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Optimal control of a generalized linear dynamic system with quadratic index of performance

by

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Resulting from application of the Bittner operational calculus the definition of a generalized linear dynamic differential system is presented. For the system the problem of synthesis of the optimal regulator with the quadratic performance index is solved.

1. Preliminaries

The Bittner operational calculus, Bittner (1974) is referred to as a system

$$CO(L^0, L^1, S, T_q, s_q, Q)\ , \tag{1}$$

where L^0 and L^1 are linear spaces over the same field Γ of scalars such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in L(L^1, L^0)$), called the (abstract) derivative, is a surjection. Moreover, a nonempty set Q is the set of indices q for the operations $T_q \in L(L^0, L^1)$ such that $ST_q f = f$, $f \in L^0$, called integrals and for the operations $s_q \in L(L^1, L^1)$ such that $s_q x = x - T_q Sx$, $x \in L^1$, called limit conditions. The kernel of S, i.e. the set $KerS := \{c \in L^1 : Sc = 0\}$, is called the space of constants for the derivative S.

2. The matrix operational calculus

In this paper we will assume that the operational calculus (1) is given after Wysocki (1994), in which:

a) L^0 is a real commutative algebra with unity e and L^1 is its subalgebra

b) the derivative S satisfies the Leibniz condition

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy\ ,\ \ x, y \in L^1$$

c) the limit conditions s_q, $q \in \tilde{Q} \subset Q$ (where $\tilde{Q}$ has at least one element) are multiplicative, i.e.

$$s_q(x \cdot y) = s_q x \cdot s_q y \ , \ q \in \tilde{Q} \ , \ x, y \in L^1 \ .$$

The mapping $I_{q_0}^{q_1} \in L(L^0, KerS)$ described by the formula

$$I_{q_0}^{q_1} f := (T_{q_0} - T_{q_1})f = s_{q_1} T_{q_0} f \ , \ q_0, q_1 \in Q \ , \ f \in L^0$$

is called operation of definite integration. It is easy to verify that the Leibniz-Newton formula

$$I_{q_0}^{q_1} Sx = R_{q_0}^{q_1} x \ , \ x \in L^1 \tag{2}$$

holds, where the operation $R_{q_0}^{q_1} \in L(L^1, KerS)$ is described by the formula

$$R_{q_0}^{q_1} x := (s_{q_1} - s_{q_0})x \ , \ q_0, q_1 \in Q \ , \ x \in L^1. \tag{3}$$

Let $Mat_{m \times n}(Z)$ denote the set of all matrices with m rows and n columns with elements belonging to the set Z. In the sets $Mat_{m \times n}(L^k), k = 0, 1$, we define the usual operations of addition of matrices and multiplication of matrix by a number. Then the sets $Mat_{m \times n}(L^k), k = 0, 1$, $Mat_{m \times n}(KerS)$ are real linear spaces such that

$$Mat_{m \times n}(KerS) \subset Mat_{m \times n}(L^1) \subset Mat_{m \times n}(L^0) \ .$$

For the elements $\hat{X} \in Mat_{m \times r}(L^k), \hat{Y} \in Mat_{r \times n}(L^k), k = 0, 1$ we define the product $\hat{X} \cdot \hat{Y}$ as the usual matrix multiplication. Then the sets $Mat_{n \times n}(L^k), k = 0, 1$, are real algebras with unity $\hat{E} := [\delta_{ij} \cdot e]_{n \times n}$, where δ_{ij} denotes the Kronecker symbol.

Let

$$S\hat{X} := [Sx_{ij}]_{m \times n} \ , \ T_q \hat{F} := [T_q f_{ij}]_{m \times n} \ , \ s_q \hat{X} := [s_q x_{ij}]_{m \times n} \ , \tag{4}$$

where $\hat{F} := [f_{ij}] \in Mat_{m \times n}(L^0)$, $\hat{X} := [x_{ij}] \in Mat_{m \times n}(L^1)$, $q \in Q$.
It is not difficult to notice that the operations (4) are linear. Moreover, using the definitions of matrix operations and the properties of the operational calculus, it is easy to prove the following relations:

$$S\hat{C} = \hat{0} \ , \ s_q \hat{C} = \hat{C} \ , \ q \in Q \ , \ \hat{C} \in Mat_{m \times n}(KerS) \tag{5}$$

$$S(\hat{X} \cdot \hat{Y}) = S\hat{X} \cdot \hat{Y} + \hat{X} \cdot S\hat{Y} \ , \ \hat{X} \in Mat_{m \times r}(L^1) \ , \ \hat{Y} \in Mat_{r \times n}(L^1) \tag{6}$$

(the Leibniz formula)

$$s_q(\hat{X} \cdot \hat{Y}) = s_q \hat{X} \cdot s_q \hat{Y} \ , \ q \in \tilde{Q} \ , \ \hat{X} \in Mat_{m \times r}(L^1) \ , \ \hat{Y} \in Mat_{r \times n}(L^1) \tag{7}$$

(the multiplication condition)

$$ST_q \hat{F} = \hat{F}, s_q \hat{X} = \hat{X} - T_q S\hat{X} \ , \ q \in Q \ , \ \hat{F} \in Mat_{m \times n}(L^0), \quad \hat{X} \in Mat_{m \times n}(L^1) \ . \tag{8}$$

It is obvious that $Mat_{n\times n}(KerS)$ is the subalgebra of algebras $Mat_{n\times n}(L^k)$, $k = 0, 1$. Moreover, $\hat{E} \in Mat_{n\times n}(KerS)$.

Let $Inv(Z)$ denote the set of invertible elements in an algebra Z. If $\hat{X} \in Inv(Mat_{n\times n}(L^1))$, then $s_q\hat{X} \in Inv(Mat_{n\times n}(KerS)), q \in \tilde{Q}$ and

$$s_q(\hat{X}^{-1}) = (s_q\hat{X})^{-1} \ , \ q \in \tilde{Q} \tag{9}$$

$$S(\hat{X}^{-1}) = -\hat{X}^{-1} \cdot S\hat{X} \cdot \hat{X}^{-1} \ . \tag{10}$$

Assume that there exists a solution $\hat{X} \in Inv(Mat_{n\times n}(L^1))$ of the abstract matrix differential equation

$$S\hat{X} = \hat{A}\hat{X} \ , \tag{11}$$

where $\hat{A} \in Mat_{n\times n}(L^0)$ is a given matrix.

