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Mathematical aspects of modelling the macroscopic behaviour of cross–ply laminates with intralaminar cracks

by

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The objective of this contribution is a study of two new effective models of cracked three-layer laminates developed in Lewiński, Telega (1994A, B). The convergence problems are discussed by using the method of Γ–convergence. For a moderately thick laminate weakened by transverse cracks of high density the convergence theorem is formulated. The convergence problems for a thin laminate weakened by transverse cracks of arbitrary density are investigated in detail. The application of the augmented Lagrangian method to solving the local problems is proposed.

Introduction

Two new effective models of three-layer laminates with intralaminar cracks have been proposed in the paper by Lewiński and Telega (1994A). The cracks were assumed to be distributed in a periodic manner. Stochastically periodic distribution has been proposed in Telega, Lewiński (1993). The study performed in Lewiński and Telega (1994A) is based on the mathematically formal method of two-scale asymptotic expansions. The first model derived in Lewiński and Telega (1994A) applies to moderately thick laminates with high density of cracks. Only an in-plane scaling is performed. In Section 4 of the present paper we will formulate the convergence theorem justifying the first model. The second model applies to thin laminates and involves rescaling of characteristic length dimensions of the laminate. Such an approach implies the rescaling of the stiffnesses and loading. Consequently, singular terms appear in the expression for the functional of the total potential energy $J_\varepsilon(\varepsilon > 0)$. Section 5 of the paper reports all the relevant results concerning the epi–convergence of the sequence of functionals $\{J_\varepsilon\}_{\varepsilon>0}$. Application of the models developed in Lewiński and Telega (1994A) to the case of laminates with aligned cracks is discussed in detail in the paper by Lewiński and Telega (1994B). To know the explicit form of the homogenized (effective) elastic potential one has to solve unilaterally constrained minimization problems. In Section 6 we will suggest an application of the augmented Lagrangian method to solving the local problems.

1. Preliminaries: in-plane deformation of symmetric three-layer laminates of moderate thickness

A new two-dimensional model of three-layer symmetric laminate, introduced in Lewiński and Telega (1993, 1994A, B), Telega and Lewiński (1993), will now be briefly described. The model is capable of describing the independent in–plane displacements of the faces and of the internal layer.

Consider a symmetric laminate composed of the faces of thickness d and the internal layer of thickness $2c$. The middle plane Ω of the internal layer is parameterized by Cartesian coordinates $x_\alpha; (x_\alpha) = x \in \Omega$. The whole laminate occupies a cylindrical domain $\mathbf{B} = \Omega \times (-h, h)$; $h = c + d$. To an arbitrary point $\underset{\sim}{x} \in \mathbf{B}$ we assign its coordinates $\underset{\sim}{x} = (x_i) = (x_\alpha, x_3 = z), z$ axis being perpendicular to the plane Ω.

The lower and upper faces $z = \mp h$ are assumed to be free of loads, whilst the lateral edge surface $S = \Gamma \times (-h, h), \Gamma = \partial\Omega$, is subjected to the tractions $p^i(s, z), s \in \Gamma$, on its part $S_\sigma = \Gamma_\sigma \times (-h, h)$. The remaining part of $S, S_w = \Gamma_w \times (-h, h)$ $(\Gamma = \bar{\Gamma}_w \cup \bar{\Gamma}_\sigma)$ is clamped. The loading p^i is assumed to have the following through-the-thickness distribution

$$p^\alpha(s,z) = \begin{cases} \dfrac{1}{2c}\bar{L}^\alpha(s)\,, & |z| < c\,, \\ \dfrac{1}{2d}(\bar{N}^\alpha(s) - \bar{L}^\alpha(s))\,, & \text{otherwise}\,; \end{cases} \tag{1.1}$$

$$p^3(s,z) = \begin{cases} \dfrac{1}{2d}(z-h)\bar{Q}(s)\,, & c < z < h\,, \\ -\dfrac{z}{2d}\bar{Q}(s)\,, & |z| < c\,, \\ \dfrac{1}{2d}(z+h)\bar{Q}(s)\,, & -h < z < -c\,. \end{cases} \tag{1.2}$$

The prescribed loading functions $\bar{N}^\alpha$, $\bar{L}^\alpha$ and $\bar{Q}$ are defined on Γ_σ. For the sake of simplicity body forces are neglected.

The through-the-thickness distribution of elastic compliances has the form

$$D_{ijkl} = \begin{cases} D^m_{ijkl} , & |z| < c , \\ D^f_{ijkl} , & \text{otherwise,} \end{cases} \tag{1.3}$$

where $\boldsymbol{D}^m$ and $\boldsymbol{D}^f$ may depend on $x \in \Gamma$. We assume that the planes $z =$ const. are the planes of material symmetry, hence

$$D^{\mathfrak{n}}_{3\alpha\beta\gamma} = D^{\mathfrak{n}}_{333\alpha} = 0 , \quad \mathfrak{n} = m \text{ or } f. \tag{1.4}$$

The tensor $\boldsymbol{D}$ satisfies the usual symmetry condition

$$D_{ijkl} = D_{jikl} = D_{klij} . \tag{1.5}$$

We make further the following assumption

$$(H) \quad \left| \begin{array}{l} D_{ijkl} \in L^\infty(\mathbf{B}) , \\ \text{there exists a constant } C > 0 \text{ such that} \\ D_{ijkl}(x) T^{ij} T^{kl} \geq C |\boldsymbol{T}|^2 , \end{array} \right.$$

for a.e. $x \in \Omega$ and for each $\mathbf{T} \in \mathbb{E}^3_s$, where $\mathbb{E}^3_s$ is the space of symmetric 3×3 matrices and

$$|\boldsymbol{T}|^2 = T^{ij} T^{ij} .$$

Throughout this paper only Cartesian coordinate systems are employed, consequently we may identify (T^{ij}) with (T_{ij}), etc. The summation convention is applied to repeated indices at the same and different levels. Moreover, C with possibly a subscript will denote a positive constant.

The three-dimensional problem of equilibrium of the laminate considered amounts to finding a stress field $\tilde{\boldsymbol{\sigma}}(\underset{\sim}{x})$ as well as a displacement field $\tilde{\boldsymbol{w}}(\underset{\sim}{x})$ for which the two-field Reissner's functional, Fung (1965)

$$\begin{aligned} I(\boldsymbol{\sigma}, \boldsymbol{w}) \;=\; & \int\limits_B \Big[\frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha})\sigma^{\alpha\beta} + (w_{\alpha,3} + w_{3,\alpha})\sigma^{\alpha 3} + \\ & + w_{3,3}\sigma^{33} - \frac{1}{2} D_{ijkl}\sigma^{ij}\sigma^{kl} \Big] \, d\underset{\sim}{x} - \\ & - \int\limits_{S_\sigma} p^i(s,z) w_i(s,z) ds\, dz \end{aligned} \tag{1.6}$$

attains its stationary value at the saddle point $(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{\sigma}})$. Towards this end one can apply Arnold and Falk version of the Brezzi's theorem (Arnold, Falk, 1987), assuming that $p^i \in L^2(S_\sigma)$. We note that in (1.6) $\boldsymbol{p} = (p^\alpha, p^3)$ is not necessarily of the form (1.1) and (1.2).

A new two-dimensional laminate model is based upon the following stress and kinematic assumptions (Lewiński, Telega, 1994A):

$$\sigma^{\alpha\beta}(\underset{\sim}{x}) = \begin{cases} \frac{1}{2c} L^{\alpha\beta}(x) \, , & |z| < c \, , \\ \frac{1}{2d}[N^{\alpha\beta}(x) - L^{\alpha\beta}(x)] \, , & \text{otherwise} \, ; \end{cases} \tag{1.7}$$

$$\sigma^{\alpha 3}(\underset{\sim}{x}) = \begin{cases} \frac{1}{2d}(z-h)Q^{\alpha}(x) \, , & c < z < h \, , \\ -\frac{z}{2c}Q^{\alpha}(x) \, , & |z| < c \, , \\ \frac{1}{2d}(z+h)Q^{\alpha}(x) \, , & -h < z < -c \, . \end{cases} \tag{1.8}$$

$$\sigma^{33}(\underset{\sim}{x}) = \begin{cases} \frac{1}{4d}(z-h)^2 R(x) \, , & c < z < h \, , \\ \frac{1}{4c}(-z^2 + ch)R(x) \, , & |z| < c \, , \\ \frac{1}{4d}(z+h)^2 R(x) \, , & -h < z < -c \, . \end{cases} \tag{1.9}$$

$$w_{\alpha}(\underset{\sim}{x}) = \begin{cases} v_{\alpha}(x) + \frac{3}{2c^2}(c^2 - z^2)u_{\alpha}(x) \, , & |z| < c \, , \\ v_{\alpha}(x) \, , & \text{otherwise} \, ; \end{cases} \tag{1.10}$$

$$w_3(\underset{\sim}{x}) = \begin{cases} \frac{1}{b}w(x) \, , & c < z < h \, , \\ \frac{z}{c}\frac{w(x)}{b} \, , & |z| < c \, , \\ -\frac{1}{b}w(x) \, , & -h < z < -c \, . \end{cases} \tag{1.11}$$

where $b = \frac{d}{2} + \frac{c}{3}$.

