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A noisy duel with two kinds of weapons. Parts IV and V[1]

by

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In the preceding parts of this paper, as well as in the present one, a noisy duel is considered in which Player I has two kinds of weapons: a gun with m bullets and a weapon which he can use when he meets the opponet. Player II has a gun with n bullets. The cases are solved $n = 1$ for any $m, m = 0$ for any n, and $m \leq 20, n \leq 5$.

In this part the cases $m = 1, 2, n = 5$ are solved.

Part IV

1. Definitions and assumptions

Let us define the game which will be called the game (m, n). Two players, I and II fight a duel. They can move as they want. The maximum speed of Player I is v_1, the maximum speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$.

Player I has two kinds of weapons: a gun with m bullets and a weapon which he can use when distance between him and the opponent is zero. Player II has only a gun with n bullets.

At the beginning of the duel the players are at distance 1 from each other. Let $P(s)$ be probability of succeeding (destroying the opponent) by Player I (II) when the distance between players is $1 - s$. The function $P(s)$ will be called accuracy function. It is assumed that it is increasing and continuous in $[0, 1]$, has continuous second derivative in $(0, 1)$ and that $P(s) = 0$ for $s \leq 0$, $P(1) = 1$.

1 Parts I, II and III were published in Control and Cybernetics, vol 22, 1993, 2, pp. 69–103.

It is also assumed that at $s = 1$ Player I succeeds surely by his short distance weapon.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero-sum game.

It is assumed that duel is noisy - player hears every shot of his opponent.

As it will be seen from the sequel, without loss of generality we can suppose that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

In the sequel we assume that the reader knows the previous parts of the paper.

Suppose now that the duel (m, n) begins when the distance between players is $1 - a$, $0 \le a < 1$. This duel will be denoted by $(m, n), \langle a \rangle$.

Moreover, suppose that between successive shots of the same player time $\hat{\varepsilon} > 0$ has to pass.

Futhermore, let (m, n), $\langle a \wedge c, a \rangle$; $0 < c \le \hat{\varepsilon}$, be the duel in which Player I has m bullets, Player II has n bullets, Player I is at the beginning of the duel at a, Player II is at 1, but if $c \le \hat{\varepsilon}$ Player II can fire from the time $\langle a \rangle$ on and Player I from the time $\langle a \rangle + c$; $\langle s \rangle$ is the first time when Player I reached the point s. If $c = \hat{\varepsilon}$ the rule is the same with the only exception that Player II is not allowed to fire $\langle a \rangle$.

Similarly we define the duel $(m, n)\langle a, a \wedge c \rangle$.

All other definitions and suppositions made for the duel (m, n) hold also for above duels.

For definitions and notions concerning duels see Karlin (1959) and Trybuła (1993).

2. Duels $(1, 5)$

Duel $(1, 5), \langle a \rangle$.

Let us consider the duel in which Player I has 1 bullet, Player II has 5 bullets and the game is beginning when Player I is at the point a.

Let $Q(a) \ge Q(\hat{a}_{15}) \cong 0.91774$. We define strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the duel $(1, 4)$, $\langle a', a' \wedge \hat{\varepsilon} \rangle$.

Strategy of Player II: Fire at $\langle a \rangle$ and play optimally the resulting duel.

"Play optimally" means: apply the strategy optimal in the limit.

Let $v_{15}(a)$ be a number defined as follows,

$$v_{15}(a) = -P(a) + Q(a) v_{14}(2, a)$$

$$= -1 + Q(a) + \begin{cases} -1 + Q(a) & \text{if } Q(a) \ge Q(\hat{a}_{13}), \\ -1 + (1 + v_{12}) Q^2(a) & \text{if } Q(\breve{a}_{14}) \le Q(a) \le Q(\hat{a}_{13}), \end{cases}$$

$$v_{15}(a) = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \ge Q(\hat{a}_{13}), \\ -1 + (1 + v_{12}) Q^3(a) & \text{if } Q(\breve{a}_{14}) \le Q(a) \le Q(\hat{a}_{13}); \end{cases}$$

$Q(s) = 1 - P(s), Q(\hat{a}_{13}) \cong 0.95572$ (see Trybuła, 1993, part II); $Q(\breve{a}_{14}) \cong 0.90920$ (see Trybuła, 1993, part III), $v_{12} = \frac{11-4\sqrt{7}}{9} \cong 0.04633$ (see Trybuła, 1993, part II).