Let

$$FM(\hat{A}) := \{ \ \hat{X} \in Inv(Mat_{n\times n}(L^1)) : S\hat{X} = \hat{A}\hat{X} \ \} \ .$$

THEOREM 1. *The differential equation* (11) *with the limit condition*

$$s_{q_0}\hat{X} = \hat{X}_0 \ , \ q_0 \in \tilde{Q} \ , \ \hat{X}_0 \in Mat_{n\times n}(KerS)$$

has a unique solution in the set $FM(\hat{A})$.

Each element $\hat{X} \in FM(\hat{A})$ is called fundamental matrix of the equation (11).

An element $\hat{X} \in FM(\hat{A})$ which is the solution of the problem

$$S\hat{X} = \hat{A}\hat{X} \ , \ s_{q_0}\hat{X} = \hat{E} \ , \ q_0 \in \tilde{Q}$$

is called normalized fundamental matrix of the equation (11) (corresponding to q_0) and denoted as $\hat{\Phi}_{q_0}(\hat{A})$.

Let

$$L_r^k := Mat_{r\times 1}(L^k) \ , \ k = 0, 1 \ , \ (KerS)_r := Mat_{r\times 1}(KerS) \ .$$

THEOREM 2. *If* $\hat{A} \in Mat_{n\times n}(L^0), \hat{B} \in Mat_{n\times m}(L^0), \bar{u} \in L_m^0$ *are given and for a certain* $q_0 \in \tilde{Q}$ *the element* $\hat{\Phi}_{q_0}(\hat{A})$ *exists, then the abstract vector-matrix differential equation*

$$S\bar{x} = \hat{A}\bar{x} + \hat{B}\bar{u} \ , \ \bar{x} \in L_n^1$$

with the limit condition

$$s_{q_0}\bar{x} = \bar{x}_0 \ , \ \bar{x}_0 \in (KerS)_n$$

has the unique solution defined by the Cauchy formula

$$\bar{x} = \hat{\Phi}_{q_0}(\hat{A})\bar{x}_0 + \hat{\Phi}_{q_0}(\hat{A})T_{q_0}[\hat{\Phi}_{q_0}^{-1}(\hat{A})\hat{B}\bar{u}] ,$$

where $\hat{\Phi}_{q_0}^{-1}(\hat{A}) := [\hat{\Phi}_{q_0}(\hat{A})]^{-1}$.

THEOREM 3. *If* $\hat{A}_1, \hat{A}_2, \hat{F} \in Mat_{n\times n}(L^0)$ *are given and for a certain* $q_0 \in \tilde{Q}$ *the elements* $\hat{\Phi}_{q_0}(\hat{A}_1), \hat{\Phi}_{q_0}(\hat{A}_2^t)$ *exist, then the abstract matrix differential equation*

$$S\hat{X} = \hat{A}_1\hat{X} + \hat{X}\hat{A}_2 + \hat{F} \ , \ \hat{X} \in Mat_{n\times n}(L^1)$$

with the limit condition

$$s_{q_0}\hat{X} = \hat{X}_0 \ , \ \hat{X}_0 \in Mat_{n\times n}(KerS)$$

has the unique solution defined by the Cauchy formula

$$\hat{X} = \hat{\Phi}_{q_0}(\hat{A}_1) \cdot \hat{X}_0 \cdot \hat{\Phi}_{q_0}^t(\hat{A}_2^t) +$$
$$+\hat{\Phi}_{q_0}(\hat{A}_1) \cdot T_{q_0}\{\hat{\Phi}_{q_0}^{-1}(\hat{A}_1) \cdot \hat{F} \cdot [\hat{\Phi}_{q_0}^{-1}(\hat{A}_2^t)]^t\} \cdot \hat{\Phi}_{q_0}^t(\hat{A}_2^t) ,$$

where the symbol "t" denotes the transposition and $\hat{\Phi}_{q_0}^t(\hat{A}_2^t) := [\hat{\Phi}_{q_0}(\hat{A}_2^t)]^t$.

3. The generalized dynamic system

Let us consider all real systems whose dynamics, after taking the suitable models of the operational calculus, is described by the dependences

$$S\bar{x} = \hat{A}\bar{x} + \hat{B}\bar{u} \ , \ \bar{u} \in L_m^0 \ , \ \bar{x} \in L_n^1 \tag{12}$$

$$\bar{y} = \hat{C}\bar{x} + \hat{D}\bar{u} \ , \ \bar{y} \in L_r^0 \ , \tag{13}$$

where the matrices $\hat{A} \in Mat_{n\times n}(L^0), \hat{B} \in Mat_{n\times m}(L^0), \hat{C} \in Mat_{r\times n}(L^0), \hat{D} \in Mat_{r\times m}(L^0)$ are given.

Model (12),(13) of these systems will be called generalized ((m, n, r)-dimensional) linear dynamic differential system with compensating constants. The given element $\bar{u}$ will be called input signal (control) of the system (12),(13), whereas the elements $\bar{x}$ and $\bar{y}$ will be called state (the state variable) and output signal (response) of the system, respectively. The abstract differential equation (12) will be called the equation of state and the equation (13) - the equation of output of the generalized dynamic system.

4. The optimal regulator

The generalized dynamic system in which the control is a function of state, i.e.

$$\bar{u} = h(\bar{x}) \tag{14}$$

will be called closed system (Fig. 1a). If the control is a function of limit state $s_{q_0}\bar{x} = \bar{x}_0 \in (KerS)_n$, i.e.

$$\bar{u} = h(\bar{x}_0)\,, \tag{15}$$

then the generalized dynamic system will be called open system (Fig. 1b).

The mapping $h : L_n^1 \longrightarrow L_m^0$ describing the dependences (14),(15) will be called regulator.

Let $L \supset L^0$ be a Mikusiński space. It is a real linear partially ordered space in which the order is introduced by a cone $K \subset L$ satisfying suitable conditions, Bittner (1974). The elements of the cone K are called non-negative elements. Let $K^0 := K \cap L^0$ and $K^0 \neq \emptyset$. Moreover, let there be given a function of control $\bar{u}$

$$J(\bar{u}) = g(\bar{x}, \bar{u}) \tag{16}$$

with values in L^0, where $\bar{x}$ is a solution of the state equation (12).

