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Existence of a Lagrange multiplier for the control problem of a variational inequality

by

Anatoly Leontiev

Institute of Hydrodynamics
630090 Novosibirsk
Russia

We shall be dealing with the following optimal control problem (P): over all triples $\{y(u), u, \xi(u)\}$ satisfying the variational inequality

$$\begin{cases} -\triangle y(u) = u + \xi(u), \qquad y(u) \in H_0^1(\Omega), \qquad u \in U_\partial, \\ y(u) \geq 0,\, \xi(u) \geq 0,\, (\xi(u), y(u)) = 0, \quad \xi(u) \in H^{-1}(\Omega) \end{cases}$$

we need to find such optimal triple $\{\overset{*}{y}, \overset{*}{u}, \overset{*}{\xi}\}$ that

$$J(\overset{*}{y}, \overset{*}{u}, \overset{*}{\xi}) \leq J(y(u), u, \xi(u)).$$

Here J is the cost functional :

$$J(y, u, \xi) = \frac{1}{2}||y - z_d||^2_{L^2(\Omega)} + \frac{1}{2}||u||^2_{L^2(\Omega)},$$

U_∂ is the nonempty closed convex subset of $L^2(\Omega)$, $z_d \in L^2(\Omega)$.

We suppose that $z_d <\cdot \inf\limits_{u \in U_\partial} y(u)$. Then the existence and uniqueness of an optimal triple can be found in Mignot (1976).

We shall prove that such a Lagrange multiplier $p \in H_0^1(\Omega)$ exists that the control problem (P) can be transformed into the lagrangian minimization problem (L) :

$$L(\overset{*}{y}, \overset{*}{u}, \overset{*}{\xi}; p) \leq L(y, u, \xi; p),$$

$$\overset{*}{y}, \overset{*}{u}, \overset{*}{\xi},\ \forall\, y, u, \xi \in \Phi,$$

where

$$L(y, u, \xi; p) \equiv J(y, u, \xi) + (\triangle y + u + \xi, p)$$

and Φ is defined with the allowance for the regularity properties of the optimal triple:

$$\Phi = \left\{ y \in H^2(\Omega) \cap H_0^1(\Omega),\, u \in U_\partial,\, \xi \in L^2(\Omega),\, |y \geq 0,\, \xi \geq 0,\, (\xi, y) = 0 \right\}.$$

Then it is possible to obtain the necessary conditions for the problem (P) via the constrained increments method, Puel (1987).

Let us consider the following penalty control problem (P^ε):

$$\begin{cases} -\triangle y^\varepsilon - \beta^\varepsilon(y^\varepsilon) = u^\varepsilon, & y^\varepsilon \in H_0^1(\Omega), \\ \inf J(y^\varepsilon, u^\varepsilon, \beta^\varepsilon(y^\varepsilon)), & u^\varepsilon \in U_\partial, \end{cases}$$

where $\beta^\varepsilon(r) = \varepsilon^{-1} r^-$ is the penalty function, $r^- = -\inf\{0, r\}$. The solution $\{\overset{*}{y^\varepsilon}, \overset{*}{u^\varepsilon}, \overset{*}{\beta^\varepsilon}\}$ (here $\overset{*}{\beta}{}^\varepsilon \equiv \beta^\varepsilon(\overset{*}{y}{}^\varepsilon)$) of this problem exists for every $\varepsilon > 0$, converges to the optimal triple at $\varepsilon \to 0$ and provides the converges of the cost functional, Barbu (1984):

$$J(\overset{*}{y^\varepsilon}, \overset{*}{u^\varepsilon}, \overset{*}{\beta^\varepsilon}) \to J(\overset{*}{y}, \overset{*}{u}, \overset{*}{\xi}).$$

We shall obtain the convergence estimates. From the cost functional we have that u^ε is bounded umiformly in ε in $L^2(\Omega)$ and therefore for some subsequence (denoted by ε):

$$\overset{*}{u^\varepsilon} \to \overset{*}{u} \qquad \text{weakly in } L^2(\Omega).$$

For every $\varepsilon > 0$: $\overset{*}{y}{}^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$. Multiplying the state equation by $-\triangle \overset{*}{y}{}^\varepsilon$ gives:

$$||\triangle \overset{*}{y^\varepsilon}||^2_{L^2(\Omega)} + \varepsilon^{-1}(\nabla(\overset{*}{y^\varepsilon})^-, \nabla(\overset{*}{y^\varepsilon})^-) = (\overset{*}{u^\varepsilon}, -\triangle \overset{*}{y^\varepsilon}).$$

