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Iteration penalty method for the contact elastoplastic problem

by

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A system of variational inequalities is considered, describing the quasi-stationary equilibrium problem of an elastoplastic plate contact with a rigid obstacle, according to the plastic flow theory. The presence of free boundaries of contact and plasticity leads to geometrical and physical restrictions of the inequality type upon the solution. The restrictions are changed by a penalty operators and the penalty problem is linearized with the help of an iteration procedure. Convergence of solutions is proved.

1. Introduction

Formulation of the problem is as follows, Khludnev (1988). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\Omega$. It is required to find functions $\omega, m = (m_{ij})$, $i,j = 1,2$ in $\Omega \times (0,T)$ satisfying following relations

$$\omega \geq \varphi, \tag{1.1}$$

$$(m_{ij,ij} + f, \bar{\omega} - \omega) \leq 0 \quad \text{for any } \bar{\omega} : \bar{\omega} \geq \varphi, \tag{1.2}$$

$$|m| \leq a, \tag{1.3}$$

$$(c_{ijkl}\dot{m}_{kl} + \dot{\omega}_{,ij}, \bar{m}_{ij} - m_{ij}) \geq 0 \quad \text{for any } \bar{m} : |\bar{m}| \leq a, \tag{1.4}$$

$$\omega = \omega_\nu = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

$$\omega = \omega^0, \quad m = m^0 \quad \text{when } t = 0. \tag{1.6}$$

Where $\varphi \in C(0,T; C^2(\bar{\Omega}))$, φ on $\partial\Omega$ is less than 0; $f \in C(0,T; L^2(\Omega))$; $c_{ijkl} \in C(0,T; L^\infty(\Omega))$, $c_{ijkl}\xi_{kl}\xi_{ij} \geq c\xi_{ij}\xi_{ij}$, $c > 0$; $w^0 \in H_0^2(\Omega)$; $m^0 \in L^2(\Omega)$; $a \in C(0,T)$, $a > 0$ are given. Brackets $(\cdot,\cdot)$ denote the scalar product in $L^2(\Omega)$, $|m|$ means $|m^2| = m_{ij}m_{ij}$, upper point denotes the derivative with respect to variable $t \in (0,T)$, ω_ν is the derivative in the direction of external normal.

$H^2(\Omega) = W_2^2(\Omega)$ is usual Sobolev space and H_0^2 means the closure of $C_0^\infty(\Omega)$ in the corresponding norm of the space. Here and in what follows, repeated lower indices mean summation from *1* to *2*, lower indices after a comma mean differentiation with respect to the corresponding space variable $(x_1, x_2) \in \Omega$. All functions under consideration are symmetric with respect to the lower indices.

In the model formulated, functions ω, m, φ, f describe the normal displacement of plate points, the bending moments, the obstacle shape and the external force, respectively. Inequality (1.1) is the geometrical condition of unpenetrable obstacle and (1.3) is the physical restriction upon moments mentioned in Temam (1983). Strict inequality in (1.3) means the elastic state of a plate and the equality holds when the plastic state is reached. Let us divide $(0, T)$ into the intervals of length Δt by points $t = s\Delta t$, $s = 0, 1, \ldots, S$ and replace continuous derivative with respect to t by its finite–difference analogy, marking function values in point $t = s\Delta t$ by the upper index s. We obtain from (1.1)–(1.6) the set of stationary problems $s = 0, 1, \ldots, S-1$

$$\omega^{s+1} \geq \varphi^{s+1}, \tag{1.7}$$

$$(m_{ij,ij}^{s+1} + f^{s+1}, \bar{\omega} - \omega^{s+1}) \leq 0 \quad \text{for any } \bar{\omega} : \bar{\omega} \geq \varphi^{s+1}, \tag{1.8}$$

$$|m^{s+1}| \leq a^{s+1}, \tag{1.9}$$

$$\left(c_{ijkl}^{s+1} \frac{m_{kl}^{s+1} - m_{kl}^{s}}{\Delta t} + \frac{\omega_{,\,ij}^{s+1} - \omega_{,\,ij}^{s}}{\Delta t}, \bar{m}_{ij} - m_{ij}^{s+1} \right) \geq 0, |\bar{m}| \leq a^{s+1}, \tag{1.10}$$

$$\omega^{s+1} = \omega_\nu^{s+1} = 0 \quad \text{on } \partial\Omega. \tag{1.11}$$

One way of studying any variational inequality is to reduce it to an extremal problem, Khludnev (1992), and another way is to use penalty method, Lions (1969).

