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A monotone follower problem

by

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A simple version of the Monotone Follower Problem is considered. We prove the following Theorem: For a given absolutely continuous finction ψ, defined on the interval $[0, T]$, and any $\varepsilon > 0$, there exist a family of Gausian processes $Z^N = \{z_t^N; 0 \le t \le T\}$, $N \in \mathbf{R}$, such that

$$\text{Prob}\left(\sup_{0 \le t \le T} \left|z_t^N - \psi(t)\right| < \varepsilon\right) \ge 1 - \delta_N,$$

where $\delta_N \to 0$ as $N \to \infty$. Moreover, the process Z^N is recursively constructed with respect to time i.e. it follows the demanded ψ.

Possible applications of this results are also indicated.

1. Introduction

The Monotone Follower Problems (MFP's) are usually considered as stochastic control problems. They consist, roughly speaking, in finding a "new" process following the "old" one in the best way, in some sense (see Beneš, Sheep and Witsenhausen, 1980, and the references given there).

In mathematical modelling of many phenomena occurring in various branches of applied sciences one can meet a somewhat different version of the MFP. This version can be described as follows: given a deterministic function ψ and $\varepsilon > 0$, find a stochastic process $Z = \{z_t; t \ge 0\}$ which follows ψ and $\text{Prob}(\sup_{0 \le t \le T} |z_t - \psi(t)| < \varepsilon)$ is close to one.

From the computational point of view it is desirable that the process Z be as simple as possible, for instance a Gaussian process.

In this paper we settle the question by constructing the process Z explicitly. Moreover, we determine Z as the unique strong solution of an appropriate stochastic differential equation (SDE).

This means that we determine Z recursively, as data are coming in i.e. the value $z_{t+\Delta}$, $\Delta > 0$, is determined on the basis of the value z_t and the "new" information $\psi(s)$, $t \leq s \leq t+\Delta$, so that the value $\psi(s)$, $s > t+\Delta$ is not required.

Applications of the problem considered here might include investment policy meant to meet demand or a description of a phenomenon whose state is continuously fluctuating around the most probable trajectory, guessed on line by experts.

2. The main results

To state our results we first introduce several notions and definitions.

Let $W = \{w_t; t \geq 0\}$ be a standard Wiener process defined on a given complete propability space $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions. Let $L[0,T]$ denote the space of Lebesgue - integrable functions on $[0,T]$ and $C_A[0,T]$ the space of absolutely continuous functions on $[0,T]$, $T \leq \infty$.

THEOREM 2.1 *For any $\psi(\cdot) \in C_A[0,T]$ and any $\varepsilon > 0$, there exist a function $\varphi(\cdot) \in L[0,T]$ and a positive number N, such that*

$$P(\sup_{0\leq t\leq T} \left| z_t^{N,\varphi} - \psi(t) \right| < \varepsilon) \geq 1 - \delta_{N,T} \tag{1}$$

where the process $\left\{ z_t^{N,\varphi};\ 0 \leq t \leq T \right\}$ is the solution of the SDE

$$dz_t = -N[z_t - \varphi(t)]dt + dw_t,\ z_0 = \psi(0) \tag{2}$$

and

$$\delta_{N,T} \leq \frac{2}{\varepsilon}\sqrt{\frac{1}{\pi N}(1 - e^{2NT}} e^{-\frac{2N\varepsilon^2}{1-e^{-2NT}}} \rightarrow_{N\rightarrow\infty} 0. \tag{3}$$

PROOF. For fixed $N > 0$, let $X^N = \{x_t^N; t \geq 0\}$ denote the solution of the SDE

$$dx_t = -Nx_t dt + dw_t,\ x_0 = 0. \tag{4}$$

It is easy to prove that X_N is a Gaussian process with zero mean and variance

$$\sigma^N(t) = \frac{1}{2N}(1 - e^{-2NT}). \tag{5}$$

Moreover, we can state the following

LEMMA 2.1 *Let $\varepsilon > 0$. Then*

$$P(\sup_{0\leq t\leq T} \left| x_t^N \right| < \varepsilon) \geq 1 - \delta_{N,T}$$

where $\delta_{N,T}$ satisfies (3).

