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Approximation of some optimization problems for systems governed by nonlinear parabolic equation

by

Anna Dębińska-Nagórska, Andrzej Just, Zdzisław Stempień

Institute of Mathematics,
Technical University of Łódź
Al. Politechniki 11
90-024 Łódź
Poland

The paper presents the Galerkin approximation of optimization problems of system governed by nonlinear partial differential equation and its convergence.

Key words: Optimal control, Galerkin approximation.

1. Introduction

We shall consider the Sobolev spaces $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V^* = H^{-1}(\Omega)$, $Y = L^2(S;V)$, $Y^* = L^2(S;V^*)$, $S = (0,T)$ for $0 < T < \infty$, where $\Omega \subset \mathbf{R}^n$ is a sufficiently regular set, Lions (1969).

We shall be concerned with the nonlinear parabolic differential equation:

$$\frac{dy}{dt} + Ay - f(y) = v \text{ on } Q = (0,T) \times \Omega \tag{1}$$

with the initial condition

$$y(0) = y_0 \tag{2}$$

where $A : Y \to Y^*$ is defined as

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial y}{\partial x_j}) + a_0(x)y$$

$a_0, a_{ij} \in L^\infty(\Omega)$ for $i,j = 1,2,\ldots,n, v \in Y*$.

We assume that:

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2 \quad \forall \xi_i, \xi_j \in \mathcal{R} \tag{3}$$

$a_0(x) \geq \alpha$ for a certain $\alpha > 0$

$f : Y \longrightarrow Y^*$ is Volterra operator (Gajewski, Gröger, Zacharias, 1974), $f(0) = 0$ and

$$||f(y_1) - f(y_2)||_{Y^*} \leq \beta ||y_1 - y_2||_Y \ \forall y_1, y_2 \in Y \tag{4}$$

for a certain $\beta > 0$.

THEOREM 1.1 *Let the assumptions (3) and (4) be satisfied. If $\alpha > \beta$ then for each $y_0 \in H, v \in Y^*$ there exists a unique $y \in W = \{\omega | \omega \in Y \wedge \frac{d\omega}{dt} \in Y^*\}$ (with a norm $||\omega||_W = ||\omega|| + ||\omega\prime||_{Y^*}$), which is the solution of the problem (1-2). Moreover, this solution is continuously depending on (y_0, v) from $H \times Y^*$ to W.*

Proof. Let $y_1, y_2 \in Y$; then from 3 and 4 we have

$$\langle (A-f)y_1 - (A-f)y_2, y_1 - y_2 \rangle = \langle A(y_1 - y_2), y_1 - y_2 \rangle$$

$$-\langle f(y_1) - f(y_2), y_1 - y_2 \rangle \geq (\alpha - \beta)||y_1 - y_2||_Y^2$$

($\langle \cdot, \cdot \rangle$ denotes the pairing between appropriate Sobolev space and its dual).

It follows from this inequality that the operator $A - f$ is strongly monotone and coercive (Deimling, 1985). Because it is continuous, too (from Y to Y^*) then the problem (1-2) has a unique solution $y \in W$ (Gajewski, Gröger, Zacharias, 1974).

Now we can check that the operation $(y_0, v) \longrightarrow y$ from $H \times Y^*$ to W is continuous.

Equation (1) can be presented in the following form:

$$\langle \frac{dy(t)}{dt}, z \rangle + \langle (Ay)(t), z \rangle - \langle (f(y)(t), z \rangle = \langle v(t), z \rangle \tag{5}$$

$\forall z \in V$ and a.a. $t \in S$.

We put in (5) $z = y(t)$ and we obtain the equality:

$$\frac{1}{2}\frac{d}{dt}||y(t)||_H^2 + \langle (Ay)(t), y(t) \rangle = \langle (f(y))(t), y(t) \rangle + \langle v(t), y(t) \rangle$$

Using assumptions (3-4) we have

$$\frac{d}{dt}||y(t)||_H^2 + (\alpha - \beta)||y(t)||_V^2 \leq \frac{1}{\alpha - \beta}||v(t)||_{V^*}^2$$

and by integration in $[0, T]$:

$$||y||_Y^2 \leq c(||y_0||_H^2 + ||v||_{Y^*}^2) \tag{6}$$

or

$$||y||_Y \leq \sqrt{c}(||y_0||_H + ||v||_{Y^*}) \tag{7}$$

for a certain constant $c > 0$.