2. Formulation of the penalty problem

Let us define

$$\beta_\varphi(\omega) = \begin{cases} 0, & \text{when } \omega \geq \varphi, \\ \omega - \varphi, & \text{when } \omega < \varphi. \end{cases}$$

Let π be the orthogonal projection of space $(L^2(\Omega))^4$ onto set $\{m \in (L^2(\Omega))^4 / \, |m| \leq a\}$. We take $\alpha_a(m) = m - \pi m$, and then

$$\alpha_a(m) = \begin{cases} 0, & \text{when } |m| \leq a, \\ (1 - a/|m|)m, & \text{when } |m| > a. \end{cases}$$

The following penalty problem with positive parameters e, q is considered

$$-m_{ij,ij}^{s+1,e,q} + \frac{1}{e}\beta_{\varphi^{s+1}}(\omega^{s+1,e,q}) = f^{s+1}, \tag{2.1}$$

$$c^{s+1}_{ijkl} m^{s+1,e,q}_{kl} + \frac{1}{q}\alpha_{a^{s+1}}(m^{s+1,e,q})_{ij} = -\omega^{s+1,e,q}_{,ij} + c^{s+1}_{ijkl} m^{s}_{kl} + \omega^{s}_{,ij},$$

$$i,j = 1,2, \tag{2.2}$$

$$\omega^{s+1,e,q} = \omega^{s+1,e,q}_{\nu} = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Weak convergence of the slution of problem (2.1)–(2.3) to the solution problem (1.7)–(1.11) as e, $q \to 0$, when (1.7)–(1.11) have a smooth solution $(\omega^{s+1}, m^{s+1}) \in H^2_0(\Omega) \times (L^2(\Omega))^4$ and f is sufficiently small, is proved in Khludnev (1988).

Now we shall rewrite (2.1)–(2.3) in the equivalent formulation. We shall not indicate dependence of functions upon the parameters e, q, s for the sake of simplicity. Let us rewrite (2.2) as follows

$$m_{ij} + \frac{1}{q}\alpha_a(m)_{ij} = g_{ij}, \quad i,j = 1,2, \tag{2.4}$$

where

$$g_{ij} = -\omega_{,ij} + (\delta_{ik}\delta_{jl} - c_{ijkl})m_{kl} + p_{ij}, \quad i,j = 1,2, \tag{2.5}$$

$$p_{ij} = c^{s+1}_{ijkl} m^{s}_{kl} + \omega^{s}_{,ij}, \quad i,j = 1,2.$$

LEMMA *Equation (2.4) is equivalent to the following equation*

$$m_{ij} = g_{ij} - \frac{1}{1+q}\alpha_a(g)_{ij}, \quad i,j = 1,2. \tag{2.6}$$

PROOF. Let (2.4) hold. Let $|m| \le a$, then $\alpha_a(m) = 0$ and $m_{ij} = g_{ij}$, therefore, $|g| \le a$ and (2.6) holds. Let $|m| > a$ then it follows from (2.4) that

$$\left(1 + \frac{1}{q}\left(1 - \frac{a}{|m|}\right)\right) m_{ij} = g_{ij}, \quad i,j = 1,2. \tag{2.7}$$

Squaring and summing up of equation (2.7) gives

$$|m| = \frac{q|g| + a}{1+q}, \tag{2.8}$$

therefore, $|g| > a$. We can obtain the following equality by substituting (2.8) into (2.7)

$$m_{ij} = g_{ij} - \frac{1}{1+q}\left(1 - \frac{a}{|g|}\right) g_{ij}, \quad i,j = 1,2.$$

and, therefore, (2.6) holds. On the other hand, (2.4) follows from (2.6) by analogy with the above reasoning. This completes the proof.