The two-dimensional model of the laminate is obtained as follows. For the sake of simplicity it is assumed that D_{ijkl} depend on $x \in \Omega$ only. We substitute the expressions (1.7) - (1.11) into the Reissner's functional (1.6) and next perform z-integration to obtain a new functional J

$$J(\boldsymbol{v}, \boldsymbol{u}, w; \boldsymbol{N}, \boldsymbol{L}, \boldsymbol{Q}, R) = I(\boldsymbol{w}, \boldsymbol{\sigma}) \tag{1.12}$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{w}$ have the form (1.7) – (1.11). Finally J can be expressed as follows

$$J(\boldsymbol{v}, \boldsymbol{u}, w; \boldsymbol{N}, \boldsymbol{L}, \boldsymbol{Q}, R) = \\ = \int_{\Omega} [N^{\alpha\beta} v_{\alpha,\beta} + L^{\alpha\beta} u_{\alpha,\beta} + Q^{\alpha}(u_{\alpha} - w_{,\alpha}) + \tag{1.13}$$

$$+Rw - W_c(x, \boldsymbol{N}, \boldsymbol{L}, \boldsymbol{Q}, R)]\,dx - \int_{\Gamma_\sigma} (\bar{N}^\alpha v_\alpha + \bar{L}^\alpha u_\alpha - \bar{Q}w)ds \ ,$$

where the complementary energy density is given by

$$W_c = \Big(D^N_{\alpha\beta\lambda\mu} N^{\alpha\beta} N^{\lambda\mu} + D^L_{\alpha\beta\lambda\mu} L^{\alpha\beta} L^{\lambda\mu} + 2D^{NL}_{\alpha\beta\lambda\mu} N^{\alpha\beta} L^{\lambda\mu} + D^Q_{\alpha\beta} Q^\alpha Q^\beta + 2D^{RL}_{\alpha\beta} RL^{\alpha\beta} + 2D^{RN}_{\alpha\beta} RN^{\alpha\beta} + D^R R^2\Big)/2 \ , \tag{1.14}$$

and

$$\begin{aligned} D^N_{\alpha\beta\lambda\mu} &= \frac{1}{2d} D^f_{\alpha\beta\lambda\mu} \ , \quad D^{LN}_{\alpha\beta\lambda\mu} = -D^N_{\alpha\beta\lambda\mu} \ , \\ D^L_{\alpha\beta\lambda\mu} &= D^N_{\alpha\beta\lambda\mu} + \frac{1}{2c} D^m_{\alpha\beta\lambda\mu} \ , \quad D^Q_{\alpha\beta} = \frac{2}{3}(dD^f_{\alpha 3\beta 3} + cD^m_{\alpha 3\beta 3}) \ , \\ D^{RL}_{\alpha\beta} &= -\frac{d}{12} D^f_{\alpha\beta 33} + \frac{1}{4}(h - \frac{c}{3}) D^m_{\alpha\beta 33} \ , \\ D^{RN}_{\alpha\beta} &= \frac{d}{12} D^f_{\alpha\beta 33} \ , \quad D^R = \frac{d^3}{40} D^f_{3333} + \frac{c}{8}(h^2 - \frac{2}{3}ch + \frac{c^2}{5} D^m_{3333}). \end{aligned} \tag{1.15}$$

The Reissner–type functional J attains its stationary value if the following are satisfied:

i. the equilibrium equations

$$-N^{\alpha\beta}{}_{,\beta} = 0 \ , \quad -L^{\alpha\beta}{}_{,\beta} + Q^\alpha = 0 \ , \quad Q^\alpha{}_{,\alpha} + R = 0 \ , \quad \text{in } \Omega \ , \tag{1.16}$$

ii. the constitutive relationships

$$\left.\begin{aligned} \varepsilon_{\alpha\beta} &= D^N_{\alpha\beta\lambda\mu} N^{\lambda\mu} + D^{NL}_{\alpha\beta\lambda\mu} L^{\lambda\mu} + D^{RN}_{\alpha\beta} R \\ \gamma_{\alpha\beta} &= D^{NL}_{\alpha\beta\lambda\mu} N^{\lambda\mu} + D^L_{\alpha\beta\lambda\mu} L^{\lambda\mu} + D^{RL}_{\alpha\beta} R \\ w &= D^{RN}_{\alpha\beta} N^{\alpha\beta} + D^{RL}_{\alpha\beta} L^{\alpha\beta} + D^R R \end{aligned}\right\} \tag{1.17}$$

$$\kappa_\alpha = D^Q_{\alpha\beta} Q^\beta \ , \tag{1.18}$$

where the deformation measures, are defined by

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta}(\boldsymbol{v}) &&= (v_{\alpha,\beta} + v_{\beta,\alpha})/2 \ , \\ \gamma_{\alpha\beta} &= \gamma_{\alpha\beta}(\boldsymbol{u}) &&= (u_{\alpha,\beta} + u_{\beta,\alpha})/2 \ , \\ \kappa_\alpha &= \kappa_\alpha(\boldsymbol{u}, w) &&= u_\alpha - w_{,\alpha} \ , \end{aligned} \tag{1.19}$$

iii. the stress-type boundary condition along the line Γ_σ

$$N_n = \bar{N}_n \ , \quad N_\tau = \bar{N}_\tau \ , \quad L_n = \bar{L}_n \ , \quad L_\tau = \bar{L}_\tau \ , \quad Q = \bar{Q} \ , \tag{1.20}$$

where

$$N_n = N^{\alpha\beta} n_\alpha n_\beta \ , \quad N_\tau = N^{\alpha\beta} n_\beta \tau_\alpha \ ,$$
$$L_n = L^{\alpha\beta} n_\alpha n_\beta \ , \quad L_\tau = L^{\alpha\beta} n_\beta \tau_\alpha \ , \quad Q = Q^\alpha n_\alpha, \tag{1.21}$$
$$\bar{N}_n = \bar{N}^\alpha n_\alpha \ , \quad \bar{N}_\tau = \bar{N}^\alpha \tau_\alpha \ , \text{etc.}$$

Here $\boldsymbol{n} = (n_\alpha)$ and $\boldsymbol{\tau} = (\tau_\alpha)$ are unit vectors: outward normal and tangent to the Γ_σ line, respectively.

The constitutive relationships (1.17) and (1.18) can be inverted to the form

$$\left.\begin{aligned} N^{\lambda\mu} &= A_v^{\lambda\mu\alpha\beta} \varepsilon_{\alpha\beta} + A_{vu}^{\lambda\mu\alpha\beta} \gamma_{\alpha\beta} + A_{vw}^{\lambda\mu} w \\ L^{\lambda\mu} &= A_{vu}^{\lambda\mu\alpha\beta} \varepsilon_{\alpha\beta} + A_u^{\lambda\mu\alpha\beta} \gamma_{\alpha\beta} + A_{uw}^{\lambda\mu} w \\ R &= A_{vw}^{\alpha\beta} \varepsilon_{\alpha\beta} + A_{uw}^{\alpha\beta} \gamma_{\alpha\beta} + A_w w \end{aligned}\right\} \tag{1.22}$$

$$Q^\alpha = H^{\alpha\beta} \kappa_\beta. \tag{1.23}$$

The properties of the compliance tensor (D_{ijkl}) imply that each element of the matrix

$$\mathsf{D} := \begin{bmatrix} \boldsymbol{D}^N & \boldsymbol{D}^{NL} & \boldsymbol{D}^{RN} \\ \boldsymbol{D}^{NL} & \boldsymbol{D}^L & \boldsymbol{D}^{RL} \\ \boldsymbol{D}^{RN} & \boldsymbol{D}^{RL} & D^R \end{bmatrix} = \boldsymbol{D}^T \tag{1.24}$$

belongs to $L^\infty(\Omega)$ and $D^Q_{\alpha\beta} \in L^\infty(\Omega)$. Lewiński and Telega (1994A) proved that the assumption (H) implies existence of a constant C such that

$$(H_1) \quad \left| \begin{array}{l} \boldsymbol{K}\boldsymbol{D}(x)\boldsymbol{K}^T \geq C(A^{\alpha\beta}A^{\alpha\beta} + B^{\alpha\beta}B^{\alpha\beta} + a^2) \ , \\ D^Q_{\alpha\beta}(x) a^\alpha a^\beta \geq C a^\alpha a^\alpha \ , \end{array} \right.$$

for a.e. $x \in \Omega$ and for all $\boldsymbol{K} = (\boldsymbol{A}, \boldsymbol{B}, a) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{E}^2$. Here $\mathbb{E}^2_s$ is the space of symmetric 2×2 matrices

Now we may write

$$W_c(x, \boldsymbol{N}, \boldsymbol{L}, \boldsymbol{Q}, R) = \frac{1}{2}(\mathsf{N}\ \mathsf{D}(x)\mathsf{N}^T + D^Q_{\alpha\beta} Q^\alpha Q^\beta), \tag{1.25}$$

where $\mathsf{N} = (\boldsymbol{N}, \boldsymbol{L}, R)$ and $\mathsf{N}^T = \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{L} \\ R \end{bmatrix}$.