We prove that strategies ξ and η are optimal in the limit and that the number $v_{15}(a)$ defined in the above is the limit value of the game for $a < \breve{a}_{15}$.

Suppose that Player II fires at $a' < \breve{a}_{14}$. For such a strategy (denote it by $\hat{\eta}$) we obtain that the payoff function satisfies the condition

$$K(\xi, \hat{\eta}) \geq -P(a') + (1 - P(a'))v_{14}(2, a') - k(\hat{\varepsilon}) =$$
$$= \begin{cases} -1 + Q^2(a') - k(\hat{\varepsilon}) \geq -1 + Q^2(a) - k(\hat{\varepsilon}), & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^3(a') - k(\hat{\varepsilon}) \geq -1 + & \\ \qquad +(1 + v_{12})Q^3(a) - k(\hat{\varepsilon}), & \text{if } Q(\breve{a}_{14}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases}$$

where $v_{14}(2, a)$ is the limit value of the game for the duel $(1,4), \langle a, a \wedge c\rangle$ and $k(\hat{\varepsilon}) \to 0$ if $\hat{\varepsilon} \to 0$.

Suppose that Player II does not fire. For such a strategy $\hat{\eta}$ we have

$$K(\xi, \hat{\eta}) = 0 \geq \begin{cases} -1 + Q^2(a) & \text{if} \quad Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^3(a) & \text{if} \quad Q(\breve{a}_{14}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

Then

$$K(\xi, \hat{\eta}) \geq v_{15}(a) - k(\hat{\varepsilon})$$

for any a such that $Q(a) \geq Q(\hat{a}_{15}) \cong 0.91774$.

On the other hand, suppose that Player I fires after $\langle a\rangle$. For such a strategy $\hat{\xi}$ we obtain

$$K(\hat{\xi}, \eta) < -P(a) + (1 - P(a))v_{14}(2, a) + k(\hat{\varepsilon}) = v_{15} + k(\hat{\varepsilon})$$

for $Q(a) \geq Q(\breve{a}_{14})$.

i) At the end, suppose that Player I fires at $\langle a\rangle$, together with Player II. For such a strategy $\hat{\xi}$ we have

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq Q^2(a)v_{04}(a) + k(\hat{\varepsilon}) \\ &= -Q^2(a)(1 - Q_3(a)) + k(\hat{\varepsilon}). \end{aligned}$$

Let $a < \hat{a}_{13}$. Then we demand that

$$-Q^2(a)(1 - Q^3(a)) + k(\hat{\varepsilon}) \leq v_{15}(a) + k(\hat{\varepsilon}) = -1 + Q^2(a) + k(\hat{\varepsilon})$$

what always holds for such a.

Let $\hat{a}_{13} \leq a \leq \breve{a}_{14}$. In this case we demand that

$$-Q^2(a)(1 - Q^3(a)) + k(\hat{\varepsilon}) \leq -1 + (1 + v_{12})Q^3(a) + k(\hat{\varepsilon})$$

or

$$S(Q) = Q^5(a) - (1 + v_{12})Q^3(a) - Q^2(a) + 1 \leq 0. \tag{1}$$

This function is decreasing in Q and is equal to zero for $Q(a) = Q(\hat{a}_{15}) \cong 0.91774$. Then, inequality holds for $Q(a) \geq Q(\hat{a}_{15})$.

This ends the proof of the assertion.

Duel $(1,5), \langle a \wedge c, a \rangle$. $Q(a) \geq Q(\breve{a}_{14}) \cong 0.90920$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player I has fired (say at a') play optimally the resulting duel $(1,4), \langle a', a' \wedge c_1 \rangle$.

Strategy of Player II: Fire before $\langle a \rangle + c$ and play optimally the resulting duel.

Proof of limit optimality of strategies ξ and η is the same as for duel $(1,5), \langle a \rangle$, with the only exception that i) is not considered here. Then the bound $Q(a) \geq Q(\hat{a}_{15})$ does not hold and we have

$$v_{15}(1,a) = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^3(a) & \text{if } 0.90920 \cong Q(\breve{a}_{14}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

Duel $(1,5), \langle a, a \wedge c \rangle$. $Q(a) \geq Q(\breve{a}_{15}) \cong 0.92409$.