Let us introduce the notation

$$a_{ijkl} = (c_{ijkl} - \delta_{ik}\delta_{jl})^{-1}. \tag{2.9}$$

Equation (2.5), together with (2.9), gives

$$m_{ij} = -a_{ijkl}(\omega_{,kl} + g_{kl} - p_{kl}), \quad i,j = 1,2. \tag{2.10}$$

We can obtain the following equations substituting (2.10) into (2.6) and (2.1)

$$(a_{ijkl} + \delta_{ik}\delta_{jl})g_{kl} - \frac{1}{1+q}\alpha_a(g)_{ij} = a_{ijkl}(p_{kl} - \omega_{,kl}) \tag{2.11}$$

$$a_{ijkl}\omega_{,klij} + \frac{1}{e}\beta_\varphi(\omega) = f + a_{ijkl}(p_{kl,ij} - g_{kl,ij}). \tag{2.12}$$

System (2.10)–(2.12) is equivalent to (2.1), (2.2) a.e. on Ω in view of the Lemma. Obtained value–boundary problem of the elliptic type with the essential non–linearity (2.10)–(2.12), (2.3) will be linearized as in the sequel.

3. The iteration penalty problem

Let (ω^0, m^0) be any functions from $H_0^2(\Omega)\times(L^2(\Omega))^4$. We organize the following iteration procedure as $n = 0, 1, 2, \ldots$.

$$a_{ijkl}\omega^{n+1}_{,klij} + \frac{1}{e}\omega^{n+1} = f + \frac{1}{e}(\omega^n - \beta_\varphi(\omega^n)) + a_{ijkl}(p_{kl,ij} - g^n_{kl,ij}), \tag{3.1}$$

$$(a_{ijkl} + \delta_{ik}\delta_{jl})g^{n+1}_{kl} = \frac{1}{1+q}\alpha_a(g^n)_{ij} + a_{ijkl}(p_{kl} - \omega^{n+1}_{,kl}), \quad i,j = 1,2, \tag{3.2}$$

$$m^{n+1}_{ij} = -a_{ijkl}(\omega^{n+1}_{,kl} + g^{n+1}_{kl} - p_{kl}), \quad i,j = 1,2, \tag{3.3}$$

$$\omega^{n+1} = \omega^{n+1}_\nu = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

Let problem (1.7)–(1.11) have smooth solution $(\omega^{s+1}, m^{s+1}) \in H_0^2(\Omega)\times(L^2(\Omega))^4$, then $p_{kl} \in L^2(\Omega)$, $k,l = 1,2$ and the r.h.s. of equation (3.1) belongs to $H^{-2}(\Omega)$, while the one of equation (3.2) belongs to $L^2(\Omega)$. Therefore, a unique solution $(\omega^{n+1}, g^{n+1}, m^{n+1}) \in H_0^2(\Omega) \times (L^2(\Omega))^4 \times (L^2(\Omega))^4$ of the linear problem (3.1)–(3.4) exists in view of the general theory of monotonous operators.

Let the following inequalities be satisfied

$$c_1(\xi_{ij}, \xi_{ij}) \le (a_{ijkl}\xi_{kl}, \xi_{ij}) \le c_2(\xi_{ij}, \xi_{ij}), \quad c_1, c_2 > 0. \tag{3.5}$$

THEOREM *Under the above conditions, the following convergence holds*

$$(\omega^n, m^n) \to (\omega, m) \quad \textit{strongly in} \quad H_0^2(\Omega) \times (L^2(\Omega))^4 \quad \textit{as} \quad n \to \infty,$$

where (ω, m) is the solution of problem (2.1)–(2.3).

PROOF. Let us denote $u^{n+1} = \omega^{n+1} - \omega^n$, $v^{n+1} = g^{n+1} - g^n$. It follows from (3.1)–(3.2) that

$$a_{ijkl}u^{n+1}_{,klij} + \frac{1}{e}u^{n+1} = \frac{1}{e}(u^n - \beta_\varphi(\omega^n) + \beta_\varphi(\omega^{n-1})) - a_{ijkl}v^n_{kl,ij}, \tag{3.6}$$

$$(a_{ijkl} + \delta_{ik}\delta_{jl})v_{kl}^{n+1} = \frac{1}{1+q}(\alpha_a(g^n)_{ij} + \alpha_a(g^{n-1})_{ij}) - a_{ijkl}u_{,kl}^{n+1},$$
$$i,j = 1,2, \tag{3.7}$$

