This statement completes the proof of Theorem 1, and finishes this paper. ■

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