We also set

$$j_c(x, \boldsymbol{N}, \boldsymbol{L}, \boldsymbol{Q}, R) = \frac{1}{2}\mathsf{N}\ \mathsf{D}(x)\mathsf{N}^T, \tag{1.26}$$

and note that $p^i \in L^2(S_\sigma)$ implies

$$\bar{N}^\alpha \in L^2(\Gamma_\sigma)\ , \quad \bar{L}^\alpha \in L^2(\Gamma_\sigma)\ , \quad \bar{Q} \in L^2(\Gamma_\sigma)\ . \tag{1.27}$$

Due to the condition (H_1) and (1.27) from the already mentioned version of Brezzi's theorem (Arnold, Falk, 1987) we conclude that the functional J, given by (1.13), possesses a unique saddle point, say $(\boldsymbol{v}, \boldsymbol{u}, w; \boldsymbol{N}, \boldsymbol{L}, R, \boldsymbol{Q})$, solving the two-dimensional equilibrium problem of the three-layer laminate in terms of the generalized fields.

For our subsequent developments we set

$$\begin{aligned} \mathsf{A} &= \mathbf{D}^{-1} = \begin{bmatrix} \boldsymbol{A}_v & \boldsymbol{A}_{vu} & \boldsymbol{A}_{vw} \\ \boldsymbol{A}_{vu} & \boldsymbol{A}_u & \boldsymbol{A}_{uw} \\ \boldsymbol{A}_{vw} & \boldsymbol{A}_{uw} & A_w \end{bmatrix} = \mathsf{A}^T\ , \\ \boldsymbol{H} &= [H^{\alpha\beta}] = (\boldsymbol{D}^Q)^{-1} \end{aligned} \tag{1.28}$$

The explicit form of the generalized stiffness matrix A can be found by using the Fenchel conjugate of $j_c(x, .\,, .)$, i.e.:

$$\begin{aligned} j_1(x, \boldsymbol{\xi}, \boldsymbol{\eta}, r) &:= j_c^*(x, \boldsymbol{\xi}, \boldsymbol{\eta}, r) = \sup\{N^{\alpha\beta}\xi_{\alpha\beta} + L^{\alpha\beta}\eta_{\alpha\beta} + Rr - \\ &- j_c(x, \boldsymbol{N}, \boldsymbol{L}, R) | (\boldsymbol{N}, \boldsymbol{L}, R) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}\} = \frac{1}{2}\boldsymbol{E}\mathsf{A}(x)\boldsymbol{E}^T, \end{aligned} \tag{1.29}$$

where $\boldsymbol{E} = (\boldsymbol{\xi}, \boldsymbol{\eta}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}$. The conditions satisfied by the generalized compliance matrices $\mathbf{D}$ and $\boldsymbol{D}^Q$ imply that a positive constant C_1 exists such that for a.e. $x \in \Omega$

$$\begin{aligned} &\boldsymbol{E}\boldsymbol{A}(x)\boldsymbol{E}^T \geq C_1(|\boldsymbol{\xi}|^2 + |\boldsymbol{\eta}|^2 + r^2)\ , \\ &H^{\alpha\beta}(x)a_\alpha a_\beta \geq C_1|\boldsymbol{a}|^2 \end{aligned} \tag{1.30}$$

for all $\boldsymbol{E} = (\boldsymbol{\xi}, \boldsymbol{\eta}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^2$.

2. Elements of the theory of epi–convergence

A detailed presentation of the theory of epi–convergence is provided by Attouch (1984) and Dal Maso (1993). In this section only essential notions will be adduced.

DEFINITION 2.1 *Let (X, τ) be a metrisable topological space and $\{G_\varepsilon\}_{\varepsilon>0}$ a sequence of functionals from X into $\bar{\mathbb{R}}$ – the extended reals.*

a. The τ–epi–limit interior, denoted also by G^i, is the functional on X defined by

$$G^i(u) = \tau - li_e G_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon)\ .$$

b. *The τ-epi-limit superior, denoted also by G^s, is the functional on X defined by*

$$G^s(u) = \tau - ls_e G_\varepsilon(u) = \min_{\{u_\varepsilon \xrightarrow{\tau} u\}} \limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) \ .$$

c. *The sequence $\{G_\varepsilon\}_{\varepsilon>0}$ is said to be τ-epi-convergent if $G^i = G^s$; we then write*

$$G = \tau - \lim_e G_\varepsilon \ .$$

PROPERTIES

Let $G_\varepsilon : (X, \tau) \to \bar{\mathbb{R}}$ be a sequence of τ-epi-convergent functionals and let $G = \tau - \lim_e G_\varepsilon$.

The following properties hold:

i. The functionals G^i and G^s are τ-lower semicontinuous ($\tau - l.s.c.$).
ii. If the functionals G_ε are convex, then $G^s = \tau - ls_e G_\varepsilon$ is also a convex functional. Hence the epi-limit $G = \tau - \lim_e G_\varepsilon$ is a τ-closed ($\tau - l.s.c.$) convex functional.
iii. If $\Phi : X \to \mathbb{R}$ is a τ -continuous functional called a perturbation functional, then

$$\tau - \lim_e (G_\varepsilon + \Phi) = \tau - \lim_e G_\varepsilon + \Phi = G + \Phi \ .$$

iv.

$$G(u) = \tau - \lim_e G_\varepsilon(u) \Leftrightarrow \begin{cases} \forall u_\varepsilon \xrightarrow{\tau} u, \ G(u) \le \liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon), \ u \in X; \\ \forall u \in X \ \exists \ u_\varepsilon \xrightarrow{\tau} u \ \text{ such that} \\ G(u) \ge \liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon). \end{cases}$$

Further characterization provides

THEOREM 2.1 *Let $G = \tau - \lim_e G_\varepsilon$ and suppose that there exists a τ-relatively compact subset $X_0 \subset X$ such that $\inf_{X_0} G_\varepsilon = \inf_X G_\varepsilon (\forall \varepsilon > 0)$. Then $\inf_X G = \lim_{\varepsilon \to 0} (\inf_X G_\varepsilon)$. Moreover, if $\{u_\varepsilon\}_{\varepsilon>0}$ is such that $G_\varepsilon - \inf_X G_\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$, then every τ - cluster point of the sequence $\{u_\varepsilon : \varepsilon \to 0\}$ minimizes G on X.*

REMARK 2.1 Of practical importance is the following sufficient condition of existence of a compact set X.

If $(X, ||\cdot||)$ is a Banach space with τ-relatively compact balls, then a sufficient condition of existence of a compact set X_0 is that the sequence $\{G_\varepsilon\}_{\varepsilon>0}$ satisfies the condition of equi-coercivity

$$\limsup_{\varepsilon} G_\varepsilon(u_\varepsilon) < +\infty \Rightarrow \limsup_{\varepsilon} ||u_\varepsilon|| < +\infty \ , \tag{2.1}$$

3. Transverse cracks in the internal layer

The three-layer laminate modelled and studied in Section 1 is an undamaged one. From now on it is assumed that the internal layer incurs transverse cracks which form a fixed layout. More detailed description of the intralaminar cracking provides our paper (Lewiński, Telega, 1994A). Here we introduce only essential notions required for the epi–convergence study.