PROOF. Since $x_t^N = e^{-Nt}\int_0^t e^{Ns}dw_s$ is Gaussian with zero mean and covariance $N^{-1}e^{-Nt}$ sh (Ns), (assuming $t > s$), then the processes $\{x_t^N;\ 0 \le t \le T\}$, $\{y_t;\ 0 \le t \le T\}$, (where $y_t \triangleq e^{-Nt}w_{\rho(t)}$, $\rho(t) \triangleq (2N)^{-1}(e^{2Nt}-1)$) are stochastically equivalent. In fact $\{y_t;\ 0 \le t \le T\}$ is again Gaussian with zero mean and covariance

$$\begin{aligned} E_{y_t y_s} &= e^{-N(t+s)}Ew_{\rho(t)}w_{\rho(s)} = e^{-N(t+s)}[\rho(t) \wedge \rho(s)] = \\ &= e^{-N(t+s)}\rho(s) = N^{-1}e^{-Nt} \text{ sh } (Ns). \end{aligned}$$

Now let $\beta_t \triangleq w_{\rho^N(t)}$. Then cov $\beta_t = E\beta_t\beta_{s'} = [\rho^N(t) \wedge \rho^N(s)] = \rho^N(s) = N^{-1}e^{-Ns}$ sh (Ns) and cov $x_t^N =$ cov $y_t \le$ cov β_t. This implies

$$\begin{aligned} P(\sup_{0\le t\le T}|x_t^N| \ge \varepsilon) &\le P(\sup_{0\le t\le T}|w_{\sigma^N(t)}| \ge \varepsilon) = \\ &= P(\sup_{0\le u\le \sigma^N(T)}|w_u| \ge \varepsilon) = \\ &= P(\sup_{0\le u\le \sigma^N(T)}\left|\sqrt{\sigma^N(T)}w_{\frac{u}{\sigma^N(T)}}\right| \ge \varepsilon) = \\ &= P(\sup_{0\le t\le 1}|w_t| \ge \frac{\varepsilon}{\sqrt{\sigma^N(T)}}) \le \\ &\le 2P(\sup_{0\le t\le 1} w_t \ge \frac{\varepsilon}{\sqrt{\sigma^N(T)}}) = \\ &= \frac{4}{\sqrt{2\pi}}\int_{\frac{\varepsilon}{\sqrt{\sigma^N(T)}}}^{\infty} e^{-x^2/2}dx \le \frac{2}{\varepsilon}\sqrt{\frac{1}{\pi N}(1-e^{-2NT})}e^{-\frac{2N\varepsilon^2}{1-e^{-2NT}}}. \end{aligned}$$

The last inequality follows from the estimation

$$\int_a^{\infty} e^{-x^2/2}dx \le \frac{1}{a}e^{-a^2/2}$$

and the third equality from the invariant property of Wiener processes, i.e. $\{w_t;\ t \ge 0\}$ and $\{\sqrt{a}w_{t/a};\ t \ge 0\}$ are both the standard Wiener processes.

The Lemma is proved. ∎

To finish the proof of the theorem, let us take the function

$$\varphi(t) = \psi(t) + \frac{1}{N}\dot{\psi}(t) \text{ for all } t \in [0,T] \tag{6}$$

and define

$$z_t = z_t^{N,\varphi} = x_t^N + \psi(t).$$

Applying Itô's formula we get

$$\begin{aligned} dz_t &= -N[z_t - \psi(t)]dt + dw_t + \dot{\psi}(t)dt = \\ &= -N[z_t - \varphi(t)]dt + dw_t,\ z_0 = \psi(0) \end{aligned}$$