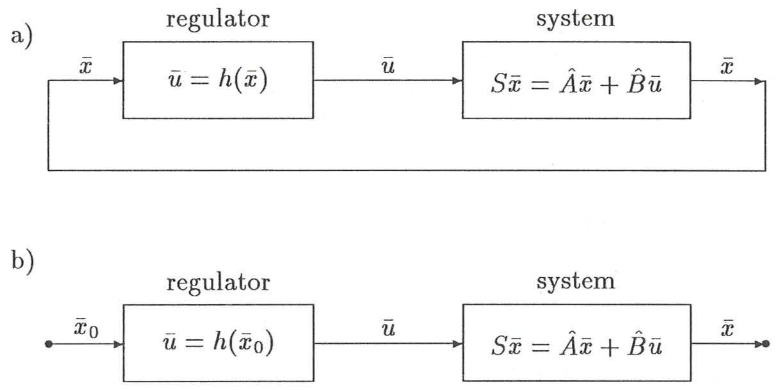

Fig.1 The block scheme of control
a) in the closed system b) in the open system

The function (16) will be called index of performance.

Assume that

$$\bigvee_{\bar{u}^* \in L_m^0} \bigwedge_{\bar{u} \in L_m^0} [\, J(\bar{u}) - J(\bar{u}^*) \in K^0 \,]\,. \tag{17}$$

From (17) it follows that $J(\bar{u}^*)$ is the least element in the set of values of the performance index (16).

The element $\bar{u}^*$ satisfying the condition (17) will be called optimal control. The problem of establishing the dependence

$$\bar{u}^* = h^*(\bar{x}) \quad \text{or} \quad \bar{u}^* = h^*(\bar{x}_0)$$

will be called synthesis of the optimal regulator.

5. The synthesis of the optimal regulator

It is easy to notice that

$$[\,(x, y \in K) \wedge (x + y = 0)\,] \Longrightarrow [\,x = y = 0\,]\,. \tag{18}$$

Suppose also that L is a commutative algebra (with unity e) such that

$$(\,x \in L\,) \Longrightarrow (\,x^2 \in K\,) \tag{19}$$

$$[\,(x \in L) \wedge (x^2 = 0)\,] \Longrightarrow [\,x = 0\,] \tag{20}$$

$$[\,(x \in K) \wedge (y \in Inv(K))\,] \Longrightarrow [\,x + y \in Inv(K)\,]\,, \tag{21}$$

where $Inv(K) := K \cap Inv(L)$.

The elements of the set $Inv(K)$ will be called positive elements.
Using (18)-(20) it is easy to verify that the mapping

$$(\cdot|\cdot) : L_n \times L_n \longrightarrow L \tag{22}$$

given by the formula

$$(\bar{x}|\bar{y}) := \bar{x}^t \cdot \bar{y} = \sum_{i=1}^{n} x_i \cdot y_i$$

has the following properties:
(i.1) $(\bar{x}_1 + \bar{x}_2|\bar{y}) = (\bar{x}_1|\bar{y}) + (\bar{x}_2|\bar{y})$
(i.2) $(\alpha\bar{x}|\bar{y}) = \alpha(\bar{x}|\bar{y})$
(i.3) $(\bar{x}|\bar{y}) = (\bar{y}|\bar{x})$
(i.4) $(\bar{x}|\bar{x}) \in K$, $[\,(\bar{x}|\bar{x}) = 0\,] \Longrightarrow [\,\bar{x} = \bar{0}\,]$,
where $\bar{x}_1, \bar{x}_2, \bar{x}, \bar{y} \in L_n := Mat_{n\times 1}(L), \alpha \in R^1$.

The mapping (22) satisfying the conditions (i.1)–(i.4) is called inner product in L_n (see Wysocki, 1989).

Further we shall assume that the operation $I_{q_0}^{q_1}, q_0, q_1 \in Q$ has the following properties:

$$I_{q_0}^{q_1}(K^0) \subset K^0 \tag{23}$$

$$I^{q_1}_{q_0}(Inv(K^0)) \subset Inv(K^0) \tag{24}$$

$$[\,(x \in K^0) \wedge (I^{q_1}_{q_0} x = 0)\,] \Longrightarrow [\,x = 0\,]\,. \tag{25}$$

From (23) and (25) it follows that

$$(\bar{x}|\bar{y})_1 := I^{q_1}_{q_0}(\bar{x}|\bar{y}) = I^{q_1}_{q_0}(\bar{x}^t \cdot \bar{y})\ ,\ \bar{x}, \bar{y} \in L^0_n$$

is the inner product in L^0_n (see Mieloszyk, 1987).

Consider the problem of synthesis of the optimal regulator for the performance index given by the formula

$$J(\bar{u}) := I^{q_1}_{q_0}(\bar{x}^t \hat{M} \bar{x} + \bar{u}^t \hat{N} \bar{u}) + s_{q_1}(\bar{x}^t \hat{P} \bar{x})\ ,\ q_0 \in Q, q_1 \in \tilde{Q}\ , \tag{26}$$

where $\hat{M} \in Mat_{n\times n}(L^0), \hat{N} \in Mat_{m\times m}(L^0), \hat{P} \in Mat_{n\times n}(KerS)$ are given symmetric matrices such that

$$\bigwedge_{\bar{x}\in L^1_n} (\ \bar{x}^t \hat{M} \bar{x} \in K^0\ ) \tag{27}$$

$$\bigwedge_{\bar{u}\in L^0_m \setminus \{0\}} (\ \bar{u}^t \hat{N} \bar{u} \in Inv(K^0)\ ) \tag{28}$$

$$\bigwedge_{\bar{c}\in (KerS)_n} (\ \bar{c}^t \hat{P} \bar{c} \in K^0\ )\,. \tag{29}$$

The matrices $\hat{M}$ and $\hat{P}$ satisfying the conditions (27) and (29) will be called non-negatively defined matrices in L^1_n and $(KerS)_n$, respectively. The matrix $\hat{N}$ satisfying the condition (28) will be called positively defined in L^0_m.