The internal layer is weakened by fissures $\varepsilon F \subset \Omega(\varepsilon > 0)$ distributed εY –periodically and of constant depth $2c$. The basic cell Y is two–dimensional and εY is homothetic to Y. We assume that F is of class C^1 and $F = \bar{F} \subset Y$, where $\bar{F}$ denotes the closure of F. When discussing the properties of the homogenized (effective) elastic potential the last hypothesis will be relaxed. The following notation is introduced for the sum of fissures such that the corresponding εY–cells are contained in Ω

$$F^\varepsilon = \bigcup_{i \in I(\varepsilon)} F_{\varepsilon,i} \quad , \quad \Omega^\varepsilon = \Omega \setminus F^\varepsilon \tag{3.1}$$

The Signorini–type conditions model the closing and opening of fissures and are given by

$$L_n^\varepsilon = \overset{1}{L}{}_n^\varepsilon = \overset{2}{L}{}_n^\varepsilon \leq 0 \,, \quad [\![u_n^\varepsilon]\!] \geq 0 \,, \; L_n^\varepsilon [\![u_n^\varepsilon]\!] = 0 \,, \; \overset{1}{L}{}_\tau^\varepsilon = \overset{2}{L}{}_\tau^\varepsilon = 0 \tag{3.2}$$

As usual, $[\![\,.\,]\!]$ denotes jump on F^ε and

$$\overset{\sigma}{L}{}_n^\varepsilon = L_\varepsilon^{\alpha\beta}{}_{|\sigma} n_\alpha n_\beta \;, \; \overset{\sigma}{L}{}_\tau^\varepsilon = L_\varepsilon^{\alpha\beta}{}_{|\sigma} n_\alpha \tau_\beta \;; \; \sigma = 1, 2, \tag{3.3}$$

are values of $L_\varepsilon^{\alpha\beta} n_\alpha n_\beta$, $L_\varepsilon^{\alpha\beta} n_\alpha \tau_\beta$ on the σ –th side of $F_{\varepsilon,i}$. Here $\boldsymbol{n} = (n_\alpha)$ is a unit vector normal to $F_{\varepsilon,i}$ and directed from the side 1 to the side 2. Moreover, $\boldsymbol{\tau} = (\tau_\alpha)$ stands for the unit tangent vector to $F_{\varepsilon,i}$. Kinematical fields $\boldsymbol{v}$ and w are assumed not to suffer jumps on F^ε, the jump $[\![u_\tau]\!]$ being unconstrained. The meaning of $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{L}_\varepsilon$ will become evident from our subsequent developments.

For later use we introduce further notations

$$\mathbb{K}_\varepsilon := \mathbb{K}(\Omega)^\varepsilon = H_{\Gamma_w}(\Omega)^2 \times K(\Omega^\varepsilon) \times H_{\Gamma_w}(\Omega) \,, \tag{3.4}$$

$$V(\Omega^\varepsilon) := H_{\Gamma_w}(\Omega)^2 \times H_{\Gamma_w}(\Omega^\varepsilon)^2 \times H_{\Gamma_w}(\Omega) \,, \tag{3.5}$$

where

$$H_{\Gamma_w}(\Omega) = \{x \in H^1(\Omega) |\, v = 0 \quad \text{on} \quad \Gamma_w\} \,, \tag{3.6}$$

$$K(\Omega^\varepsilon) = \{\boldsymbol{u} \in H^1(\Omega^\varepsilon)^2 |\; \boldsymbol{u} = \boldsymbol{O} \;\text{ on }\; \Gamma_w \;\text{ and }\; [\![u_n]\!] \geq 0 \text{ on } F^\varepsilon\} \,. \tag{3.7}$$

4. Moderately thick laminate weakened by transverse cracks of high density

Applicability of the model based on an in–plane scaling is discussed in our paper (Lewiński, Telega, 1994A). The set of kinematically admissible fields is specified by (3.4) and for a fixed $\varepsilon > 0$ the equilibrium problem is formulated as

PROBLEM $(P^1_{\Omega^\varepsilon})$

$$J_1^\varepsilon(\boldsymbol{v}^\varepsilon,\boldsymbol{u}^\varepsilon,w^\varepsilon)=\inf\{J_1^\varepsilon(\boldsymbol{v},\boldsymbol{u},w)|(\boldsymbol{v},\boldsymbol{u},w)\in\mathbb{K}_\varepsilon\}\,,$$

where the functional of the total potential energy J_1^ε is given by

$$\begin{aligned}J_1^\varepsilon(\boldsymbol{v},\boldsymbol{u},w) &= \int\limits_{\Omega^\varepsilon}[j_1(\boldsymbol{x},\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\gamma}(\boldsymbol{u}),w)+\\ &+\frac{1}{2}H^{\alpha\beta}(\boldsymbol{x})\kappa_\alpha(\boldsymbol{u},w)\kappa_\beta(\boldsymbol{u},w)]d\boldsymbol{x}-f(\boldsymbol{v},\boldsymbol{u},w)\,,\end{aligned} \tag{4.1}$$

where

$$f(\boldsymbol{v},\boldsymbol{u},w)=\int\limits_{\Gamma_\sigma}(\bar{N}^\alpha v_\alpha+\bar{L}^\alpha u_\alpha-\bar{Q}w)ds\,. \tag{4.2}$$

The functional J_1^ε is coercive on the space $V(\Omega^\varepsilon)$ provided that $F=\bar{F}\subset Y$. It is also coercive on the closed and convex set $\mathbb{K}_\varepsilon$. To prove that the functional J_1^ε is coercive, the Korn's inequality for the highly irregular domain Ω^ε is needed. The relevant inequality was derived in our paper (Telega, Lewiński, 1988), cf. also Telega (1990B). Obviously, $(\boldsymbol{v}^\varepsilon,\boldsymbol{u}^\varepsilon,w^\varepsilon)\in\mathbb{K}_\varepsilon$ solving problem $(P^1_{\Omega^\varepsilon})$ is unique.

The case when F intersects the boundary ∂Y of Y is more complicated, cf. Fig. 4 in Lewiński and Telega (1994A). We still assume that $Y\setminus F$ is a connected set. By combining the results concerning extension theorems, presented in Chapter I of the book by Oleinik et al. (1990), with our approach (Telega, Lewiński, 1988) one can demonstrate that Korn's inequality still holds true. Consequently the existence and uniqueness result remains valid also in this case.

The fundamental result of this section concerns the epi–convergence of the sequence $\{J_1^\varepsilon\}_{\varepsilon>0}$.

THEOREM 4.1 *The sequence of functionals $\{J_1^\varepsilon\}_{\varepsilon>0}$ is epi–convergent in the topology $\tau=(w-H^1(\Omega)^2)\times(s-L^2(\Omega))\times(w-H^1(\Omega))$ to*

$$J_1^h(\boldsymbol{v},\boldsymbol{u},w)=\int\limits_\Omega U_h[\boldsymbol{x},\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\gamma}(\boldsymbol{u}),\boldsymbol{\kappa}(\boldsymbol{u},w),w]d\boldsymbol{x}-f(\boldsymbol{v},\boldsymbol{u},w)\,, \tag{4.3}$$

The functional J_1^h is coercive on the space $H_{\Gamma_w}(\Omega)^2\times H_{\Gamma_w}(\Omega)^2\times H_{\Gamma_w}(\Omega)$ and $U_h=U_1+U_2$, where

$$U_1(\boldsymbol{x},\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h)= \tag{4.4}$$
$$=\inf\left\{\frac{1}{|Y|}\int\limits_{YF}j_1[\boldsymbol{x},\boldsymbol{\varepsilon}^h+\boldsymbol{\varepsilon}^y(\boldsymbol{v}),\boldsymbol{\gamma}^h+\boldsymbol{\gamma}^y(\boldsymbol{u}),w^h]dy|(\boldsymbol{v},\boldsymbol{u})\in H^1_{per}(Y)\times K_{YF}\right\}$$

$$U_2(\boldsymbol{x},\boldsymbol{\kappa}^h) = \frac{1}{2}H^{\alpha\beta}(\boldsymbol{x})\kappa^h_\alpha\kappa^h_\beta\,. \tag{4.5}$$

Here $\varepsilon^h, \gamma^h, \kappa^h \in \mathbb{E}^2_s,\ w^h \in \mathbb{R}$ *; moreover*

$$\varepsilon^y_{\alpha\beta}(\boldsymbol{v}) = \left(\frac{\partial v_\alpha}{\partial y_\beta} + \frac{\partial v_\beta}{\partial y_\alpha}\right)/2, \tag{4.6}$$

and similarly for $\gamma^y(\boldsymbol{u})$. *Furthermore we set*

$$H^1_{per}(Y) = \{v \in H^1(Y) | v \quad \text{assumes equal values at opposite sides of } Y\}\ , \tag{4.7}$$

$$K_{YF} = \{\boldsymbol{u} \in H^1_{per}(\dot{Y}F)^2 \mid [\![u_\nu]\!] \geq 0 \ \text{ on } \ F\}\ . \tag{4.8}$$

where $\boldsymbol{\nu}$ *stands for the unit vector normal to* F *and directed from the side 1 to 2 of* F.