From (1) and these inequalities follows, Lions (1983):

$$\frac{dy}{dt} = v + f(y) - Ay \in Y^*$$

Furthermore, from (3,4) and (6) we obtain

$$||y'||_{Y^*} \leq c_1(||y_0||_H + ||v||_{Y^*}) \tag{8}$$

for a certain constant $c_1 > 0$.

From (7) and (8) it follows that the solution of problem (1-2) continuously depends on y_0 and v.

LEMMA 1.1 *Let the assumptions (3) and (4) be satisfied for $\alpha > \beta$, let $y_0 \in H$ and $v \in Y^*$. Let $f : Y \to Y^*$ be demicontinuous on Y with the norm from $L^2(S; H)$. Let $(v_n)_{n \in \mathcal{N}}$ be a sequence of elements in Y^* and $(y_n)_{n \in \mathcal{N}}$a sequence of solutions of (1-2) for $v_n \in Y^*$.*

If $v_n \to_{n\to\infty} \bar{v}$ weakly in Y^*, then $y_n \to_{n\to\infty} \bar{y}$ weakly in W and $y_n \to_{n\to\infty} \bar{y}$ strongly $L^2(Q)$ where $\overline{y}$ is the unique solution of problem (1-2) for $\overline{v}$.

Proof. From Theorem 1.1 we know that the equation

$$\begin{aligned}\langle \frac{dy_n(t)}{dt}, z\rangle + \langle (Ay_n)(t), z\rangle - \langle (f(y_n))(t), z\rangle = \\ = \langle v_n(t), z\rangle \forall z \in V\end{aligned} \tag{9}$$

with the initial condition

$$y_n(0) = y_0 \tag{10}$$

has for each $n \in \mathbf{N}$ exactly one solution $y_n = y(v_n) \in W$. We take in (9) $z = y_n(t) \in V$ and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||y_n(t)||_H^2 + \langle (Ay_n)(t), y_n(t)\rangle - \langle (f(y_n))(t), y_n(t)\rangle = \\ = \langle v_n(t), y_n(t)\rangle \text{ for a.a. } t \in [0, T].\end{aligned}$$

From (3-4) and by integration in $[0,T]$ we have, Malanowski(1975)

$$||y_n(T)||_H^2 + (\alpha - \beta)||y_n||_Y^2 \leq ||y_0||_H^2 + \frac{1}{\alpha - \beta}||v_n||_{Y^*}^2.$$

From this inequality and from the assumption $v_n \rightarrow_{n\rightarrow\infty} \overline{v}$ in Y^* we see that

$$||y_n||_Y \leq c_1 \tag{11}$$

for a certain $c_1 > 0$.

From (9) we have directly:

$$y_n\prime = v_n + f(y_n) - Ay_n \in Y^*$$

Hence, it is clear that from (3,4) and (11) we have the estimation

$$||y_n||_W \leq c_2 \text{ for a certain } c_2 > 0.$$

It follows that the sequence $(y_n)_{n\in\mathbf{N}}$ is bounded in W and thus there exists a subsequence, which we also denote $(y_n)_{n\in\mathbf{N}}$converging to an element $\overline{y}$ weakly in W, so, Lions (1969), strongly in $L^2(Q)$:

$$y_n \rightarrow_{n\rightarrow\infty} \overline{y} \text{ weakly in } W$$

$$y_n \rightarrow_{n\rightarrow\infty} \overline{y} \text{ strongly in } L^2(Q).$$

Now we can prove that $\overline{y}$ is the solution of problem (1-2) for $\overline{v}$.