Since $||\triangle \overset{*}{y^\varepsilon}||_{L^2(\Omega)} \geq ||\overset{*}{y^\varepsilon}||_{H^2(\Omega)}$ for $\overset{*}{y^\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$, the above relation yields:

$$||\overset{*}{y^\varepsilon}||_{H^2(\Omega)} \leq C \qquad \text{uniformly in } \varepsilon,$$

and therefore we may choose such a subsequence (denoted by ε) that

$$\overset{*}{y}{}^\varepsilon \to \overset{*}{y} \qquad \text{weakly in } H^2(\Omega) \cap H_0^1(\Omega).$$

Then $\overset{*}{\beta}{}^\varepsilon \to \overset{*}{\xi}$ weakly in $L^2(\Omega)$, and convergences of $\overset{*}{u}{}^\varepsilon$ and $J(\overset{*}{y^\varepsilon}, \overset{*}{u^\varepsilon}, \overset{*}{\beta^\varepsilon})$ give us that

$$\overset{*}{u^\varepsilon} \to \overset{*}{u} \qquad \text{strongly in } L^2(\Omega).$$

The optimality conditions for the penalty control problem are in the existence of the adjoint state $\overset{*}{p^\varepsilon} \in H_0^1(\Omega)$ satisfying, together with $\overset{*}{y^\varepsilon}$, $\overset{*}{u^\varepsilon}$ and $\overset{*}{\beta^\varepsilon}$, the optimality system, Barbu (1984) :

$$\begin{cases} -\triangle \overset{*}{y^\varepsilon} - \overset{*}{\beta} (\overset{*}{y^\varepsilon}) = \overset{*}{u^\varepsilon}, & \overset{*}{y^\varepsilon} \in H_0^1(\Omega), \\ -\triangle \overset{*}{p^\varepsilon} + \dot{\beta}^\varepsilon (\overset{*}{y^\varepsilon}) \overset{*}{p^\varepsilon} = \overset{*}{y^\varepsilon} - z_d, & \overset{*}{p^\varepsilon} \in H_0^1(\Omega), \\ (\overset{*}{p^\varepsilon} + \overset{*}{u^\varepsilon}, u - \overset{*}{u^\varepsilon}) \geq 0, & \overset{*}{u^\varepsilon}, \ \forall u \in U_\partial. \end{cases}$$

Here $\dot{\beta}^\varepsilon(r) = \frac{d}{dr}\beta^\varepsilon(r)$, $\dot{\beta}^\varepsilon(r) \leq 0$, $\forall \varepsilon > 0$. Then the uniform convergence $\overset{*}{y}^\varepsilon \to \overset{*}{y}$ (by the compact imbedding $H^2(\Omega)$ in $C(\bar{\Omega})$) and the condition for z_d give :

$$\overset{*}{y^\varepsilon} - z_d > 0 \qquad \text{for all } \varepsilon < \varepsilon^\circ.$$

By virtue of the maximum principle for the adjoint state equation, we have :

$$\overset{*}{p^\varepsilon} \geq 0 \qquad \text{in } \bar{\Omega} \text{ for all } \varepsilon < \varepsilon^\circ.$$

This equation also gives the estimate:

$$\| \overset{*}{p^\varepsilon} \|_{H_0^1(\Omega)} \leq C \qquad \text{uniformly in } \varepsilon.$$

Choosing a subsequence (denoted by ε), we may assume that

$$\overset{*}{p^\varepsilon} \to \overset{*}{p} \qquad \text{weakly in } H_0^1(\Omega).$$

The function $\overset{*}{p}$ is called the adjoint state for the problem (P).

Let us consider the minimization problem (L^ε) :

$$L(\bar{y^\varepsilon}, \bar{u^\varepsilon}, \bar{\beta^\varepsilon}; \overset{*}{p^\varepsilon}) \leq L(y^\varepsilon, u^\varepsilon, \beta^\varepsilon; \overset{*}{p^\varepsilon})$$

on the convex set

$$\Phi^\varepsilon \equiv \left\{ y^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega),\ u^\varepsilon \in U_\partial,\ \beta^\varepsilon \equiv \beta^\varepsilon(y^\varepsilon) \right\}.$$