The proof follows that of Th. 4.1 given in Telega (1993) and is omitted here. We only observe that the functional

$$\phi(\boldsymbol{v}, \boldsymbol{u}, w) = \frac{1}{2}\int\limits_\Omega H^{\alpha\beta}(x)\kappa_\alpha(\boldsymbol{u}, w)\kappa_\beta(\boldsymbol{u}, w)dx - f(\boldsymbol{v}, \boldsymbol{u}, w)\ , \tag{4.9}$$

is continuous in the topology τ and plays thus the role of a perturbation functional. Next, we set

$$G^\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) = \begin{cases} \displaystyle\int\limits_{\Omega^\varepsilon} j_1[x, \varepsilon(\boldsymbol{v}(x)), \gamma(\boldsymbol{u}(x)), w(x)]dx, \\ \qquad\qquad \text{if } \boldsymbol{v} \in H^1(\Omega)^2, w \in H^1(\Omega) \text{ and } \boldsymbol{u} \in K^\varepsilon\,; \\ +\infty\ , \qquad \text{otherwise} \end{cases} \tag{4.10}$$

where

$$K^\varepsilon = \{\boldsymbol{u} \in H^1(\Omega^\varepsilon)^2 \mid [\![u_n]\!] \geq 0 \ \text{ on } \ F^\varepsilon\}\ . \tag{4.11}$$

The demonstration reduces to proving that the sequence of functionals $\{G^\varepsilon(v, \cdot, w)\}_{\varepsilon>0}$ epi–converges in the strong topology of $L^2(\Omega)^2$ to

$$G(\boldsymbol{v}, \boldsymbol{u}, w) = \begin{cases} \displaystyle\int\limits_\Omega U_1[x, \varepsilon(\boldsymbol{v}(x)), \gamma(\boldsymbol{u}(x)), w(x)], \\ \qquad\qquad \text{if } \boldsymbol{v}, \boldsymbol{u} \in H^1(\Omega)^2, w \in H^1(\Omega)\,; \\ +\infty\ , \qquad \text{otherwise.} \end{cases} \tag{4.12}$$

REMARK 4.1 The partial effective elastic potential U_2 coincides with the corresponding potential for the virgin material. To characterize the behaviour of the effective potential U_h it is thus sufficient to examine the partial potential U_1, cf. Lewiński, Telega (1994A).

PROPERTIES OF U_1

i. $U_1(x,\cdot,\cdot,\cdot)$ is a strictly convex function provided that F does not separate Y into two (say) disjoint subdomains. Otherwise it is a convex function.

ii. $U_1(x,\cdot,\cdot,\cdot)$ is of class C^1.

iii. There exists a constant $C^1 > 0$ such that

$$U_1(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h) \le C_1(|\boldsymbol{\varepsilon}^h|^2 + |\boldsymbol{\gamma}^h|^2 + |w^h|^2),$$

for a.e. $x \in \Omega$ and all $(\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$.

iv. There exists a constant $C_0 > 0$ such that

$$U_1(x,\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h) \ge C_0(|\boldsymbol{\varepsilon}^h|^2 + |\boldsymbol{\gamma}^h|^2 + |w^h|^2),$$

for a.e. $x \in \Omega$ and all $(\boldsymbol{\varepsilon}^h,\boldsymbol{\gamma}^h,w^h) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \times \mathbb{R}$, provided that F does not separate Y into disjoint parts.

5. Thin laminate weakened by transverse cracks of arbitrary density: refined scaling

5.1. Preliminaries

In this section we are studying a thin three-layer laminate with transverse cracks in the internal layer by imposing the following natural rescaling, cf. Lewiński, Telega (1994A):

$$c \longrightarrow \varepsilon c\ , \qquad d \longrightarrow \varepsilon d\ , \qquad h \longrightarrow \varepsilon h\ . \tag{5.1}$$

If ε tends to zero, the thickness of the laminate also diminishes to zero. To compensate for this degeneracy we scale the loading

$$\bar{N}^\alpha \longrightarrow \varepsilon \bar{N}^\alpha\ , \qquad \bar{L}^\alpha \longrightarrow \varepsilon \bar{L}^\alpha\ , \qquad \bar{Q} \longrightarrow \bar{Q}. \tag{5.2}$$

The length scales scaling (5.1) implies the following scaling of the stiffnesses involved in the constitutive relationships (1.22) and (1.23)

$$\begin{aligned}
&(A_v^{\alpha\beta\lambda\mu}\, ,\, A_{vu}^{\alpha\beta\lambda\mu}\, ,\, A_u^{\alpha\beta\lambda\mu}) \longrightarrow (\varepsilon A_v^{\alpha\beta\lambda\mu}\, ,\, \varepsilon A_{vu}^{\alpha\beta\lambda\mu}\, ,\, \varepsilon A_u^{\alpha\beta\lambda\mu})\ ,\\
&(A_{vw}^{\alpha\beta}\, ,\, A_{uw}^{\alpha\beta}) \longrightarrow \left(\frac{1}{\varepsilon}A_{vw}^{\alpha\beta}\, ,\, \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}\right)\ ,\\
&A_w \longrightarrow \frac{1}{\varepsilon^3}A_w\ , \quad H^{\alpha\beta} \longrightarrow \frac{1}{\varepsilon}H^{\alpha\beta}\ .
\end{aligned} \tag{5.3}$$

The constitutive relations become

$$\left.\begin{aligned}
N_\varepsilon^{\alpha\beta} &= \varepsilon A_v^{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}(\boldsymbol{v}^\varepsilon) + \varepsilon A_{vu}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\boldsymbol{u}^\varepsilon) + \frac{1}{\varepsilon}A_{vw}^{\alpha\beta}w^\varepsilon\\
L_\varepsilon^{\alpha\beta} &= \varepsilon A_{vu}^{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}(\boldsymbol{v}^\varepsilon) + \varepsilon A_u^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\boldsymbol{u}^\varepsilon) + \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}w^\varepsilon\\
R_\varepsilon &= \frac{1}{\varepsilon}A_{vw}^{\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{v}^\varepsilon) + \frac{1}{\varepsilon}A_{uw}^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{u}^\varepsilon) + \frac{1}{\varepsilon^3}A_w w^\varepsilon
\end{aligned}\right\} \tag{5.4}$$

$$Q_\varepsilon^\alpha = \frac{1}{\varepsilon} H^{\alpha\beta} \kappa_\rho(\boldsymbol{u}^\varepsilon, w^\varepsilon) . \tag{5.5}$$

We set

$$\mathsf{A}_p^\varepsilon = \begin{bmatrix} \varepsilon \boldsymbol{A}_v & \varepsilon \boldsymbol{A}_{vu} & \frac{1}{\varepsilon}\boldsymbol{A}_{vw} \\ \varepsilon \boldsymbol{A}_{vu} & \varepsilon \boldsymbol{A}_u & \frac{1}{\varepsilon}\boldsymbol{A}_{uw} \\ \frac{1}{\varepsilon}\boldsymbol{A}_{vw} & \frac{1}{\varepsilon}\boldsymbol{A}_{uw} & \frac{1}{\varepsilon^3} A_w \end{bmatrix} , \tag{5.6}$$

$$j_p^\varepsilon(x, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{a}, r) = \frac{1}{2} \dot{\boldsymbol{E}} \boldsymbol{A}_p^\varepsilon(x) \boldsymbol{E}^T + \frac{1}{2\varepsilon} H^{\alpha\beta}(x) a_\alpha a_\beta , \tag{5.7}$$

where $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{a}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}^2 \times \mathbb{R}$. For a fixed $\varepsilon > 0$ the functional of the total potential energy is given by

$$\begin{aligned} & J_p^\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) = \\ & = \int_{\Omega^\varepsilon} j_p^\varepsilon[x, \boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\gamma}(\boldsymbol{u}), \kappa(\boldsymbol{u}, w), w] dx - \int_{\Gamma_\sigma} (\varepsilon \bar{N}^\alpha v_\alpha + \varepsilon \bar{L}^\alpha u_\alpha - \bar{Q} w) ds , \end{aligned} \tag{5.8}$$

where $(\boldsymbol{v}, \boldsymbol{u}, w) \in \mathbb{K}_\varepsilon$. In fact, the functional J_p^ε is well defined for $(\boldsymbol{v}, \boldsymbol{u}, w) \in H^1(\Omega)^2 \times H^1(\Omega^\varepsilon)^2 \times H^1(\Omega)$. Since the functional J_p^ε is convex, the minimization problem

$$(P_p^\varepsilon) \,|\, J_p^\varepsilon(\tilde{\boldsymbol{v}}^\varepsilon, \tilde{\boldsymbol{u}}^\varepsilon, \tilde{w}^\varepsilon) = \inf \{ J_p^\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) \,|\, (\boldsymbol{v}, \boldsymbol{u}, w) \in \mathbb{K}_\varepsilon \}, \tag{5.9}$$

is equivalent to solving the following variational inequality, cf. Kikuchi, Oden (1988), Telega (1987)

$$\left| \begin{array}{l} \text{Find } (\tilde{\boldsymbol{v}}^\varepsilon, \tilde{\boldsymbol{u}}^\varepsilon, \tilde{w}^\varepsilon) \in \mathbb{K}_\varepsilon \quad \text{such that} \\ b^\varepsilon(\tilde{\boldsymbol{v}}^\varepsilon, \tilde{\boldsymbol{u}}^\varepsilon, \tilde{w}^\varepsilon ; \boldsymbol{v}, \boldsymbol{u} - \tilde{\boldsymbol{u}}^\varepsilon, w) \geq g^\varepsilon(\boldsymbol{v}, \boldsymbol{u} - \tilde{\boldsymbol{u}}^\varepsilon, w) \ \forall (\boldsymbol{v}, \boldsymbol{u}, w) \in \mathbb{K}_\varepsilon . \end{array} \right. \tag{5.10}$$