Multiplying equation (9) by an arbitrary function $\varphi \in C^1([0,T])$ which satisfies $\varphi(T) = 0$ and integrating with the integration-by-parts theorem for Bochner integral over the domain $[0,T]$ we obtain

$$\begin{aligned}-\int_0^T \langle y_n(t), z\rangle\varphi'(t)dt + \int_0^T \langle (Ay_n)(t), z\rangle\varphi(t)dt + \\ -\int_0^T \langle (f(y_n)(t), z\rangle\varphi(t)dt = \int_0^T \langle v_n(t), z\rangle\varphi(t)dt + \langle y_0, z\rangle\varphi(0)\end{aligned}$$

Now we are able to pass to the limit with $n \rightarrow \infty$:

$$\begin{aligned}-\int_0^T \langle \overline{y}(t), z\rangle\varphi'(t)dt + \int_0^T \langle (A\overline{y})(t), z\rangle\varphi(t) - \\ -\int_0^T \langle (f(\overline{y})(t), z\rangle\varphi(t)dt = \\ = \int_0^T \langle \overline{v}(t), z\rangle\varphi(t)dt + \langle y_0, z\rangle\varphi(0).\end{aligned} \tag{12}$$

This equation is verified for any $\varphi \in C^1([0,T]), \varphi(T) = 0$.

This implies that from definition of distributional derivate, for $\varphi \in D(0,T)$ too we obtain:

$$\begin{aligned} \int_0^T \langle \overline{y}'(t), z\rangle dt + \int_0^T \langle (A\overline{y})(t), z\rangle dt- \\ - \int_0^T \langle (f(\overline{y}))(t), z\rangle dt = \\ = \int_0^T \langle \overline{v}(t), z\rangle dt \end{aligned} \tag{13}$$

Hence $\overline{y}$ verifies equation (1) for $v = \overline{v}$.

By theorem of integration by parts for Bochner integrals from (12) and (13), $\varphi \in C^1([0,T])$ being arbitrary we can conclude that $\overline{y}(0) = y_0$.

So $\overline{y}$ is a solution of equation (1-2) for $\overline{v}$.

From the fact that there is only one solution of (1-2) we deduce that not only the subsequence but the whole sequence $(y_n)_{n\in\mathbf{N}}$ converges weakly to $\overline{y} = y(\overline{v})$ in W and strongly in $L^2(Q)$. ■

LEMMA 1.2 *Let the assumptions of Lemma 1.1 be satisfied. If* $v_n \rightarrow_{n\rightarrow\infty} \overline{v}$ *strongly in* Y^* *then* $y_n \rightarrow_{n\rightarrow\infty} \overline{y}$ *strongly in* Y *where* $\overline{y}$ *is the unique solution of problem (1-2) for* $\overline{v}$.

Proof. From Lemma 1.1 we have immediately that $y_n \rightarrow_{n\rightarrow\infty} \overline{y}$ weakly in W and $y_n \rightarrow_{n\rightarrow\infty} \overline{y}$ strongly in $L^2(Q)$ where $\overline{y}$ is the unique solution of problem (1-2) for $\overline{v}$.

Since

$$\begin{aligned} &\frac{1}{2}\int_0^T \frac{d}{dt}\|y_n(t) - \overline{y}(t)\|_H^2 dt + \int_0^T \langle (A(y_n - \overline{y}))(t), y_n(t) - \overline{y}(t)\rangle dt \\ &- \int_0^T \langle (f(y_n))(t) - (f(\overline{y}))(t), y_n(t) - \overline{y}(t)\rangle dt = \\ &= \int_0^T \langle v_n(t), y_n(t)\rangle dt + \frac{1}{2}\langle \overline{y}(T), \overline{y}(T)\rangle - \frac{1}{2}\langle y_n(T), \overline{y}(T)\rangle + \\ &-\frac{1}{2}\langle \overline{y}(T), y_n(T)\rangle + \frac{1}{2}\langle y_0, y_0\rangle + \\ &+ \int_0^T [\langle (A\overline{y})(t), \overline{y}(t)\rangle + \langle (Ay_n)(t), \overline{y}(t)\rangle - \langle (A\overline{y})(t), y_n(t)\rangle] dt + \\ &+ \int_0^T [\langle (f(\overline{y}))(t), y_n(t)\rangle + \langle (f(y_n))(t), \overline{y}(t)\rangle - \langle (f(\overline{y}))(t), \overline{y}(t)\rangle] dt = P_n \end{aligned}$$

then from (3) and (4) we have

$$0 \leq (\alpha - \beta)\|y_n - \overline{y}\|_Y^2 \leq P_n.$$

Now for $n \to \infty$ we see that $P_n \to 0$, hence $y_n \to \overline{y}$ strongly in Y. ■

