Control and Cybernetics

vol. **23** (1994) No. 4

Approximate controllability for linear parabolic equations with rapidly oscillating coefficients

by

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We consider linear parabolic equations with rapidly oscillating coefficients in a bounded domain Ω of $\mathbb{R}^n$ with Dirichlet type homogeneous boundary conditions. Under some natural assumptions on the coefficients we prove that, for any fixed positive time, the system may be approximately controlled in an uniform way with respect to the oscillation parameter with controls supported in any open subset of Ω. More precisely, we prove that the controls remain bounded when the oscillation parameter tends to zero and that they converge strongly in L^2 to a control for the homogenized parabolic system.

1. Introduction and main results

Let Ω be a bounded domain of $\mathbb{R}^n (n \geq 1)$ with boundary $\Gamma = \partial\Omega$ of class C^2. Let ω be an open and nonempty subset of Ω and $T > 0$. We denote by χ_ω the characteristic function of ω. For $T > 0$ given we consider the following parabolic equation with rapidly oscillating coefficients and with a control supported in ω:

$$\begin{cases} u_t - \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u\right) = f\chi_\omega & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = u^0(x) & \text{in } \Omega \end{cases} \tag{1}$$

where $\varepsilon > 0$ and $a \in C^1(\mathbb{R}^n)$ is such that

$$\begin{cases} 0 < a_0 \leq a(x) \leq a_1 \quad \text{a. e. in } \mathbb{R}^n \\ a \text{ is periodic of period 1 in each variable } x_i, i = 1, ..., n. \end{cases} \tag{2}$$

The initial data u^0 is supposed to be in $L^2(\Omega)$ and the control f in $L^2(\Omega \times (0,T))$.

Let us recall that the homogenized heat equation is given by (see, for instance, Bensoussan, Lions and Papanicolaou, 1978 and Brahim-Otsmane, Frankfort and Murat, 1992):

$$\begin{cases} u_t - \operatorname{div}(A\nabla u) = f\chi_\omega & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = u^0(x), & \text{in } \Omega \end{cases} \tag{3}$$

where A is the homogenized constant matrix. More precisely, we have

THEOREM A (Brahim-Otsmane, Frankfort, Murrat, 1992) *Let us consider in* (1) *a sequence of initial data* $u_\varepsilon^0 \in L^2(\Omega)$ *and a sequence of right hand sides* $f_\varepsilon \in L^2(\omega \times (0,T))$. *Then,*

i) If u_ε^0 *(resp.* f_ε*) weakly converges in* $L^2(\Omega)$ *(resp.* $L^2(\omega \times (0,T))$*) to* u^0 *(resp.* f*) as* $\varepsilon \to 0$*, the solutions* u_ε *of* (1) *satisfy*

$u_\varepsilon \to u$ *weakly-* in* $L^\infty(0,T;L^2(\Omega))$

as $\varepsilon \to 0$*, where* u *is the solution of the limit system* (3).

ii) If u_ε^0 *(resp.* f_ε*) strongly converges in* $L^2(\Omega)$ *(resp.* $L^2(\omega \times (0,T))$*) to* u^0 *(resp.* f*) as* $\varepsilon \to 0$*, the solutions* u_ε *of* (1) *satisfy*

$u_\varepsilon \to u$ *strongly* $C([0,T];L^2(\Omega))$

as $\varepsilon \to 0$*, where* u *is the solution of the limit system* (3).

We consider the following approximate controllability problem for system (1): Given u^0 and u^1 in $L^2(\Omega)$ and $\alpha > 0$, to find a control $f_\varepsilon \in L^2(\Omega \times (0,T))$ such that the solution $u_\varepsilon = u_\varepsilon(x,t)$ of (1) satisfies

$$\|u_\varepsilon(T) - u^1\|_{L^2(\Omega)} \le \alpha. \tag{4}$$

We also study the uniform boundedness of the control f_ε in $L^2(\Omega \times (0,T))$ and the solution u_ε in $L^\infty((0,T);L^2(\Omega))$ and its eventual convergence to a control and a solution of the limiting homogenized heat equation (3) as $\varepsilon \to 0$.

For ε fixed, the approximate controllability of system (1) is a direct consequence of the unique continuation for solutions of the homogeneous equation:

$$\begin{cases} -\varphi_t - \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla\varphi\right) = 0 & \text{in } \Omega \times (0,T) \\ \varphi = 0 & \text{on } \Gamma \times (0,T) \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega \end{cases} \tag{5}$$

More precisely, for ε fixed, since $\varphi \equiv 0$ in $\omega \times (0,T)$ implies $\varphi^0 \equiv 0$ in Ω (see, for instance, Mizohata, 1958 and Saut, Scheurer, 1987), by Hahn–Banach Theorem, the approximate controllability of system (1) holds. However, this proof of the approximate controllability does not provide any information on the dependence of the control on the initial and final data and on the parameter ε.