Here

$$\begin{aligned} b_p^\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w; \boldsymbol{s}, \boldsymbol{t}, z) \;=\; & \int_{\Omega^\varepsilon} \{ [\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\gamma}(\boldsymbol{u}), w] \mathsf{A}_p^\varepsilon [\boldsymbol{\varepsilon}(\boldsymbol{s}), \boldsymbol{\gamma}(\boldsymbol{t}), z]^T + \\ & + \frac{1}{\varepsilon} H^{\alpha\beta} \kappa_\alpha(\boldsymbol{u}, w) \kappa_\beta(\boldsymbol{t}, z) \} dx, \end{aligned}$$

$$(\boldsymbol{v}, \boldsymbol{u}, w), (\boldsymbol{s}, \boldsymbol{t}, z) \in H^1(\Omega)^2 \times H^1(\Omega^\varepsilon)^2 \times H^1(\Omega) ,$$

$$g^\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) = \int_{\Gamma_\sigma} (\varepsilon \bar{N}^\alpha v_\alpha + \varepsilon \bar{L}^\alpha u_\alpha - \bar{Q} w) ds .$$

Variational problem (5.10) transforms readily to

$$\left|\begin{array}{l}\text{Find } (\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}_\varepsilon \quad \text{such that} \\ a^\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon; \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{u}^\varepsilon, w^\varepsilon) \geq f^\varepsilon(\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{u}^\varepsilon, w) \ \ \forall(\boldsymbol{v}, \boldsymbol{u}, w) \in \mathbb{K}_\varepsilon, \end{array}\right. \tag{5.11}$$

where $a^\varepsilon = \frac{1}{\varepsilon} b^\varepsilon$, $f^\varepsilon = \frac{1}{\varepsilon} g^\varepsilon$. The last problem is equivalent to solving

Problem (P_ε)

$$\left|\begin{array}{l}\text{Find} \\ J_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon) = \inf\{J_\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) \,|\, (\boldsymbol{v}, \boldsymbol{u}, w) \in \mathbb{K}_\varepsilon\} \ , \end{array}\right. \tag{5.12}$$

where $J_\varepsilon = \frac{1}{\varepsilon} J_p^\varepsilon$ and

$$J_\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) = \int\limits_{\Omega^\varepsilon} j^\varepsilon[x, \boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\gamma}(\boldsymbol{u}), \boldsymbol{\kappa}(\boldsymbol{u}, w), w] dx - g^\varepsilon(\boldsymbol{v}, \boldsymbol{u}, w) \ , \tag{5.13}$$

$$j^\varepsilon(x, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{a}, r) = \frac{1}{2} \boldsymbol{E} \mathsf{A}^\varepsilon(x) \boldsymbol{E}^T + \frac{1}{2\varepsilon^2} H^{\alpha\beta}(x) a_\alpha a_\beta \ . \tag{5.14}$$

Here

$$\mathsf{A}^\varepsilon = \frac{1}{\varepsilon} \mathsf{A}_p^\varepsilon = \begin{bmatrix} \boldsymbol{A}_v & \boldsymbol{A}_{vu} & \frac{1}{\varepsilon^2} \boldsymbol{A}_{vw} \\ \boldsymbol{A}_{vu} & \boldsymbol{A}_u & \frac{1}{\varepsilon^2} \boldsymbol{A}_{uw} \\ \frac{1}{\varepsilon^2} \boldsymbol{A}_{vw} & \frac{1}{\varepsilon^2} \boldsymbol{A}_{uw} & \frac{1}{\varepsilon^4} \boldsymbol{A}_w \end{bmatrix} . \tag{5.15}$$

We observe that the functional J_p^ε models the physical problem for $\varepsilon > 0$. The mathematical situation is described by J_ε. As we already known, they are interrelated by $J_p^\varepsilon = \varepsilon J_\varepsilon$.

For the domain Ω^ε a trace theorem seems not to be available. To obtain some estimations required in the study of epi–convergence of the sequence $\{J_\varepsilon\}_{\varepsilon > 0}$ we assume

$$\bar{N}^\alpha \in L^2(\Gamma_\sigma) \ , \qquad \bar{L}^\alpha \in L^\infty(\Gamma_\sigma) \ , \qquad \bar{Q} \in L^2(\Gamma_\sigma) \ , \tag{5.16}$$

and preserve the properties satisfied by the matrices A and $\boldsymbol{H}$, cf. Section 1. Then we have

THEOREM 5.1 *Under the assumption (5.16) a solution* $(\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon) \in \mathbb{K}_\varepsilon$ *to problem* (P_ε) *exists and is unique.*

PROOF. The functional J_ε is strictly convex and the cone $K(\Omega^\varepsilon)$ is closed in $H^1(\Omega^\varepsilon)^2$. To demonstrate the coercivity of $J_\varepsilon (0 < \varepsilon < 1$ and fixed) we use two following lemmas.

LEMMA 5.1 (TELEGA (1990A)) *Let $BD(\Omega)$ be the space of functions of with bounded deformation, Telega (1990B). The injection $H^1(\Omega^\varepsilon)^2 \subset BD(\Omega)$ is continuous.*

LEMMA 5.2 (TELEGA (1990B), TELEGA, LEWIŃSKI (1988)) *Suppose that meas $\Gamma_w > 0$. For each $\boldsymbol{u} \in H_{\Gamma_w}(\Omega^\varepsilon)^2$ Korn's inequality holds true*

$$\|\boldsymbol{u}\|_{1,\Omega^\varepsilon} \leq (C\varepsilon + C_1)\|\gamma(\boldsymbol{u})\|_{0,\Omega^\varepsilon}.$$

5.2. The macroscopic elastic potential and its dual

We set

$$j(x, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{a}, r) = j_1(x, \boldsymbol{\xi}, \boldsymbol{\eta}, r) + \frac{1}{2}H^{\alpha\beta}(x)a_\alpha a_\beta \ , \tag{5.17}$$

where $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{a}, r) \in \mathbb{E}_s^2 \times \mathbb{E}_s^2 \times \mathbb{R}^2 \times \mathbb{R}$. The homogenized (effective) potential is given by, cf. Lewiński, Telega (1994A)

$$W_h(x, \boldsymbol{e}) = \inf\left\{\frac{1}{|Y|}\int\limits_{YF} j[x, \boldsymbol{e} + \boldsymbol{\varepsilon}^y(\boldsymbol{v}), \boldsymbol{\gamma}^y(\boldsymbol{u}), w]dy \,|\, (\boldsymbol{v}, \boldsymbol{u}, w) \in \mathbb{K}_{YF}\right\}, \tag{5.18}$$

where $\boldsymbol{e} \in \mathbb{E}_s^2$ and

$$\mathbb{K}_{YF} = H^1_{\text{per}}(Y)^2 \times K_{YF} \times H^1_{\text{per}}(Y) \tag{5.19}$$

PROPERTIES OF W_h.

a. $W_h(x, \cdot)$ is strictly convex and of class C_1.
b. There exist constants $C_1 > C_0 > 0$ such that for a.e. $x \in \Omega$

$$C_0|e|^2 \leq W_h(x, \boldsymbol{e}) \leq C_1|e|^2 \ , \qquad \text{for all } \boldsymbol{e} \in \mathbb{E}_s^2 \ . \tag{5.20}$$

These properties are preserved even when F separates the basic cell Y into disjoint parts.