2. Optimal control problems

Let there be given a space of controls $U = L^2(Q)$ and a given element $v_0 \in U$.

The optimal control problem (I) can be stated as follows: find a control $v^o \in U$ which minimizes the integral functional

$$J(y, v) = \|y\|_{L^2(Q)}^2 + \|v - v_0\|_{L^2(Q)}^2 \tag{14}$$

where $y = y(v)$ is a solution of (1-2) for $v \in U$.

We put $\phi(v) = J(y(v), v)$.

THEOREM 2.1 *(Siedman, Zhou, 1982) Let $f : Y \to Y^*$ be demicontinuous on Y with the norm from $L^2(S; H), \alpha > \beta$ and the assumptions (3) and (4) be satisfied. Then the optimal control problem (I) has at least one solution $v^o \in L^2(Q)$ such that $\Phi(v^0) = \inf_{v \in U} \Phi(v)$.*

Proof. Let $(v_n)_{n \in \mathbf{N}}$ be a minimizing sequence for the function Φ:

$$\forall_{n \in \mathbf{N}} v_n \in U \text{ and } \lim_{n \to \infty} \Phi(v_n) = \inf_{v \in U} \Phi(v).$$

Φ is radially unbounded, then $(v_n)_{n \in \mathbf{N}}$ is bounded in U. It follows that there exists a subsequence, which we also denote by $(v_n)_{n \in \mathbf{N}}$ such that $v_n \to_{n \to \infty} \overline{v}$ weakly in U.

Let $y_n = y(v_n)$ and $y^0 = y(v^0)$. From Lemma 1.1 we know that the sequence $(y_n)_{n \in \mathbf{N}}$ is weakly convergent in W to $\overline{y}$ and the pair $(\overline{v}, \overline{y})$ satisfies the equation (1-2). Because function (14) is weakly lower semicontinuous in $L^2(Q) \times L^2(Q)$, then

$$\inf_{v \in U} \Phi(v) = \lim_{n \to \infty} \Phi(v_n) = \lim_{n \to \infty} \inf J(y_n, v_n) \geq J(\overline{y}, \overline{v})$$

From this $J(\overline{y}, \overline{v}) = \inf_{v \in U} \Phi(v) = J(y^0, v^0)$, which proves the theorem. ■

The optimal control problem (II) can be stated as follows: find a control $v^0 \in U_{ad}$, U_{ad} being a closed, convex, non-empty set of U, which minimizes the functional (14), where $v_0 \in U$ and $y = y(v)$ is a solution of (1-2) for $v \in U_{ad}$.

THEOREM 2.2 *(Dębińska-Nagórska, Just, Stempień, to appear) Under the assumption from Th.2.1 if $U_a d \subset U$ is closed, convex and non-empty, then the optimal control problem (II) has at least one solution $v^0 \in U_{ad}$ such that $\Phi(v^0) = \inf_{v \in U_{ad}} \Phi(v)$.*

Proof. The theorem can be proved in the same way as theorem 2.1.

3. Approximation of the control problems

Consider a family $\{V_h\}_{h\in G}$ of finite - dimensional subspaces of V (Malanowski, 1975; Deimling, 1985), which satisfies the following conditions:

$$\forall h_1, h_2 \in G(h_1 > h_2 \Rightarrow V_{h_1} \subset V_{h_2}) \tag{15}$$
$$\overline{\bigcup_{h\in G} V_h} = V$$

where the set $G \subset (0,1]$ of parameters h has an accumulation point at 0.