In this paper we adapt the techniques developped by C. Fabre, J. P. Puel and the author in Fabre, Puel, Zuazua (1992A,B, 1993A,B) to obtain uniform bounds on the controls and solutions as $\varepsilon \to 0$ and to prove its convergence to a control and a controlled solution for the homogenized limit problem. Let us recall that when $u^0 \equiv 0$ the control $f_\varepsilon \equiv \overline{\varphi}_\varepsilon$ where $\overline{\varphi}_\varepsilon$ solves (5) with initial data $\overline{\varphi}_\varepsilon^0$, the minimizer of the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \alpha||\varphi^0||_{L^2(\Omega)} - \int_\Omega u^1\varphi^0 dx \tag{6}$$

over $L^2(\Omega)$, is such that (4) holds (see Fabre, Puel, Zuazua, 1992A–1993B).

The adjoint system associated to the homogenized system (3) is given by

$$\begin{cases} -\varphi_t - \operatorname{div}(A\nabla\varphi) = 0 & \text{in } \Omega \times (0,T) \\ \varphi = 0 & \text{on } \Gamma \times (0,T) \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega \end{cases} \tag{7}$$

Let us also introduce the functional J corresponding to (3) and (7):

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \alpha||\varphi^0||_{L^2(\Omega)} - \int_\Omega u^1\varphi^0 dx \tag{8}$$

where φ is the solution of (7) with final data φ^0.

For simplicity, we consider first the case where $u^0 \equiv 0$ and $||u^1||_{L^2(\Omega)} \geq \alpha$. Our main result is as follows:

THEOREM 1 *If $u^0 \equiv 0$ and $\alpha > 0$ the approximate controls f_ε obtained by minimizing J_ε over $L^2(\Omega)$ are uniformly bounded in $C([0,T];L^2(\Omega))$. Moreover, they strongly converge in $C([0,T];L^2(\Omega))$ as $\varepsilon \to 0$ to the control f associated to the minimizer of the limit functional J, which is an approximate control for the homogenized system* (3).
On the other hand, the solutions u_ε of (1) *converge strongly in $C([0,T];L^2(\Omega))$ as $\varepsilon \to 0$ to the solution u of the limit problem* (3).

Let us consider now the case where u^0 is non–zero. We set $v_\varepsilon^1 = u_\varepsilon(T)$ where u_ε is the solution of (1) with $f \equiv 0$. It is easy to check that v_ε^1 is uniformly bounded in $H_0^1(\Omega)$. By the classical homogenization theory we know that v_ε^1 weakly converges in $H_0^1(\Omega)$ to $v^1 = u(T)$ where u is the solution of (3) with $f \equiv 0$. Given $u^1 \in L^2(\Omega)$ we set $u_\varepsilon^1 = u^1 - v_\varepsilon^1$. In this way, the problem is reduced to the case where $u^0 \equiv 0$ but, instead of having a fixed target u^1 we have a sequence of targets u_ε^1 that converge weakly in $H_0^1(\Omega)$. In this case, in the definition of the functional J_ε we have to replace u^1 by u_ε^1.

We have the following result:

THEOREM 2 *Suppose that $u^0 \equiv 0$, $\alpha > 0$ and consider a sequence of final data u_ε^1 in $L^2(\Omega)$ such that, as $\varepsilon \to 0$, they strongly converge in $L^2(\Omega)$ to $u^1 \in L^2(\Omega)$. Then, the conclusions of Theorem 1 hold.*

Theorem 1 is a particular case of Theorem 2. Thus we will focus in the proof of Theorem 2.

The main ingredients of the proof of these results are the techniques developped in Fabre, Puel, Zuazua (1992A–1993B) that allow us to prove the uniform coercivity of the functionals J_ε and the uniform boundedness of the minimizers and the classical homogenization theory for parabolic equations.

ACKNOWLEDGEMENTS. This work was partly done while the author was visiting the Laboratoire d'Analyse Numérique of the Université Pierre et Marie Curie. The author has been supported by the Project PB90-0245 of the DGICYT (Spain) and by the Grant·SC1*-CT91-0732 of the CEE.

2. Proof of the main result

Let us recall that, in the setting of Theorem 2, the functional J_ε is given by:

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega |\varphi|^2 dxdt + \alpha\|\varphi^0\|_{L^2(\Omega)} - \int_\Omega u_\varepsilon^1 \varphi^0 dx. \tag{9}$$

We set

$$M_\varepsilon = \inf_{\varphi^0 \in L^2(\Omega)} J_\varepsilon(\varphi^0) \tag{10}$$

The following Lemma is a consequence of the results of Fabre, Puel, Zuazua (1992A–1993B) :

LEMMA 1 *For every $\varepsilon > 0$ we have*

$$\liminf_{\|\varphi^0\|_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{L^2(\Omega)}} \geq \alpha. \tag{11}$$

The functional J_ε achieves its minimum M_ε in $L^2(\Omega)$ at a unique $\overline{\varphi}_\varepsilon^0$ and $\overline{\varphi}_\varepsilon^0 \equiv 0$ if and only if $\|u_\varepsilon^1\|_{L^2(\Omega)} \leq \alpha$.