The Fenchel conjugate of W_h permits to find the complementary elastic potential W_h^* :

$$W_h^*(x, \boldsymbol{N}_h) = \sup\{N_h^{\alpha\beta}e_{\alpha\beta} - W_h(x, \boldsymbol{e}) \,|\, \boldsymbol{e} \in \mathbb{E}_s^2\} \ ; \quad \boldsymbol{N}_h \in \mathbb{E}_s^2 \ . \tag{5.21}$$

Taking account of (5.18) and proceeding similarly as in Telega (1992), after lengthy calculations we finally get

$$W_h^*(x, \boldsymbol{N}_h) = \tag{5.22}$$
$$= \frac{1}{|Y|}\inf\left\{\int\limits_{Y\setminus F} W_c[x, \boldsymbol{N}_h + \mathfrak{n}(y), \mathfrak{l}(y), \boldsymbol{q}(y), \mathfrak{r}(y)]dy \,\middle|\, (\mathfrak{n}, \mathfrak{l}, \boldsymbol{q}, \mathfrak{r}) \in S_{\text{per}}\right\},$$

where

$$S_{\text{per}} = \Big\{ (\mathfrak{n}, \mathfrak{l}, \boldsymbol{q}, \mathfrak{r}) \in L^2(Y, \mathbb{E}^2_s) \times L^2(Y \setminus F, \mathbb{E}^2_s) \times L^2(Y, \mathbb{R}^2) \times$$
$$\times L^2(Y) \mid \mathrm{div}_y \mathfrak{n} = 0 \text{ in } Y;\ \mathrm{div}_y \mathfrak{l} - \boldsymbol{q} = 0\ ,\ \text{in } Y \setminus F\ ;$$
$$\mathrm{div}_y \boldsymbol{q} + \mathfrak{r} = 0\ ,\ \text{in } Y\ ; \mathfrak{n}\boldsymbol{\mu}\ ,\ \mathfrak{l}\boldsymbol{\mu} \text{ and } \boldsymbol{q}\cdot\boldsymbol{\mu}$$
$$\text{assume opposite values on opposite sides of } Y\ ;$$
$$\mathfrak{l}_\nu \le 0 \text{ and } \mathfrak{l}_T = 0 \text{ on } F\ ; \int_Y \mathfrak{n}(y)dy = 0 \Big\}\ . \tag{5.23}$$

Here $\boldsymbol{\mu}$ stands for the outward unit normal to ∂Y, whilst $\boldsymbol{T}$ denotes the unit tangent vector to F ; moreover

$$\mathfrak{n}\boldsymbol{\mu} = (\mathfrak{n}^{\alpha\beta}\mu_\beta)\ ,\ \mathfrak{l}\boldsymbol{\mu} = (\mathfrak{l}^{\alpha\beta}\mu_\beta)\ ,\ \boldsymbol{q}\cdot\boldsymbol{\mu} = q^\alpha \cdot \mu_\alpha\ ,$$

$$\mathfrak{l}_T = \mathfrak{n}^{\alpha\beta}\nu_\alpha T_\beta\ ,\ \mathfrak{l}_\nu = \mathfrak{l}^{\alpha\beta}|_1 \nu_\alpha\nu_\beta\ .$$

REMARK 5.1

i. The local equilibrium equation

$$\mathrm{div}_y \boldsymbol{q} + \mathfrak{r} = 0\ ,\ \text{in } Y\ , \tag{5.24}$$

yields

$$\int_Y \mathfrak{r}(y)dy = -\int_Y \mathrm{div}_y \boldsymbol{q}(y) = -\int_{\partial Y} q^\alpha \mu_\alpha ds = 0\ , \tag{5.25}$$

because $\boldsymbol{q}\cdot\boldsymbol{\mu}$ assumes opposite values on opposite sides of Y.

ii. By using the local equilibrium equation

$$\mathrm{div}_y \mathfrak{l} - \boldsymbol{q} = 0\ ,\ \text{in } Y \setminus F\ , \tag{5.26}$$

we write

$$\int_Y q_\alpha(y)dy = \int_{Y F} \frac{\partial \mathfrak{l}^{\alpha\beta}}{\partial y_\beta} dy = \\ = \int_{\partial Y} \mathfrak{l}^{\alpha\beta}\mu_\beta ds + \int_F (\mathfrak{l}_{\alpha|1} - \mathfrak{l}_{\alpha|2})dF = 0 \tag{5.27}$$

since $\mathfrak{l}\boldsymbol{\mu}$ assumes opposite values on opposite sides of Y and, according to the action and reaction principle, $\mathfrak{l}_{\alpha|1} = \mathfrak{l}^{\alpha\beta}_{|1}\nu_\beta = -\mathfrak{l}_{\alpha|2}$.

iii. Local equilibrium equations (5.24) and (5.26) imply

$$\mathrm{div}_y \mathrm{div}_y \mathfrak{l} = \mathrm{div}_y \boldsymbol{q} = -\mathfrak{r}, \tag{5.28}$$

and thus $\mathrm{div}_y\mathrm{div}_y \mathfrak{l} \in L^2(Y \setminus F)$. Consequently the following expression makes sense

$$\int_{Y\setminus F} [\mathrm{div}_y(\mathrm{div}_y \mathfrak{l} - \boldsymbol{q})] w\, dy = 0 \qquad \forall w \in H^1_{\text{per}}(Y).$$

Integrating by parts we infer that $(\mathrm{div}_y \mathbf{l}) \cdot \boldsymbol{\mu}$ assumes opposite values on opposite sides of Y.

For our subsequent developments we make the following assumption:

$$\text{the scalar local field } \mathbf{r} \text{ is } Y-\text{periodic.} \tag{5.29}$$

Hence, $\mathrm{div}_y \boldsymbol{q}$ and $\mathrm{div}_y \mathrm{div}_y \mathbf{l}$ are $Y-$ periodic.

iv. Performing the rescaling $y \to x/\varepsilon$, from the local equilibrium equations one readily gets

$$\frac{1}{\varepsilon}\mathrm{div}\mathbf{n}\left(\frac{\cdot}{\varepsilon}\right) = \dot{0}\ , \qquad \text{in} \quad \Omega\ , \tag{5.30}$$

$$\frac{1}{\varepsilon}\mathrm{div}\mathbf{l}\left(\frac{\cdot}{\varepsilon}\right) - \boldsymbol{q}\left(\frac{\cdot}{\varepsilon}\right) = 0\ , \qquad \text{in} \quad \Omega^{\varepsilon}\ , \tag{5.31}$$

$$\frac{1}{\varepsilon}\mathrm{div}\boldsymbol{q}\left(\frac{\cdot}{\varepsilon}\right) + \mathbf{r}\left(\frac{\cdot}{\varepsilon}\right) = 0\ , \qquad \text{in} \quad \Omega\ , \tag{5.32}$$

$$\frac{1}{\varepsilon^2}\mathrm{div}\,\mathrm{div}\mathbf{l}\left(\frac{\cdot}{\varepsilon}\right) = \frac{1}{\varepsilon}\mathrm{div}\boldsymbol{q}\left(\frac{\cdot}{\varepsilon}\right) = -\mathbf{r}\left(\frac{\cdot}{\varepsilon}\right)\ , \qquad \text{in} \quad \Omega^{\varepsilon}\ . \tag{5.33}$$

5.3. Convergence

The basic result of this Section is formulated as

THEOREM 5.2 *The sequence of functionas $\{J_\varepsilon\}_{\varepsilon>0}$ epi–converges to J_h , $J_h = \tau - \lim_e J_\varepsilon$, where $\tau = s - L^2(\Omega)^2 \times L^2(\Omega)^2 \times L^2(\Omega)$ and*

$$J_h(\boldsymbol{v}) = \int_\Omega W_h[x, \boldsymbol{\varepsilon}(\boldsymbol{v}(x))]dx - \int_{\Gamma_\sigma} \bar{N}^\alpha v_\alpha ds\ ,\ \boldsymbol{v} \in H_{\Gamma_w}(\Omega)^2\ . \tag{5.34}$$

PROOF. It is similar to the proof of Th. 5.1 in Telega (1992) and is omitted here. Apart from the auxiliary results formulated in previous two subsections one has to use some duality argument, as in Telega (1992), and the following lemmas.