Corollary 1. *$s_{q_1}(\bar{x}^t \hat{P} \bar{x}) \in K^0$ for each $\bar{x} \in L^1_n$.*

Proof. As $s_{q_1}\bar{x}^t = (s_{q_1}\bar{x})^t$, so from multiplicativity of the operation s_{q_1} and from the property (5) it follows that

$$s_{q_1}(\bar{x}^t \hat{P} \bar{x}) = (s_{q_1}\bar{x})^t \cdot \hat{P} \cdot s_{q_1}\bar{x}\ .$$

Moreover, $s_{q_1}\bar{x} \in (KerS)_n$. Hence and from (29) we obtain the statement. ■

Corollary 2. *$\hat{N} \in Inv(Mat_{m\times m}(L^0))$.*

Proof. From the condition (28) it follows that

$$(\ \hat{N}\bar{u} = \bar{0}\ ) \Longleftrightarrow (\ \bar{u} = \bar{0}\ )\ ,$$

meaning the existence of $\hat{N}^{-1}$. ■

COROLLARY 3. *If* $\bar{u} \in L_m^0 \setminus \{\bar{0}\}$, *then* $J(\bar{u}) \in Inv(K^0)$.

PROOF. This property follows from the conditions (27),(28),(21),(24) and from Corollary 1. ■

LEMMA 1. *If the matrices* $\hat{A} \in Mat_{n\times n}(L^0)$, $\hat{B} \in Mat_{n\times m}(L^0)$, $\hat{R} \in Mat_{n\times n}(L^1)$ *are given, then for the elements* $\bar{u} \in L_n^0, \bar{x} \in L_n^1$ *satisfying the equation*

$$S\bar{x} = \hat{A}\bar{x} + \hat{B}\bar{u} \tag{30}$$

the equality

$$I_{q_0}^{q_1}([\bar{u}^t, \bar{x}^t] \begin{bmatrix} \hat{0} & \vdots & \hat{B}^t\hat{R} \\ \cdots & \cdot & \cdots\cdots\cdots\cdots \\ \hat{R}\hat{B} & \vdots & S\hat{R} + \hat{A}^t\hat{R} + \hat{R}\hat{A} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \cdots \\ \bar{x} \end{bmatrix}) - R_{q_0}^{q_1}(\bar{x}^t\hat{R}\bar{x}) = 0 \tag{31}$$

holds.

PROOF. From the Leibniz formula (6) we have

$$I_{q_0}^{q_1} S(\bar{x}^t\hat{R}\bar{x}) = I_{q_0}^{q_1}(S\bar{x}^t \cdot \hat{R}\bar{x} + \bar{x}^t S\hat{R} \cdot \bar{x} + \bar{x}^t \hat{R} S\bar{x}) .$$

Because

$$S\bar{x}^t = (S\bar{x})^t = \bar{x}^t\hat{A}^t + \bar{u}^t\hat{B}^t ,$$

what follows from (30), so

$$I_{q_0}^{q_1} S(\bar{x}^t\hat{R}\bar{x}) = I_{q_0}^{q_1}\{(\bar{x}^t\hat{A}^t + \bar{u}^t\hat{B}^t)\hat{R}\bar{x} + \bar{x}^t S\hat{R} \cdot \bar{x} + \bar{x}^t\hat{R}(\hat{A}\bar{x} + \hat{B}\bar{u})\} . \tag{32}$$

From the Leibniz-Newton formula (2) it follows that

$$I_{q_0}^{q_1} S(\bar{x}^t\hat{R}\bar{x}) - R_{q_0}^{q_1}(\bar{x}^t\hat{R}\bar{x}) = 0 .$$

Hence and from (32) we obtain the expanded form of the formula (31). ■

The abstract matrix differential equation

$$S\hat{R} = -\hat{A}^t\hat{R} - \hat{R}\hat{A} + \hat{R}\hat{B}\hat{N}^{-1}\hat{B}^t\hat{R} - \hat{M} \tag{33}$$

will be called Riccati equation.

Assume that the equation (33) with the limit condition

$$s_{q_1}\hat{R} = \hat{P} \tag{34}$$

has a unique solution.

COROLLARY 4. *The solution* $\hat{R}$ *of the problem* (33),(34) *is a symmetric matrix.*

PROOF. It is easy to notice that the matrix $\hat{B}\hat{N}^{-1}\hat{B}^t$ is symmetric. Therefore from (33) we obtain

$$(S\hat{R})^t = S\hat{R}^t = -\hat{A}^t\hat{R}^t - \hat{R}^t\hat{A} + \hat{R}^t\hat{B}\hat{N}^{-1}\hat{B}^t\hat{R}^t - \hat{M} \ . \tag{35}$$

Moreover,

$$s_{q_1}\hat{R}^t = (s_{q_1}\hat{R})^t = \hat{P}^t = \hat{P} \ . \tag{36}$$

From (35) and (36) it follows that matrices $\hat{R}$ and $\hat{R}^t$ are the solutions of the same differential equation (33) with the same limit condition (34). From the uniqueness of the solution of that problem it follows that $\hat{R} = \hat{R}^t$. ■

THEOREM 4. *Assume that the generalized dynamic system with the state equation*

$$S\bar{x} = \hat{A}\bar{x} + \hat{B}\bar{u}$$

and the limit condition

$$s_{q_0}\bar{x} = \bar{x}_0 \ , \quad q_0 \in Q \ , \ \bar{x}_0 \in (KerS)_n$$

is given. There exists the uniquely defined optimal control $\bar{u}^*$ represented by the formula:
– in the closed system

$$\bar{u}^* = -\hat{N}^{-1}\hat{B}^t\hat{R}\bar{x} \ , \tag{37}$$

if $\bar{u}^* \in L_m^0 \setminus \{\bar{0}\}$.
– in the open system

$$\bar{u}^* = -\hat{N}^{-1}\hat{B}^t\hat{R}\hat{\Phi}_{q_0}(\hat{A} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{R})\bar{x}_0 \ , \tag{38}$$

if $q_0 \in \tilde{Q}, \bar{u}^* \in L_m^0 \setminus \{\bar{0}\}$ and the element $\hat{\Phi}_{q_0}(\hat{A} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{R})$ exists, where $\hat{R}$ is the solution of the Riccati equation (33) *with the limit condition* (34).
Moreover,

$$J(\bar{u}^*) = s_{q_0}(\bar{x}^t\hat{R}\bar{x}) \ , \ if \ \ q_0 \in Q \tag{39}$$

$$J(\bar{u}^*) = \bar{x}_0^t s_{q_0}\hat{R} \cdot \bar{x}_0 \ , \ if \ \ q_0 \in \tilde{Q} \ . \tag{40}$$