Moreover, if $f = \overline{\varphi}_\varepsilon$ where $\overline{\varphi}_\varepsilon$ solves (5) *with data $\overline{\varphi}_\varepsilon^0$, the solution of* (1) *satisfies* (4).

The following Lemma establishes the uniform boundedness of the minimizers:

LEMMA 2 *We have*

$$\liminf_{\substack{\|\varphi^0\|_{L^2(\Omega)}\to\infty \\ \varepsilon\to 0}} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{L^2(\Omega)}} \geq \alpha. \tag{12}$$

Furthermore, the minimizers $\{\overline{\varphi}_\varepsilon^0\}_{\varepsilon\geq 0}$ are uniformly bounded in $L^2(\Omega)$.

PROOF OF LEMMA 2. Let us consider sequences $\varepsilon_j \to 0$ and $\varphi^0_{\varepsilon_j} \in L^2(\Omega)$ such that $\|\varphi^0_{\varepsilon_j}\|_{L^2(\Omega)} \to \infty$ as $j \to \infty$.

Let us introduce the normalized data

$$\psi^0_{\varepsilon_j} = \frac{\varphi^0_{\varepsilon_j}}{\|\varphi^0_{\varepsilon_j}\|_{L^2(\Omega)}}$$

and the corresponding solutions of (5):

$$\psi_{\varepsilon_j} = \frac{\varphi_{\varepsilon_j}}{\|\varphi^0_{\varepsilon_j}\|_{L^2(\Omega)}}.$$

We have

$$I_j = \frac{J_{\varepsilon_j}(\varphi^0_{\varepsilon_j})}{\|\varphi^0_{\varepsilon_j}\|_{L^2(\Omega)}} = \frac{1}{2}\|\varphi^0_{\varepsilon_j}\|_{L^2(\Omega)} \int_0^T \int_\omega |\psi_{\varepsilon_j}|^2 dxdt + \alpha - \int_\Omega u^1_{\varepsilon_j} \psi^0_{\varepsilon_j} dx.$$

We distinguish the following two cases:

Case 1: $\liminf_{j\to\infty} \int_0^T \int_\omega |\psi_{\varepsilon_j}|^2 dxdt > 0$. In this case, we have clearly $\liminf_{j\to\infty} I_j = \infty$.

Case 2: $\liminf_{j\to\infty} \int_0^T \int_\omega |\psi_{\varepsilon_j}|^2 dxdt = 0$. In this case we argue by contradiction.

Suppose there exists a subsequence (still denoted by the index j) such that

$$\int_0^T \int_\omega |\psi_{\varepsilon_j}|^2 dxdt \to 0 \tag{13}$$

and

$$\lim_{j\to\infty} I_j < \alpha. \tag{14}$$

By extracting a subsequence (denoted by the index j) we have

$$\psi^0_{\varepsilon_j} \rightharpoonup \psi^0 \text{ weakly in } L^2(\Omega).$$

The homogenization theory (see Theorem A) guarantees that

$$\psi_{\varepsilon_j} \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;L^2(\Omega))$$

where ψ is the solution of (7) with initial data ψ^0. In view of (13) we have

$$\psi \equiv 0 \text{ in } \omega \times (0,T)$$

and by Hölmgren's Uniqueness Theorem this implies that $\psi^0 \equiv 0$. Thus

$$\psi^0_{\varepsilon_j} \rightharpoonup 0 \text{ weakly in } L^2(\Omega)$$

and therefore

$$\liminf_{j\to\infty} I_j \geq \liminf_{j\to\infty}(\alpha - \int_\Omega u_j^1 \psi_j^0 dx) = \alpha$$

since u_j^1 converges strongly in $L^2(\Omega)$. This is in contradiction with (14) and concludes the proof of (12).

On the other hand, it is obvious that $I_\varepsilon \leq 0$ for all $\varepsilon > 0$. Thus, (11) implies the uniform boundedness of the minimizers in $L^2(\Omega)$.

Concerning the convergence of the minimizers we have the following Lemma:

LEMMA 3 *The minimizers $\overline{\varphi}_\varepsilon^0$ of J_ε converge strongly in $L^2(\Omega)$ as $\varepsilon \to 0$ to the minimizer $\overline{\varphi}^0$ of J and M_ε converges to*

$$M = \inf_{\varphi^0 \in L^2(\Omega)} J(\varphi^0)\ . \tag{15}$$

Moreover, the corresponding solutions $\overline{\varphi}_\varepsilon$ of (5) converge in $C([0,T]; L^2(\Omega))$ to the solution $\overline{\varphi}$ of (7) as $\varepsilon \to 0$.