The approximation of space $L^2(S;V)$ is understood here as a family of spaces $\{L^2(S;V_h)\}_{h\in G}$. As an approximate solution of (1-2) we assume the function $y_h \in L^2(S;V_h)$ which is the solution of the equation:

$$\langle y_h{\prime}(t), z_h\rangle + \langle (Ay_h)(t), z_h\rangle - \langle f(y_h)(t), z_h\rangle = \langle v(t), z_h\rangle \quad \forall z_h \in V_h \tag{16}$$

with the initial condition

$$y_h(0) = y_{0h} \tag{17}$$

where y_{0h} is the orthogonal projection of y_0 onto V_h with the norm from H.

From the assumptions (3) and (4) it is obvious (Gajewski, Gröger, Zacharias, 1974) that problem (16-17) for each $h \in G$ has the unique solution $y_h \in L^2(S;V_h)$. Moreover $y_h \in W$.

As an approximation of control space U (Malanowski, 1975) we assume a family of finite dimensional subspaces $\{U_k\}_{k\in K}$, which satisfy the following conditions:

$$\forall k_1, k_2 \in K(k_1 > k_2 \Rightarrow U_{k_1} \subset U_{k_2}) \tag{18}$$

$$\overline{\bigcup_{k\in K} U_k} = U \tag{19}$$

where the set $K \subset (0,1]$ of parameters K has an accumulation point at 0.

We shall study the following optimisation problem (I_h): find a control which minimizes the cost functional:

$$\Phi(v_k) = J(y_{hk}, v_k) = ||y_{hk}||^2_{L^2(Q)} + ||v_k - v_{0k}||^2_{L^2(Q)} \tag{20}$$

where $y_{hk} = y_h(v_k)$ is the solution of the equation

$$\langle y_h{\prime}(t), z_h\rangle + \langle (Ay_h)(t), z_h\rangle - \langle (f(y_h))(t), z_h\rangle = \langle v_k(t), z_h\rangle \quad \forall z_h \in V_h \tag{21}$$

with the initial condition (17) for a control $v_k \in U_k$, v_{0k} being the orthogonal projection of v_0 onto U_k.

THEOREM 3.1 *Under the assumption from Th.2.1 the optimal control problem* (I_h) *has at least one solution* $v^0_{kh} \in U_K$ *such that*

$$\Phi(v^0_{kh}) = \inf_{v_k \in U_k} \Phi(v_k) \text{ where } \Phi(v_k) = J(y_{hk}, v_k).$$

Proof. The theorem can be proved in the same way as Theorem 2.1.

LEMMA 3.1 *Let* $(v_k)_{k\in K}$ *be a sequence of elements in* U_k *and* $(y_{hk})_{h\in G,\, k\in K}$ *a sequence of solutions of (21) with the initial condition (17) for the above. Let* $f : Y \to Y^*$ *be demicontinuous on* Y *with the norm from* $L^2(S;H)$, $\alpha > \beta$ *and the assumptions (3) and (4) be satisfied. Then the following conditions hold:*
a) if $v_k \to_{k\to 0} \overline{v}$ *weakly in* U *then* $y_{hk} \to_{k,h\to 0} \overline{y}$ *weakly in* W *and* $y_{hk} \to_{k,h\to 0} \overline{y}$ *strongly in* $L^2(Q)$ *where* $\overline{y}$ *is the unique solution of problem (1-2) for* $\overline{v}$.
b) if $v_k \to_{k\to 0} \overline{v}$ *strongly in* U *then* $y_{hk} \to_{k,h\to 0} \overline{y}$ *strongly in* $L^2(S;V)$ *where* $\overline{y}$ *is the unique solution of problem (1-2) for* $\overline{v}$.