Thus $z_t^{N,\varphi} = x_t^N + \psi(t)$ satisfies the Eq. (2). Applying Lemma 2.1 to $z_t^{N,\varphi} - \psi(t) = x_t^N$ we get (1) and (3). Finally, note, that (6) implies that $\varphi(\cdot) \in L[0,T]$, thus the Itô's uniqueness conditions hold for the Eq. (2), i.e. Eq. (2) admits the unique strong solution defined and continuous for all $t \in [0,T]$, see Lipcer, Shiryayev (1977); Ikeda, Watanabe (1981). The Theorem is proved. ■

COROLLARY 2.1 *For any $\psi(\cdot) \in C_A[0,\infty)$ and any $\varepsilon > 0$, there exist a function $\varphi(\cdot) \in L[0,\infty)$ and a positive number N, such that*

$$P(\sup_{t\geq 0}\left|z_t^{N,\varphi} - \psi(t)\right| < \varepsilon) \geq 1 - \delta_N$$

$$\delta_N \leq \frac{2}{\varepsilon\sqrt{\pi N}} e^{-2N\varepsilon^2} \rightarrow_{N\rightarrow\infty} 0.$$

COROLLARY 2.2 *Let $r > 0$ be a fixed number. For any $\psi(\cdot) \in C_A[0,\infty)$ and any $\varepsilon > 0$, there exist a function $\varphi(\cdot) \in L[0,\infty)$ such that*

$$P(\sup_{t\geq 0}|z_t^{\varepsilon,\varphi} - \psi(t)| < \varepsilon) \geq 1 - \delta_\varepsilon$$

where $\{z_t^{\varepsilon,\varphi};\ t \geq 0\}$ solves the Eq. (2) with $N = \varepsilon^{-(2+r)}$, and

$$\delta_\varepsilon \leq 2\pi^{-1/2}\varepsilon^{r/2}e^{-2/\varepsilon^r} \rightarrow_{\varepsilon\rightarrow 0} 0.$$

The above Corollaries are easy consequence of Theorem 2.1, and therefore the proofs are omitted.

3. Remarks

1. Corollary 2.2 shows that if the demanded $\varphi(\cdot)$ is an absolutely continuous function then one can follow it on the whole $\mathbf{R}^+ = [0,\infty)$ with arbitrarily high accuracy.
2. Possible application in pursuit – evasion games. Let $\{\psi(t);\ t \geq 0\}$ represent a trajectory of the evader. It can be given for example by the simple equation of motion

$$\dot{\psi}(t) = v(t),\ v(\cdot) \in L[0,\infty).$$

The aim of the pursuer which moves according to the equation

$$dz_t = u_t dt + dw_t,\ u(\cdot) \in L[0,\infty)$$

is to follow the evader as close as possible in spite of random disturbances. The value u_t has to be determined only by means of the following information: the trajectory $\{\psi(s);\ 0 \leq s \leq t\}$ and the evader control v_t. If the initial states $\psi(0)$, z_0 are equal or close together, then our results are applicable with

$$u_t = N[\psi(t) - z_t] + \dot{\psi}(t).$$

3. Possible applications in economics and finance. The process Z^N defined in (2) is an Ornstein–Uhlenbeck process. Such a process is a fairly good model of the short term interest rate's behaviour, see Vasicek (1977); Merton (1973); Merton (1974); Abikhalil, Dupont and Janssen (1985).

Let $\psi(t)$, $0 \leq t \leq T$, be the prediction of the interest rate behaviour on $[0, T]$, made by experts at time $t = 0$. Theorem 2.1 indicates that if $\psi \in C_A[0, T]$ then the model (2) can be as much in agreement with the experts' anticipations, as one could wish. Moreover, since the process Z is recursively constructed, the above statement is also true if the initial time $t = 0$ is replaced by the current time, $\psi \in C_A[t, t+h]$, where h is now the prediction horizon.

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