PROOF. Suppose that $\bar{u} \in L_m^0 \setminus \{\bar{0}\}$. Representing the performance index (26) in the vector-matrix form and availing of Lemma 1 we obtain

$$J(\bar{u}) = I_{q_0}^{q_1}([\bar{u}^t, \bar{x}^t] \begin{bmatrix} \hat{N} & \vdots & \hat{0} \\ \cdots & \cdot & \cdots \\ \hat{0} & \vdots & \hat{M} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \cdots \\ \bar{x} \end{bmatrix}) + s_{q_1}(\bar{x}^t\hat{P}\bar{x}) +$$

$$+I_{q_0}^{q_1}([\bar{u}^t, \bar{x}^t] \begin{bmatrix} \hat{0} & \vdots & \hat{B}^t\hat{R} \\ \cdots & \cdot & \cdots\cdots\cdots \\ \hat{R}\hat{B} & \vdots & S\hat{R} + \hat{A}^t\hat{R} + \hat{R}\hat{A} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \cdots \\ \bar{x} \end{bmatrix}) - R_{q_0}^{q_1}(\bar{x}^t\hat{R}\bar{x}) \, .$$

On the basis of the multiplicativity of s_{q_1}, the formulas (3),(5) and

$$s_{q_1}\hat{R} = \hat{P} \ , \ S\hat{R} + \hat{A}^t\hat{R} + \hat{R}\hat{A} + \hat{M} = \hat{R}\hat{B}\hat{N}^{-1}\hat{B}^t\hat{R}$$

we obtain

$$J(\bar{u}) = I_{q_0}^{q_1}([\bar{u}^t, \bar{x}^t] \begin{bmatrix} \hat{N} & \vdots & \hat{B}^t\hat{R} \\ \cdots & \cdot & \cdots\cdots\cdots \\ \hat{R}\hat{B} & \vdots & \hat{R}\hat{B}\hat{N}^{-1}\hat{B}^t\hat{R} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \cdots \\ \bar{x} \end{bmatrix}) + s_{q_0}(\bar{x}^t\hat{R}\bar{x}) \, .$$

Using Corollary 4 it is not difficult to verify that the last equality takes the form

$$J(\bar{u}) = I_{q_0}^{q_1}[(\bar{u} + \hat{N}^{-1}\hat{B}^t\hat{R}\bar{x})^t\hat{N}(\bar{u} + \hat{N}^{-1}\hat{B}^t\hat{R}\bar{x})] + s_{q_0}(\bar{x}^t\hat{R}\bar{x}) \, .$$

If $\bar{u} + \hat{N}^{-1}\hat{B}^t\hat{R}\bar{x} \in L_m^0 \setminus \{\bar{0}\}$, then the first addend of that expression is the positive element, what follows from (28) and (24). Therefore the condition (17) will be satisfied for $\bar{u}^* \in L_m^0 \setminus \{\bar{0}\}$ such that

$$J(\bar{u}^*) = s_{q_0}(\bar{x}^t\hat{R}\bar{x}) \, . \tag{41}$$

From Corollary 3 we have $J(\bar{u}^*) \in Inv(K^0)$. The element $J(\bar{u}^*)$ has the form (41) if and only if the optimal control in the closed system is represented by the formula

$$\bar{u}^* = -\hat{N}^{-1}\hat{B}^t\hat{R}\bar{x} \, . \tag{42}$$

Assume that $q_0 \in \tilde{Q}$. Then

$$J(\bar{u}^*) = \bar{x}_0^t s_{q_0}\hat{R} \cdot \bar{x}_0 \, ,$$

as the operation s_{q_0} is multiplicative, $s_{q_0}\bar{x}^t = (s_{q_0}\bar{x})^t$ and $s_{q_0}\bar{x} = \bar{x}_0$. Putting the element (42) into the equation $S\bar{x} = \hat{A}\bar{x} + \hat{B}\bar{u}$ we obtain the optimal state equation

$$S\bar{x} = (\hat{A} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{R})\bar{x} \, .$$

Assume that there exists the normalized fundamental matrix $\hat{\Phi}_{q_0}(\hat{A} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{R})$. Then on the basis of Theorem 2 we get

$$\bar{x}^* = \hat{\Phi}_{q_0}(\hat{A} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{R})\bar{x}_0 \, .$$

Putting $\bar{x}^*$ into (42) we obtain the form (38) of the optimal control in the open system. ∎

The index of performance corresponding to the generalized dynamic system with the state equation

$$S\bar{x} = \hat{A}\bar{x} + \hat{B}\bar{u}$$

and the output equation

$$\bar{y} = \hat{C}\bar{x}$$

is determined by the formula

$$J(\bar{u}) := I_{q_0}^{q_1}(\bar{y}^t\hat{F}\bar{y} + \bar{u}^t\hat{N}\bar{u}) + s_{q_1}(\bar{x}^t\hat{P}\bar{x}) \ , \ q_0 \in Q \ , \ q_1 \in \tilde{Q} \ ,$$

where the matrix $\hat{F} \in Mat_{r\times r}(L^0)$ is non-negatively defined in L_r^0 and $\hat{N}, \hat{P}$ are matrices defined as previously. This performance index takes the form (26) if we admit $\hat{M} := \hat{C}^t\hat{F}\hat{C}$.