Proof. The proof is analogous to the proof of Lemmas 1.1 and 1.2.
Taking in equation (21) $z_h = y_h(t) \in V_h$ we obtain:

$$\frac{1}{2}\frac{d}{dt}||y_h(t)||^2_H + \langle (Ay_h)(t), y_h(t)\rangle - \langle f(y_h)(t), y_h(t)\rangle = \langle v_k(t), y_h(t)\rangle$$

Integrating this equality in $[0,T]$ from (3) and (4) we have:

$$||y_h(T)||^2_H + (\alpha-\beta)||y_h||^2_Y \le ||y_{0h}||^2_H + \frac{1}{\alpha-\beta}||v_k||^2_U$$

from this inequality and from the assumption $v_k \to_{k\to 0} \overline{v}$

$$||y_{hk}||_Y \le c_2 \text{ for a certain } c_2 > 0$$

where $y_{hk} = y_h(v_k)$.
From (21) we have directly

$$y_{h'} = v_k + f(y_h) - Ay_h \in Y^*$$

and from this $||y_{hk}||_W \le c_3$ for a certain $c_3 > 0$.

It follows that the sequence $(y_{hk})_{h\in G,\, k\in K}$ is bounded in W. Hence there exists a subsequence which we also denote by $(y_{hk})_{h\in G,\, k\in K}$, converging to $\overline{y}$ weakly in W, so - strongly in $L^2(Q)$ (Lions, 1969): $y_{hk} \to \overline{y}$ in W, $y_{hk} \to \overline{y}$ in $L^2(Q)$ where $k,h \to 0$.

Now we can prove that $(\overline{v}, \overline{y})$ is the solution of (1-2).

Function y_{hk} verifies the equation:

$$\langle y_{hk}{}'(t), z_r\rangle + \langle (Ay_{hk})(t), z_r\rangle - \langle (f(y_{hk})(t), z_r\rangle = \langle v_k(t), z_r\rangle$$

with the initial condition:

$$y_{hk}(0) = y_{0h}$$

where z_r is an element of V_h (also $V_{\tilde{h}}$ for $\tilde{h}(h)$

Multiplying this equation by any function $\varphi \in C^1[0,T]$, $\varphi(T) = 0$, integrating in $[0,T]$ we obtain in the limit

$$-\int_0^T \langle \overline{y}(t), z_r\rangle \varphi'(t)dt + \int_0^T \langle (A\overline{y}(t), z_r\rangle \varphi(t)dt =$$
$$= \int_0^T \langle f(\overline{y})(t), z_r\rangle \varphi(t)dt + \int_0^T \langle \overline{v}(t), z_r\rangle \varphi(t)dt + \langle y_0, z_r\rangle \varphi(0).$$

This equation is verified for any $\varphi \in C^1([0,T])$, in particular for $\varphi \in D(0,T)$ and from the definition of distributional derivate we have:

$$\int_0^T \langle \overline{y}'(t), z_r\rangle \varphi(t)dt + \int_0^T \langle (A\overline{y})(t), z_r\rangle \varphi(t)dt =$$
$$= \int_0^T \langle f(\overline{y})(t), z_r\rangle \varphi(t)dt + \int_0^T \langle \overline{v}(t), z_r\rangle \varphi(t)dt.$$

By theorem of integration-by-parts for Bochner integrals (Gajewski, Gröger, Zacharias, 1974), free choice of $\varphi \in D(0,T)$, $z_r \in V_h$ and the condition $\overline{\bigcup_{h\in G} V_h} = V$ we can conclude that $\overline{y}(0) = y_0$. So $\overline{y}$ is the solution of equation (1-2) for $\overline{v}$.

From the fact that problem (1-2) has only one solution it follows that the whole sequence $(y_{hk})_{h\in G,\, k\in K}$, not only the subsequence is weakly convergent in W to $\overline{y} = y(\overline{v})$. ■

The proof of part (b) is identical to proof of Lemma 1.2.

Let us now consider the problem of convergence of the approximation.