PROOF OF LEMMA 3. By extracting a subsequence (that we still denote by the index ε) we have

$$\overline{\varphi}_\varepsilon^0 \rightharpoonup \psi^0 \text{ weakly in } L^2(\Omega)$$

as $\varepsilon \to 0$. It is sufficient to check that $\psi^0 \equiv \overline{\varphi}^0$ or, equivalently,

$$J(\psi^0) \leq J(\varphi^0) \text{ for all } \varphi^0 \in L^2(\Omega). \tag{16}$$

From Theorem A we know that

$$\overline{\varphi}_\varepsilon \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T; L^2(\Omega))$$

where ψ is the solution of (7) with initial data ψ^0. By lower semicontinuity and taking into account that u_ε^1 converges strongly to u^1 in $L^2(\Omega)$ we deduce that

$$J(\psi^0) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}_\varepsilon^0). \tag{17}$$

On the other hand, for each $\varphi^0 \in L^2(\Omega)$ we have

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}_\varepsilon^0) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(\varphi^0). \tag{18}$$

But for $\varphi^0 \in L^2(\Omega)$ fixed, Theorem A ensures that the solutions φ_ε of (5) converge strongly to the solution φ of (7) in $C([0,T]; L^2(\Omega))$. Thus

$$\liminf_{\varepsilon\to 0} J_\varepsilon(\varphi^0) = J(\varphi^0)$$

and (16) holds.

This concludes the proof of the weak convergence of the minimizers and it also shows that

$$M \leq \liminf_{\varepsilon \to 0} M_\varepsilon. \tag{19}$$

On the other hand, in view of (18) we have

$$M = J(\overline{\varphi}^0) = \limsup_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}^0) \geq \limsup_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon) = \limsup_{\varepsilon \to 0} M_\varepsilon. \tag{20}$$

From (19) and (20) we deduce (15).

Observe that (15) combined with the weak (resp. strong) convergence of $\overline{\varphi}^0_\varepsilon$ (resp. u^1_ε) in $L^2(\Omega)$ implies that

$$\lim_{\varepsilon \to 0} \left(\frac{1}{2} \int_0^T \int_\omega |\overline{\varphi}_\varepsilon|^2 dx dt + \alpha \|\overline{\varphi}^0_\varepsilon\|_{L^2(\Omega)} \right) = \frac{1}{2} \int_0^T \int_\omega |\overline{\varphi}|^2 dx dt + \alpha \|\overline{\varphi}^0\|_{L^2(\Omega)}.$$

This identity, combined with the weak convergence of $\overline{\varphi}^0_\varepsilon$ (resp. $\overline{\varphi}_\varepsilon$) to $\overline{\varphi}^0$ (resp. $\overline{\varphi}$) in $L^2(\Omega)$ (resp. $L^2(\omega \times (0,T))$) implies that

$$\varphi^0_\varepsilon \to \varphi^0 \text{ strongly in } L^2(\Omega). \tag{21}$$

Theorem A implies then that

$$\varphi_\varepsilon \to \varphi \text{ strongly in } C([0,T]; L^2(\Omega))$$

as $\varepsilon^0 \to 0$.

This concludes the proof of Lemma 3.

In view of (21), the strong convergence in $C([0,T]; L^2(\Omega))$ of u_ε is a consequence of Theorem A.

3. Comments

1. The main result of this paper can be easily extended to more general parabolic equations of the form
   $\rho_\varepsilon(x)u_t - div(A_\varepsilon(x)\nabla u) = f\chi_\omega$
   as those considered in Bensoussan, Lions, Papanicolaou (1978), and Brahim-Otsmane, Frankfort, Murat (1992) provided we have unique continuation for the solutions of the corresponding adjoint system.
   Note that we have assumed a to be C^1. This is precisely to guarantee the unique continuation (see Fabre, Puel, Zuazua, 1993B).
2. The controls we have considered are those that minimize the L^2-norm among all the admissible controls satisfying (4). Following Fabre, Puel, Zuazua (1992A–1993B) we may also consider the quasi bang–bang controls minimizing the L^∞-norm. The results of this paper can easily be adapted to this case too.

3. In Cioranescu et al. (1991) — Cioranescu, Donato, Zuazua (1991) we have developed a different approach to the problems of controllability and homogenization in the context of the wave equation. In these works we show that under some additional assumptions on the oscillatory character of the coefficients (no oscillations on a neighborhood of the control region), if the limit problem is exactly controllable, one may let simultaneously ε and α to zero and still get a bounded sequence of controls and solutions.

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