6. The explicit form of solution of the Riccati equation

A. In the case when $\hat{M} = \hat{0}$, i.e. when the performance index (26) is represented by the formula

$$J(\bar{u}) = I_{q_0}^{q_1}(\bar{u}^t\hat{N}\bar{u}) + s_{q_1}(\bar{x}^t\hat{P}\bar{x})$$

we are able to obtain the solution of the Riccati equation in an explicit form. For $\hat{M} = \hat{0}$ the matrix $\hat{R}$ is the solution of the problem

$$S\hat{R} = -\hat{A}^t\hat{R} - \hat{R}\hat{A} + \hat{R}\hat{B}\hat{N}^{-1}\hat{B}^t\hat{R} \tag{43}$$

$$s_{q_1}\hat{R} = \hat{P} \ . \tag{44}$$

Assume that $\hat{R} \in Inv(Mat_{n\times n}(L^1))$.Then $\hat{P} \in Inv(Mat_{n\times n}(KerS))$.Moreover, from (43) we obtain

$$-\hat{R}^{-1}S\hat{R}\cdot\hat{R}^{-1} = \hat{R}^{-1}\hat{A}^t + \hat{A}\hat{R}^{-1} - \hat{B}\hat{N}^{-1}\hat{B}^t \ .$$

Because $S\hat{R}^{-1} = -\hat{R}^{-1}S\hat{R}\cdot\hat{R}^{-1}$, so

$$S\hat{R}^{-1} = \hat{R}^{-1}\hat{A}^t + \hat{A}\hat{R}^{-1} - \hat{B}\hat{N}^{-1}\hat{B}^t \ . \tag{45}$$

We also have

$$s_{q_1}\hat{R}^{-1} = \hat{P}^{-1} \tag{46}$$

what follows from (9). Applying Theorem 3 we obtain the solution $\hat{R}^{-1} = \hat{H}$ of the problem (45),(46). Hence $\hat{R} = \hat{H}^{-1}$.

COROLLARY 5 (cf Th.3 Bittner, 1974). *If $\hat{A} = \hat{0}$, then $\hat{R}^{-1} = \hat{P}^{-1} - T_{q_1}(\hat{B}\hat{N}^{-1}\hat{B}^t)$.*

B. The abstract differential equations

$$S\bar{x} = \hat{A}\bar{x} - \hat{B}\hat{N}^{-1}\hat{B}^t\bar{\psi}$$
$$S\bar{\psi} = -\hat{M}\bar{x} - \hat{A}^t\bar{\psi} \ , \quad \bar{x}, \bar{\psi} \in L^1_n \tag{47}$$

will be called canonical Hamilton equations corresponding to the generalized dynamic system with the state equation (12) and the performance index (26). In the vector-matrix notation the system (47) takes the form

$$S\begin{bmatrix} \bar{x} \\ \cdots \\ \bar{\psi} \end{bmatrix} = \begin{bmatrix} \hat{A} & \vdots & -\hat{B}\hat{N}^{-1}\hat{B}^t \\ \cdots & \cdot & \cdots\cdots\cdots \\ -\hat{M} & \vdots & -\hat{A}^t \end{bmatrix} \begin{bmatrix} \bar{x} \\ \cdots \\ \bar{\psi} \end{bmatrix} .$$

Let

$$\hat{\Phi}_{q_1}(\hat{G}) = \begin{bmatrix} \hat{\Phi}_{11} & \vdots & \hat{\Phi}_{12} \\ \cdots & \cdot & \cdots \\ \hat{\Phi}_{21} & \vdots & \hat{\Phi}_{22} \end{bmatrix}$$

be the normalized fundamental matrix corresponding to the matrix

$$\hat{G} = \begin{bmatrix} \hat{A} & \vdots & -\hat{B}\hat{N}^{-1}\hat{B}^t \\ \cdots & \cdot & \cdots\cdots\cdots \\ -\hat{M} & \vdots & -\hat{A}^t \end{bmatrix} . \tag{48}$$

Therefore $S\hat{\Phi}_{q_1}(\hat{G}) = \hat{G}\hat{\Phi}_{q_1}(\hat{G})$ and

$$s_{q_1}\hat{\Phi}_{ij} = \delta_{ij}\hat{E} \ , \quad i,j = 1,2 . \tag{49}$$

THEOREM 5. *If* $\hat{\Phi}_{11} + \hat{\Phi}_{12}\hat{P} \in Inv(Mat_{n\times n}(L^1))$, *then the matrix*

$$\hat{R} := (\hat{\Phi}_{21} + \hat{\Phi}_{22}\hat{P})(\hat{\Phi}_{11} + \hat{\Phi}_{12}\hat{P})^{-1} \tag{50}$$

is the solution of the Riccati equation (33) *with the limit condition* (34).

PROOF. As the operation s_{q_1} is multiplicative then from the properties (5),(9) and (49) we obtain $s_{q_1}\hat{R} = \hat{P}$, i.e. the limit condition (34).

Define $\hat{X}, \hat{\Psi} \in Mat_{n\times n}(L^1)$ as follows:

$$\hat{X} := \hat{\Phi}_{11} + \hat{\Phi}_{12}\hat{P} \ , \quad \hat{\Psi} := \hat{\Phi}_{21} + \hat{\Phi}_{22}\hat{P} .$$

Therefore

$$\begin{bmatrix} \hat{X} \\ \cdots \\ \hat{\Psi} \end{bmatrix} = \hat{\Phi}_{q_1}(\hat{G}) \cdot \begin{bmatrix} \hat{E} \\ \cdots \\ \hat{P} \end{bmatrix}$$

and

$$S\begin{bmatrix} \hat{X} \\ \cdots \\ \hat{\Psi} \end{bmatrix} = \hat{G}\cdot\begin{bmatrix} \hat{X} \\ \cdots \\ \hat{\Psi} \end{bmatrix} .$$

Hence

$$\begin{aligned} S\hat{X} &= \hat{A}\hat{X} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{\Psi} \\ S\hat{\Psi} &= -\hat{M}\hat{X} - \hat{A}^t\hat{\Psi} . \end{aligned}$$

As $\hat{R} = \hat{\Psi}\hat{X}^{-1}$, so from the last equalities we obtain

$$\begin{aligned} S\hat{R} &= S\hat{\Psi}\cdot\hat{X}^{-1} + \hat{\Psi}S\hat{X}^{-1} = S\hat{\Psi}\cdot\hat{X}^{-1} - \hat{\Psi}\hat{X}^{-1}S\hat{X}\cdot\hat{X}^{-1} = \\ &= (-\hat{M}\hat{X} - \hat{A}^t\hat{\Psi})\hat{X}^{-1} - \hat{\Psi}\hat{X}^{-1}(\hat{A}\hat{X} - \hat{B}\hat{N}^{-1}\hat{B}^t\hat{\Psi})\hat{X}^{-1} = \\ &= -\hat{M} - \hat{A}^t\hat{R} - \hat{R}\hat{A} + \hat{R}\hat{B}\hat{N}^{-1}\hat{B}^t\hat{R} \end{aligned}$$

what means that the matrix $\hat{R}$, defined by the formula (50), is the solution of the Riccati equation (33). ∎