THEOREM 3.2 *Let the assumptions of lemma 3.1 be satisfied. Then there exist weak condensation points of a set of solutions of the optimisation problem* (I_h) *in* $U \times W$ *and each of these points is the solution of the optimisation problem (I).*

Proof. Function (20) is radially unbounded, so that the sequence $(v^0_{kh})_{h\in G,\, k\in K}$ is bounded in U. It follows that there exists a subsequence which we also denote by $(v^0_{kh})_{h\in G,\, k\in K}$ such that $v^0_{kh} \to_{k,h\to 0} \tilde{v}$ weakly in U. Then

Lemma 3.1 implies that $y^0_{hk} \rightarrow_{k,h\rightarrow 0} \tilde{y}$ weakly in W and $y^0_{hk} \rightarrow_{k,h\rightarrow 0} \tilde{y}$ strongly in $L^2(Q)$, where $(\tilde{v},\tilde{y})$ verifies (1-2). The function J is weakly lower semi-continuous in $L^2(Q)\times L^2(Q)$. Then, because $(v^0_{kh})_{h\in G,\, k\in K}$ is a minimising sequence, Lions (1983) we have:

$$\inf_{v\in U} \Phi(v) = \lim_{k,h\rightarrow 0} \Phi(v^0_{kh}) = \lim_{k,h\rightarrow 0} \inf J(v^0_{kh}, y^0_{kh}) \geq J(\tilde{v},\tilde{y}).$$

This implies that $(\tilde{v},\tilde{y})$ is one of the solutions of the optimisation problem (I), since $\tilde{v} = v^0$ and $\tilde{y} = y^0$.

THEOREM 3.3 *Let the assumption of Lemma 3.1 be satisfied and*

$$(v^0_{kh} - v_{0k}, v_k - v^0_{kh})_{L^2(Q)} + (y^0_{hk}, y_{hk} - y^0_{hk})_{L^2(Q)} \geq 0 \forall v_k \in U_k \tag{22}$$

then there exist strong condensation points of problem (I_h) in $U\times L^2(Q)$ and each of these points is a solution of the optimisation problem (I).

Proof. From Theorem 3.2 and Lemma 3.1 it follows that the appropriately chosen subsequence $(v^0_{kh})_{h\in G,\, k\in K}$ converges to v^0 weakly in $L^2(Q)$ and the adequate sequence $(y^0_{hk})_{h\in G,\, k\in K}$ where $y^0_{hk} = y_h(v^0_{kh})$ converges to y^0 weakly in W and strongly in $L^2(Q)$. According to (18) and (19) for v^0 there exists a sequence $(v_{k0})_{k\in K}$ such that $v_{k0} \rightarrow_{k\rightarrow 0} v^0$ strongly in U and $v_{k0}\in U_k$ $\forall k \in$ K.

From (15) for $v_k = v_{k0}$ and $y_{hk0} = y_h(v_{k0})$ we have

$$0 \leq ||v^0_{kh} - v_{k0}||^2_{L^2(Q)} \leq (y^0_{hk}, y_{hk0} - y^0_{hk})_{L^2(Q)} - (v_{k0} - v_{0k}, v^0_{kh} - v_{k0})_{L^2(Q)}$$

Now we can find the limit for $k, h \rightarrow 0$ of this inequality and thus we obtain:

$v^0_{kh} \rightarrow v_{k0}$ strongly in U.

From the above and the inequality:

$$||v^0_{kh} - v^0||_U \leq ||v^0_{kh} - v_{k0}||_U + ||v_{k0} + v^0||_U$$

it follows that $v^0_{kh} \rightarrow v^0$ strongly in U. Consequently, that result, together with lemma 3.1, prove the theorem. ■

We can now take into consideration the optimisation problem (II).

We shall study the following optimisation problem (II_h): find a control $v^0_{kh} \in U_{adk} = U_{ad}\cap U_k$ which minimizes the cost functional (20) where $y_{hk} = y_h(v_k)$ is a solution of equation (21) with the initial condition (17) for a control $v_k \in U_{adk}$ and v_{0k} is the orthogonal projection of v_0 onto U_{adk}.

Similarly to Theorems 3.1, 3.2 and 3.3 we can prove three theorems.