LEMMA 5.3 (TELEGA, LEWIŃSKI, 1988) *For each $\varepsilon > 0$ there exists a linear and continuous operator*

$$\mathbb{Q}^\varepsilon : H^1(\Omega^\varepsilon)^2 \longrightarrow H^1(\Omega)^2$$

satisfying the conditions:

a. $\quad ||\mathbb{Q}^\varepsilon \boldsymbol{u}||_{0,\Omega} \le C||\boldsymbol{u}||_{0,\Omega}\ ,$

b. $\quad ||\gamma(\mathbb{Q}^\varepsilon \boldsymbol{u})||_{0,\Omega} \le C||\gamma(\boldsymbol{u})||_{0,\Omega^\varepsilon}\ ,$

c. $\quad ||\mathbb{Q}^\varepsilon \boldsymbol{u} - \boldsymbol{u}||_{0,\Omega} \le \varepsilon C\,||\gamma(\boldsymbol{u})||_{0,\Omega^\varepsilon}\ ,$

For an arbitrary sequence $\{\boldsymbol{u}^\varepsilon\}_{\varepsilon>0}$ such that

$$\sup_{\varepsilon>0} ||\boldsymbol{u}^\varepsilon||_{1,\Omega^\varepsilon} < \infty\ ,$$

the sequence $\{\mathbb{Q}^\varepsilon \boldsymbol{u}^\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1(\Omega)^2$ and

$$||\mathbb{Q}^\varepsilon \boldsymbol{u}^\varepsilon - \boldsymbol{u}^\varepsilon||_{0,\Omega} \longrightarrow 0 \ \text{ when } \ \varepsilon \longrightarrow 0\ .$$

LEMMA 5.4 *Let*

$$\{\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon\}_{\varepsilon>0} \subset H^1(\Omega)^2 \times H^1(\Omega^\varepsilon)^2 \times H^1(\Omega) ,$$

be such that

$$\sup_{\varepsilon>0}\{\|\boldsymbol{v}^\varepsilon\|_{1,\Omega} + \|\boldsymbol{u}^\varepsilon\|_{1,\Omega^\varepsilon} + \|w^\varepsilon\|_{1,\Omega}\} < \infty .$$

Suppose a constant $C \geq 0$ independent of ε exists and satisfying

$$J_\varepsilon(\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon) \leq C . \tag{5.35}$$

Denote by $(\boldsymbol{v}, \boldsymbol{u}, w)$ the limit of a convergent subsequence in $(w - H^1(\Omega)^2) \times (s - L^2(\Omega)^2) \times (w - H^1(\Omega))$. Then $\boldsymbol{u} = \nabla w = \boldsymbol{O}$ and $w = 0$.

REMARK 5.2 If $(\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon)$ solves problem $(\mathcal{P}_\varepsilon)$ then $C = 0$ because $J_\varepsilon(\boldsymbol{O}, \boldsymbol{O}, 0) = 0$.

LEMMA 5.5 *Let $(\mathfrak{n}, \mathfrak{l}, \boldsymbol{q}, r) \in S_{\text{per}}$, $\varphi \in \mathbf{D}(\Omega)$ and*

$$\{\boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon, w^\varepsilon\}_{\varepsilon>0} \subset H^1(\Omega)^2 \times H^1(\Omega^\varepsilon)^2 \times H^1(\Omega)$$

be a sequence strongly convergent to $(\boldsymbol{v}, \boldsymbol{u}, w)$ in $L^2(\Omega)^2 \times L^2(\Omega)^2 \times L^2(\Omega)$. Then we have

$$\begin{aligned} R &= \lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon} \varphi(x) \left[\mathfrak{n}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) \varepsilon_{\alpha\beta}(\boldsymbol{v}^\varepsilon) + \mathfrak{l}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) \gamma_{\alpha\beta}(\boldsymbol{u}^\varepsilon) + \right. \\ &\quad \left. + q^\alpha\left(\frac{x}{\varepsilon}\right) \kappa_\alpha(\boldsymbol{u}^\varepsilon, w^\varepsilon) + r\left(\frac{x}{\varepsilon}\right) w^\varepsilon\right] dx = \\ &= -\frac{1}{|Y|} \int_{Y\setminus F} \mathfrak{l}^{\alpha\beta}(y) dy \int_\Omega \varphi_{,\beta}(x) u_\alpha(x) dx. \end{aligned}$$

REMARK 5.3 Of interest in the study of epi-convergence is the case $\boldsymbol{u} = 0$, see Lemma 5.4; then $R = 0$.

6. The augmented Lagrangian method for solving local problems

The minimization problems occurring in (4.4) and (5.18) are unilaterally constrained ones. In this section we are sketching an application of the general results due to Ito and Kunisch (1990) to minimization problem (5.18). Similar procedure is valid for (4.4).

The minimization problem occurring in (5.18) can be written in the following way

$$(\mathcal{P}) \quad \inf_{g(\boldsymbol{u})\leq 0} \left\{\frac{1}{2} a_{Y^F}(\boldsymbol{U}, \boldsymbol{U}) - \mathbf{L}(\boldsymbol{U}) | \boldsymbol{U} \in \tilde{H}_{\text{per}}\right\} \tag{6.1}$$

where $\boldsymbol{U} = (\boldsymbol{v}, \boldsymbol{u}, w)$, $\tilde{H}_{\text{per}} \subset H_{\text{per}}$ is the subspace of periodic functions with zero mean value over $Y; H_{\text{per}} = H^1_{\text{per}}(Y)^2 \times H^1_{\text{per}}(YF)^2 \times H^1_{\text{per}}(Y)$, $a_{YF}(\cdot,\cdot)$ is the bilinear form which is coercive on $\tilde{H}_{\text{per}}$, L is the linear and continuous functional and $g(\boldsymbol{u}) = -[\![u_\nu]\!] \in L^2(F)$.

According to Ito, Kunisch (1990) there exists $(\boldsymbol{U}^*, \lambda^*) \in \tilde{H}_{\text{per}} \times H$, $H = L^2(F)$, such that $\boldsymbol{U}^*$ is the solution to $(\mathcal{P})$, $\lambda^* \geq 0$ and

i. $< \lambda^*, g(\boldsymbol{u}^*) >_{L^2 \times L^2} = 0$

ii. $a_{YF}(\boldsymbol{U}^*, \boldsymbol{U}) - \mathsf{L}(\boldsymbol{U}) - < \lambda^*, [\![u_\nu]\!] >_{L^2 \times L^2} = 0$

for all $\boldsymbol{U} \in \tilde{H}_{\text{per}}$. If $\mathfrak{n}_\nu$ stands for normal contact stresses on F, then $\lambda^* = -j^{-1}(\mathfrak{n}_\nu)$, $\mathfrak{n}_\nu \in H^{-1/2}(F)$, j = Riesz map, $\mathsf{j} : H^{1/2}(F) \longrightarrow H^{-1/2}(F)$.

Now we define a family of augmented Lagrangian problems by

$$(\mathcal{P})_{m,\lambda} \quad \min\{L_m(\boldsymbol{U}, \lambda) \,|\, \boldsymbol{U} \in \tilde{H}_{\text{per}}\} \ ,$$

where

$$L_m(\boldsymbol{U}, \lambda) = \frac{1}{2} a_{YF}(\boldsymbol{U}, \boldsymbol{U}) - \mathsf{L}(\boldsymbol{U}) + < \lambda, \hat{g}(\boldsymbol{u}, \lambda, m) > + \frac{m}{2} \|\hat{g}(\boldsymbol{u}, \lambda, m)\|^2_H$$

and

$$\hat{g}(\boldsymbol{u}, \lambda, m) = \sup\left(g(\boldsymbol{u}), \frac{\lambda}{m}\right) , \ m > 0 \ , \quad \lambda \in H \ .$$

The augmented Lagrangian algorithm

(1) Choose $\lambda_1 \in H$, $\lambda_1 \geq 0$ and $m > 0$.
(2) Put $n = 1$.
(3) Solve $(\mathcal{P})_{m,\lambda_n}$ for $\boldsymbol{U}^n$.
(4) Put $\lambda_{n+1} = \lambda_n + m\hat{g}(\boldsymbol{u}^n, \lambda_n, m) = \sup(0, \lambda_n + mg(\boldsymbol{u}^n))$.
(5) Take $n = n + 1$ and return to step (3).

This algorithm consists of a sequence of unconstraint minimization problems $(\mathcal{P})_{m,\lambda_n}$ whose solutions converge to the solution of $(\mathcal{P})$, provided that they exist. Moreover we have

$$C\|\boldsymbol{U}^* - \boldsymbol{U}^n\|^2_{\tilde{H}_{\text{per}}} + \frac{1}{2m}\|\lambda_{n+1} - \lambda^*\|^2_H \leq \frac{1}{2m}\|\lambda_n - \lambda^*\|^2_H \ .$$

The positive constant C follows from the obvious inequality

$$a_{YF}(\boldsymbol{U}, \boldsymbol{U}) \geq C\|\boldsymbol{U}\|^2_{\tilde{H}_{\text{per}}} \qquad \text{for all} \quad \boldsymbol{U} \in \tilde{H}_{\text{per}} \ .$$

In the case of the augmented Lagrangian algorithm with variable stepsize, step (4) is to be replaced by

(4') Put $\lambda_{n+1} = \lambda_n + \delta m \hat{g}(\boldsymbol{u}^n, \lambda_n, m)$, $\delta \in (0, 1)$.

Then we have

$$C\|\boldsymbol{U}^* - \boldsymbol{U}^n\|^2_{\tilde{H}_{\text{per}}} + \frac{m}{2}(1 - \delta)\|\hat{g}(\boldsymbol{u}^n, \lambda_n, m)\|^2_H +$$

$$\frac{1}{2m\delta}\|\lambda_{n+1} - \lambda^*\|^2_H \leq \frac{1}{2m\delta}\|\lambda_n - \lambda^*\|^2_H \ .$$

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