7. Examples

A. Let

$$L^0 := C^0(Q, R^1) ,\ L^1 := C^1(Q, R^1)$$

and

$$S := \frac{d}{dt} ,\ T_q := \int_q^t ,\ s_q := |_{t=q} ,$$

where $q \in Q := [t_0, t_k] \subset R^1$.
With the usual multiplication of functions, the spaces L^0, L^1 are commutative algebras with unity $e = \{1\}$, the derivative S satisfies the Leibniz condition and the operations s_q are multiplicative.
With the cone

$$K := \{f \in L^0 : f(t) \geq 0, t \in Q\}$$

and the modulus

$$|f| := \{|f(t)|\} ,\ f = \{f(t)\} \in L^0$$

$L := L^0$ is the Mikusiński space.
In the considered model of the operational calculus the equation (12) is the state equation of a nonstationary dynamic system with compensating constants, i.e.

$$\bar{x}'(t) = \hat{A}(t)\bar{x}(t) + \hat{B}(t)\bar{u}(t) .$$

The performance index (26) (for $q_0 = t_0, q_1 = t_k$) takes the form

$$J(\bar{u}) = \int_{t_0}^{t_k} [\bar{x}^t(\tau)\hat{M}(\tau)\bar{x}(\tau) + \bar{u}^t(\tau)\hat{N}(\tau)\bar{u}(\tau)]d\tau + \bar{x}^t(t_k)\hat{P}\bar{x}(t_k) .$$

The formulated classical problem of synthesis of the optimal regulator can be solved by means of the Pontryagin maximum principle or the Bellman dynamic programming. In Kalman, Falb, Arbib (1969) that problem has been solved by elementary methods using the linearity of the system and the elementary properties of quadratic forms. The solution described in this work has been obtained in an elementary way by reference to the operational calculus in algebras and the basic properties of partially ordered spaces. Moreover, the obtained general method can be used in other models of the operational calculus.

B. Consider the generalized dynamic system

$$S^2 y = u \tag{51}$$

with the limit conditions

$$s_{q_0} y = c_0 \ , \ s_{q_0} S y = c_1 \ , \tag{52}$$

where $y \in L^2 := \{y \in L^1 : Sy \in L^1\}$, $u \in L^0$, $c_0, c_1 \in KerS$, $q_0 \in Q$. Admitting

$$x_1 := y \ , \ x_2 := Sy \tag{53}$$

the problem (51),(52) can be represented in the vector-matrix form

$$S\bar{x} = \hat{A}\bar{x} + \hat{B}u \ , \ y = \hat{C}\bar{x} \ , \ s_{q_0}\bar{x} = \bar{x}_0 \ , \tag{54}$$

where

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \ \hat{A} = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} , \ \hat{B} = \begin{bmatrix} 0 \\ e \end{bmatrix} , \ \hat{C} = [e, 0] \ , \ \bar{x}_0 = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \tag{55}$$

and e is a unity in an algebra L^0.

The non-homogeneous differential Euler equation of second order

$$t^2\ddot{y} + t\dot{y} = u \tag{56}$$

with initial conditions

$$y(1) = d_0 \ , \ \dot{y}(1) = d_1 \tag{57}$$

can be reduced into the form (54) if we consider an operational calculus in which

$$L^0 := C^0(Q, R^1) \ , \ L^1 := C^1(Q, R^1)$$

and

$$Sy := \{t\frac{dy}{dt}\}\ ,\ T_q f := \{\int_q^t \frac{f(\tau)}{\tau}d\tau\}\ ,\ s_q y := \{y(q)\}\ ,$$

where $f = \{f(t)\} \in L^0, y = \{y(t)\} \in L^1, q \in Q := [1,2]$.
Then the initial conditions (57) determine the limit conditions (52) with $q_0 = 1$. Namely,

$$c_0 = \{d_0\}\ ,\ c_1 = \{d_1\}\ .$$

Assume that the algebras L^0, L^1 and the Mikusiński space L are defined as in Example A. Then the derivative S satisfies the Leibniz condition and the operations s_q are multiplicative.

Let us consider the problem of synthesis of the optimal regulator for the problem (56),(57) with the index of performance

$$J(u) = \int_1^2 \frac{1}{\tau}u^2(\tau)d\tau + y^2(2)\ . \tag{58}$$

Taking into consideration the matrix representation (54),(55) of this problem and the form of integrals T_q we have here $q_0 = 1, q_1 = 2$ and

$$\hat{M} = \hat{0}\ ,\ \hat{N} = [1]\ ,\ \hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ .$$

It is not difficult to check that

$$\hat{\Phi}_2(\hat{G}) = \hat{\Phi}(t) = \left[\begin{array}{cc:cc} 1 & \ln\frac{t}{2} & \frac{1}{6}\ln^3\frac{t}{2} & -\frac{1}{2}\ln^2\frac{t}{2} \\ 0 & 1 & \frac{1}{2}\ln^2\frac{t}{2} & -\ln\frac{t}{2} \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 0 & -\ln\frac{t}{2} & 1 \end{array}\right]$$

is the normalized matrix, i.e.

$$t\frac{d}{dt}\hat{\Phi}(t) = \hat{G}\hat{\Phi}(t)\ ,\ \hat{\Phi}(2) = \hat{E}\ ,$$

where $\hat{G}$ is the matrix of the form (48) corresponding to our problem. Moreover, $\hat{\Phi}_{11} + \hat{\Phi}_{12}\hat{P}$ is a non-singular matrix in the interval $[1,2]$. Therefore from the formula (50) we have

$$\hat{R} = \frac{3}{3 - \ln^3\frac{t}{2}}\begin{bmatrix} 1 & -\ln\frac{t}{2} \\ -\ln\frac{t}{2} & \ln^2\frac{t}{2} \end{bmatrix}\ .$$

Using the formula (37) we obtain

$$u^* = \frac{3}{3-\ln^3\frac{t}{2}}(x_1\ln\frac{t}{2} - x_2\ln^2\frac{t}{2}),$$

i.e. the optimal control depended on the state vector $\bar{x}$. Taking into consideration the form of the derivative S and the formulas (53) we have

$$u^* = \frac{3}{3-\ln^3\frac{t}{2}}(y\ln\frac{t}{2} - t\dot{y}\ln^2\frac{t}{2}). \tag{59}$$