THEOREM 3.4 *Let $f: Y \rightarrow Y^*$ be demicontinuous on Y with the norm from $L^2(S;H)$, $\alpha > \beta$ and the assumptions (3) and (4) be satisfied. Then the optimal control problem (II_h) has at least one solution $v^0_{kh} \in U_{adk}$.*

THEOREM 3.5 *Let $(v_k)_{k\in K}$ be a sequence of elements in U_{adk} and $(y_{hk})_{h\in G,\, k\in K}$ a sequence of solutions of (21) with the initial condition (17). Let f be demicontinuous from $L^2(S;V)$ with the norm from $L^2(S;H)$ to $L^2(S,V*)$, $\alpha > \beta$ and the assumptions (3) and (4) be satisfied. Then there exist weak condensation points of a set of solutions of the optimisation problem (II_h) in $U \times W$ and each of these points is the solution of the optimisation problem (II).*

THEOREM 3.6 *Let the assumptions of theorem 3.5 be satisfied and the inequality (22) hold. Then there exist strong condensation points of problem (II_h) in $U \times L^2(Q)$ and each of these points is the solution of optimisation problem (II).*

4. An example

We introduce the operator $A : Y \to Y^*$:

$$Ay = -\sum_{i=1}^{n} \frac{\partial^2 y}{\partial x_i^2} + y$$

and $f : Y \to Y^*$:

$$(f(y))(t) = g(t)||y(t)||_{L^2(\Omega)}$$

where g is the given function from $L^2(Q)$ such that

$$||g||_{L^2(Q)} \leq \beta < 1.$$

The optimisation problem can be formulated as follows: find a control $v^0 \in L^2(Q)(v^0 \in U_{ad} \subset L^2(Q))$ which minimizes the functional:

$$J(y, v) = ||y||^2_{L^2(Q)} + ||v - v_0||^2_{L^2(Q)}$$

where y is the solution of the equation

$$\begin{cases} \frac{\partial y}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2 y}{\partial x_i^2} + y - g||y||_{L^2(\Omega)} = v \text{ on } Q \\ y(0) = y_0 \end{cases}$$

and $v \in L^2(Q)$ $(v \in U_{ad} \subset L^2(Q))$.

The operations A and f satisfy all the assumptions of Theorem 2.1.

Using the notations from paragraph 3 we transform this problem into the one of control for the system of non-linear ordinary differential equations:

$$\begin{cases} \langle \frac{dy_h(t)}{dt}, z_h \rangle + a(y_h(t), z_h) + ||y_h(t)||_{L^2(\Omega)}(g(t), z_h)_{L^2(\Omega)} = \\ (v_k(t), z_h)_{L^2(\Omega)} \ \forall z_h \in V \\ y_h(0) = y_{h0} \end{cases}$$

with cost functional:

$$J(y_h, z_h) = ||y_h||^2_{L^2(Q)} + ||v_k - v_{k0}||^2_{L^2(Q)}$$

where, from Lax-Milgram lemma (see Lions, 1968)

$$a(y_h, z_h) = \int_\Omega \sum_{i=1}^{n} \frac{\partial y_h}{\partial x_i} \frac{\partial z_h}{\partial x_i} dx + \int_\Omega y_h z_h dx.$$

References

K. Deimling (1985) *Nonlinear Functional Analysis.* Springer - Verlag, Berlin.

Dębińska-Nagórska A., Just A., Stempień Z. (to appear) On Galerkin approximation for some minimisation problem in a set of solutions of non-linear differential equation in Banach spaces. *Postępy Cybernetyki.*

Gajewski H., Gröger K., Zacharias K. (1974) *Nichtlineare operatorgleichungen und operatordifferentialgleichungen.* Akademie-Verlag, Berlin.

Lions J.L. (1983) Contrôle des systèmes distribués singuliers. Gauthier - Villars, Paris.

Lions J.L. (1969) Quelques méthodes de résolution des problèmes aux limites non linéares. Dunod, Paris.

Lions J.L. (1968) Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles. Dunod, Paris.

Malanowski K. (1975) O aproksymacji pewnego zadania sterowania optymalnego dla układów opisanych równaniami parabolicznymi. *Archiwum Automatyki i Telemechaniki* **20**, No. 4.

Seidman T., Zhou H.-Z. (1982) Existence end uniqueness of optimal controls for a quasilinear parabolic equation. *SIAM J.Control & Optimization* Vol. 20, No. 6.