The monotonicity of penalty operators gives

$$\begin{aligned} &|\alpha_a(g^n) - \alpha_a(g^{n-1})| \le |g^n - g^{n-1}|, \\ &|\omega^n - \omega^{n-1} - (\beta_\varphi(\omega^n) - \beta_\varphi(\omega^{n-1})| \le |\omega^n - \omega^{n-1}|. \end{aligned} \tag{3.8}$$

Further, we can introduce the scalar product in $L^2(\Omega)$, thanks to condition (3.5), as follows

$$\langle s_{ij}^1, s_{ij}^2\rangle = (a_{ijkl}s_{kl}^1, s_{ij}^2)$$

and define equivalent norms in $(L^2(\Omega))^4$ and $H_0^2(\Omega)$, respectively,

$$[v]_0^2 = \langle v_{ij}, v_{ij}\rangle, \quad [u]_2^2 = \langle u_{,ij}, u_{,ij}\rangle.$$

We write the estimates of the norms using (3.5)

$$||u||_0^2 \le ||u||_2^2 \le \frac{1}{c_1}[u]_2^2, \quad [v]_0^2 \le c_2 \sum_{i,j} ||v_{ij}||_0^2, \tag{3.9}$$

where $||\cdot||_k$ is the usual norm in $H^k(\Omega)$. We multiply (3.6) by u^{n+1} and (3.7) by v^{n+1}, then we integrate over Ω and sum. We can obtain the following inequality using Holder inequality and estimates (3.8), (3.9)

$$\begin{aligned} &[u^{n+1}]_2^2 + \frac{1}{e}||u^{n+1}||_0^2 + [v^{n+1}]_0^2 + \frac{1}{1+q}\sum_{i,j}||v_{ij}^{n+1}||_0^2 \le \\ &\varrho\left([u^n]_2^2 + \frac{1}{e}||u^n||_0^2 + [v^n]_0^2 + \frac{1}{1+q}\sum_{i,j}||v_{ij}^n||_0^2\right) \le \dots \\ &\varrho^n\left([u^1]_2^2 + \frac{1}{e}||u^1||_0^2 + [v^1]_0^2 + \frac{1}{1+q}\sum_{i,j}||v_{ij}^1||_0^2\right), \end{aligned}$$

where constant $\varrho < 1$ depends on s, e, q, c_1, c_2. Thus, we have the strong convergence as follows

$$(u^n, v^n) \to (0,0) \quad \text{strongly in} \quad H_0^2(\Omega) \times (L^2(\Omega))^4 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Element $(\omega, g) \in H_0^2(\Omega) \times (L^2(\Omega))^4$ exists, by virtue of convergence of the geometrical progression series with exponent $\varrho < 1$, such that the following convergence holds

$$(\omega^n, g^n) \to (\omega, g) \quad \text{strongly in} \quad H_0^2(\Omega) \times (L^2(\Omega))^4 \quad \text{as} \quad n \to \infty. \tag{3.11}$$

Let us pass to the limit in (3.1)–(3.3) as $n \to \infty$ using convergence (3.10), (3.11) and penalty operators continuity. Therefore, we shall obtain that (ω, m), where $m_{ij} = -a_{ijkl}(\omega_{,kl} + g_{kl} - p_{kl})$, $i,j = 1,2$, is the solution of problem (2.10)–(2.12), (2.3). This concludes the proof. ■

References

KHLUDNEV A.M. (1988) On variational inequalities in contact plastic problems, *Differentsial'nye Uravneniya*, **24**, 9, 1622–1628, (in Russian).

KHLUDNEV A.M. (1992) An optimal control problem for a fourth-order variational inequality, *Partial Differential Equations. Banach Center Publications*, **27**, Warszawa, 225–231.

LIONS J.L. (1969) *Quelques Mèthodes de Rèsolution des Problèmes aux Limites Non Linèaires,* Paris, Dunod.

TEMAM R. (1983) *Problèmes Mathèmatiques en Plasticitè*, Paris, Gauthier–Villars.