Putting (59) into (56) we obtain the equation of optimal trajectory of the system

$$t^2\ddot{y} + (t + \frac{3t\ln^2\frac{t}{2}}{3-\ln^3\frac{t}{2}})\dot{y} - \frac{3\ln\frac{t}{2}}{3-\ln^3\frac{t}{2}}y = 0 \tag{60}$$

with initial conditions (57). It is easy to prove that the function

$$y^* = a_1\ln\frac{t}{2} - a_2(6+\ln^3\frac{t}{2}),$$

where

$$a_1 = \frac{6d_1 - 3d_0\ln^2 2 - d_1\ln^3 2}{6+2\ln^3 2}, \quad a_2 = -\frac{d_0 + d_1\ln 2}{6+2\ln^3 2},$$

is the solution of the problem (60),(57). Putting this function and its derivative into (59) we obtain

$$u^* = -6a_2\ln\frac{t}{2}, \tag{61}$$

i.e. the explicit form of the optimal control of the system (56),(57) minimizing the performance index (58). Using the formula (40) or putting directly the function (61) and $y^*(2) = -6a_2$ into the functional (58) we obtain

$$J(u^*) = 12a_2^2(3+\ln^3 2) = \frac{3(d_0 + d_1\ln 2)^2}{3+\ln^3 2}.$$

C. Let a distributed parameters system be given the dynamics of which is described by the partial differential equation

$$\frac{\partial x}{\partial z} + \frac{\partial x}{\partial t} = u(z,t). \tag{62}$$

Moreover, let

$$x(z,0) = \varphi(z), \quad \varphi \in C^2(R^1, R^1). \tag{63}$$

Assume that we are to find such functions $u^*(z,t), x^*(z,t)$, corresponding to the system (62),(63), for which the functional

$$J[u(z_0,t_0)] = \int_0^1 u^2(z_0 - t_0 + \tau, \tau)d\tau + x^2(z_0 - t_0 + 1, 1) \tag{64}$$

attains its minimum at a given point $(z_0,t_0) \in R^1 \times [0,1]$.

This problem can be solved as the problem of synthesis of the optimal regulator in a given point of the surface $R^1 \times [0,1]$ if we consider the operational calculus with the derivative

$$Sx := \left\{ \frac{\partial x}{\partial z} + \frac{\partial x}{\partial t} \right\},$$

the integrals

$$T_q f := \left\{ \int_q^t f(z - t + \tau, \tau) d\tau \right\}$$

and the limit conditions

$$s_q x := \{x(z - t + q, q)\},$$

where

$$f = \{f(z,t)\} \in L^0 := C^1(R^1 \times [0,1], R^1),$$

$$x = \{x(z,t)\} \in L^1 := \{x \in L^0 : Sx \in L^0\}, \; q \in Q := [0,1]$$

(see Bittner, Mieloszyk, 1982) Then with the usual multiplication of functions of two variables, the spaces L^0, L^1 are commutative algebras with unity $e = \{1\}$, the derivative S satisfies the Leibniz condition and the operations s_q are multiplicative.

With the cone

$$K := \{ f \in C^0(R^1 \times [0,1], R^1) : \; f(z,t) \geq 0, \; (z,t) \in R^1 \times [0,1] \}$$

and the modulus

$$|f| := \{|f(z,t)|\}, \; f = \{f(z,t)\} \in C^0(R^1 \times [0,1], R^1)$$

$L := C^0(R^1 \times [0,1], R^1)$ is the Mikusiński space such that $L^0 \subset L$.

In the considered model of the operational calculus the system (62) takes the operational form $Sx = u, y = x$ whence it follows that $\hat{A} = [0]$, $\hat{B} = [1]$, $\hat{C} = [1]$, whereas the Cauchy condition (63) determines the limit state of the system with $q_0 = 0$. Namely,

$$s_{q_0} x = x_0 = \{\varphi(z - t)\}. \tag{65}$$

For $\hat{M} = [0], \hat{N} = [1], \hat{P} = [1]$ and $q_1 = 1$ the performance index (26) takes the form (64) in every point $(z_0, t_0) \in R^1 \times [0, 1]$.
Using Corollary 5 we have

$$R^{-1} = 1 - \int_1^t d\tau = 2 - t \, .$$

Hence and from (37) we obtain the form of the optimal control in the closed system

$$u^* = -\frac{1}{2-t} x(z,t) \, . \tag{66}$$

Therefore the optimal state variable satisfies the partial equation

$$\frac{\partial x}{\partial z} + \frac{\partial x}{\partial t} = -\frac{1}{2-t} x(z,t) \tag{67}$$

and the Cauchy condition (63) which induces the limit condition (65).
As

$$\Phi_{q_0}(a) = \exp[\int_{q_0}^t a(z - t + \tau, \tau) d\tau]$$

is the normalized fundamental function corresponding to the function $a = \{a(z,t)\} \in L^0$, i.e. $S\Phi_{q_0}(a) = a\Phi_{q_0}(a), s_{q_0}\Phi_{q_0}(a) = 1$ (see, Mieloszyk 1987), so from Theorem 2 the solution of the problem (67),(63) takes the form

$$x^* = \exp[-\int_0^t \frac{d\tau}{2-\tau}] \cdot \varphi(z-t) = \frac{2-t}{2}\varphi(z-t) \, .$$

Putting this function in (66) we obtain

$$u^* = -\frac{1}{2}\varphi(z-t) \, ,$$

i.e. the explicit form of the optimal control of the system (62),(63).
Moreover, from (40) we get

$$J[u^*(z_0, t_0)] = \frac{1}{2}\varphi^2(z_0 - t_0) \, ,$$

where $(z_0, t_0) \in R^1 \times [0, 1]$ is an arbitrary point.

8. Conclusions

The Bittner operational calculus has been applied to give an algebraic description of a group of problems known as the optimal regulator synthesis. In this approach we generalize the ideas from Kalman, Falb, Arbib (1969). This has led to the solutions which cover some particular cases as Examples A and B, solvable in the classical way (cf Kalman, Falb, Arbib, 1969; Kwakernaak, Sivan, 1972). Moreover, in a unified form we can cover some cases of the optimal regulator synthesis for systems described by partial differential equations (see Example C).

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