Since $\overset{*}{p^\varepsilon}$ is non-negative on $\bar{\Omega}$ and

$$\beta^\varepsilon(\lambda w_1 + (1-\lambda)w_2) \leq \lambda\beta^\varepsilon(w_1) + (1-\lambda)\beta^\varepsilon(w_2) \quad \text{for every } \lambda,\ 0 \leq \lambda \leq 1$$

the functional $L(y^\varepsilon, u^\varepsilon, \beta^\varepsilon; \overset{*}{p^\varepsilon})$ is strictly convex on Φ^ε. Then the following system gives the necessary and sufficient conditions for the problem (L^ε) :

$$\begin{cases} -\triangle \overset{*}{p^\varepsilon} - \dot{\beta}^\varepsilon (\bar{y^\varepsilon}) \overset{*}{p^\varepsilon} = \bar{y^\varepsilon} - z_d, & \bar{y^\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega), \\ (\overset{*}{p^\varepsilon} + \bar{u^\varepsilon}, u - \bar{u^\varepsilon}) \geq 0, & \bar{u^\varepsilon}, \ \forall u \in U_\partial. \end{cases}$$

Comparing it with the optimality system of (P^ε), we establish that the triple $\{\overset{*}{y^\varepsilon}, \overset{*}{u^\varepsilon}, \overset{*}{\beta^\varepsilon}\}$ gives a solution of the problem (L^ε) :

$$L(\overset{*}{y^\varepsilon}, \overset{*}{u^\varepsilon}, \overset{*}{\beta^\varepsilon}; \overset{*}{p^\varepsilon}) \leq L(y^\varepsilon, u^\varepsilon, \beta^\varepsilon; \overset{*}{p^\varepsilon}).$$

Let us show that for all $\{y, u, \xi\} \in \Phi$ on can construct such a sequence $\{y^\varepsilon, u^\varepsilon, \beta^\varepsilon\} \in \Phi^\varepsilon$ that

$$\begin{array}{ll} y^\varepsilon \to y & \text{strongly in } H^2(\Omega) \cap H_0^1(\Omega), \\ u^\varepsilon \to u & \text{strongly in } L^2(\Omega), \\ \beta^\varepsilon \to \xi & \text{strongly in } L^2(\Omega). \end{array}$$

By using the smoothing functions $\xi_n(x) \equiv \int_\Omega \xi(y)\omega^n(|x-y|)dy$, where $\omega^n(r)$ is a mollifier in R^2, $\omega^n \geq 0$, we may build for all $\xi \in L^2(\Omega)$, $\xi \geq 0$ such a sequence $\xi^\varepsilon \in \dot{C}^\infty(\Omega)$, $\xi^\varepsilon \geq 0$ that

$$\begin{array}{ll} \xi^\varepsilon \to \xi & \text{strongly in } L^2(\Omega), \\ \varepsilon\xi^\varepsilon,\ \varepsilon D\xi^\varepsilon \to 0 & \text{strongly in } L^2(\Omega). \end{array}$$

Let us define the functions $y^\varepsilon \equiv y - \varepsilon\xi^\varepsilon$, $u^\varepsilon \equiv u$, $\beta^\varepsilon \equiv \beta^\varepsilon(y^\varepsilon) = \varepsilon^{-1}(y - \varepsilon y^\varepsilon)^-$. Since $||\xi^\varepsilon - \xi||^2_{L^2(\Omega)} = ||\xi^\varepsilon - \xi||^2_{L^2(\Omega+)} + ||\xi^\varepsilon||^2_{L^2(\Omega^\circ)}$, where Ω° is a subset of $\Omega : y(x) \not\equiv 0,\ x \in \Omega^\circ$; $\Omega^+ \equiv \Omega \backslash \Omega^\circ$, we have: $||\xi^\varepsilon - \xi||_{L^2(\Omega+)}$, $||\xi^\varepsilon||_{L^2(\Omega^\circ)} \to 0$, if ε tend to zero. Moreover,

$$\begin{array}{lll} \text{in} & \Omega^+ : & \varepsilon^{-1}(y - \varepsilon\xi^\varepsilon)^- \equiv \xi^\varepsilon, \\ \text{in} & \Omega^\circ : & 0 \leq \varepsilon^{-1}(y - \varepsilon\xi^\varepsilon)^- \leq \xi^\varepsilon. \end{array}$$

Hence, $\beta^\varepsilon \to \xi$ strongly in $L^2(\Omega)$. The function y^ε has also the required convergence property.

Using these results and the convergence of $\overset{*}{y^\varepsilon}$, $\overset{*}{u^\varepsilon}$, $\overset{*}{\beta^\varepsilon}$ and $\overset{*}{p^\varepsilon}$, it is possible to pass to the limit in (L^ε) and obtain the function $\overset{*}{y}$, $\overset{*}{u}$ and $\overset{*}{\xi}$ as the solution of the problem (L) for $p \equiv \overset{*}{p}$. Therefore, the existence of the desired Lagrange multiplier has been proved, and it has been established that this multiplier is the adjoint state function.

References

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