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the resulting duel $(1,4), \langle a', a' \wedge \hat{\varepsilon} \rangle$.

Strategy of Player II: If Player I did not fire before, fire at $\langle a \rangle + c$ and play optimally the duel $(1,4), \langle \alpha', a' \wedge c_1 \rangle$, $\alpha' = \langle a \rangle + c$. If he has fired play optimally the resulting duel.

The sign $\dot{s}$ denotes the time when Player I is at the point s (not necessarily the first time, which is denoted by $\langle s \rangle$).

Here also the proof is nearly the same as for the duel $(1,5), \langle a \rangle$. Here an additional case has to be considered when Player I fires before $\langle a \rangle + c$. For such strategy $\hat{\xi}$ we have

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq P(a) - (1 - P(a))(1 - (1 - P(1))^4) + k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^5(a) + k(\hat{\varepsilon}). \end{aligned}$$

Let $a \leq \hat{a}_{13}$. We demand that

$$1 - 2Q(a) + Q^5(a) \leq -1 + Q^2(a)$$

which is always satisfied for considered a since the multinomial

$$S_1(Q) = Q^5(a) - Q^2(a) - 2Q(a) + 2$$

has one minimum for $0 \leq Q \leq 1$ and $S_1(Q(\hat{a}_{13})) \cong S_1(0.95572) \cong -0.02748$, $S_1(1) = 0$.

Let $\hat{a}_{13} \leq a \leq \check{a}_{14}$. We demand that

$$1 - 2Q(a) + Q^5(a) \leq -1 + (1 + v_{12})Q^3(a)$$

or

$$S_2(Q) = Q^5(a) - (1 + v_{12})Q^3(a) - 2Q(a) + 2 \leq 0.$$

This function is decreasing in Q and $S_2(Q(\check{a}_{15})) = 0$. Hence, $S_2(Q(a)) \leq 0$ for $a \leq \check{a}_{15}$.

Then strategies ξ and η are optimal in the limit for $a \leq \check{a}_{15}$, where $Q(\check{a}_{15}) \cong 0.92409$.

3. Results for the duels $(1, 5)$

Let $v_{15}(1, a)$, $v_{15}(a)$, $v_{15}(2, a)$ be the limit values of the game for the duels $(1, 5), \langle a \wedge c, a \rangle$; $(1, 5)$, $\langle a \rangle$; $(1, 5), \langle a, a \wedge c \rangle$, respectively. From the results of previous section we have

$$v_{15}(1, a) = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^3(a) & \text{if } 0.90920 \cong Q(\check{a}_{14}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases}$$

$$v_{15}(a) = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^3(a) & \text{if } 0.91774 \cong Q(\hat{a}_{15}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases}$$

$$v_{15}(2, a) = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -1 + (1 + v_{12})Q^3(a) & \text{if } 0.92409 \cong Q(\check{a}_{15}) \leq Q(a) \leq Q(\hat{a}_{13}). \end{cases}$$

4. Duels $(2, 5)$

Duel $(2, 5), \langle a \rangle$

CASE 4.1 $Q(a) \geq Q(a_{25}) \cong 0.99804$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the duel $(2, 4), \langle a', a' \wedge \hat{\varepsilon} \rangle$.

Strategy of Player II: If Player I did not reach the point a_{25} defined by (2) further on and did not fire, do not fire neither. If he had reached this point and did not fire, fire at $\langle a_{25} \rangle$ and play optimally the resulting duel. If Player I had fired before he reached the point a_{25} (say at a') play optimally the duel $(1, 5)$, $\langle a' \wedge \hat{\varepsilon}, a' \rangle$.

We prove that for $a \leq a_{25}$ strategies ξ and η are optimal in the limit and the limit value of the game is

$$v_{25}(a) = 0.$$

Suppose that Player II fires at $a' \le a$. For such a strategy (denote it by $\hat\eta$) we have

$$\begin{aligned} K(\xi,\hat\eta) &\ge -P(a') + (1-P(a'))v_{24}(2,a') - k(\hat\varepsilon) \\ &= -1 + (1+P^2(\hat a_{13}))Q(a') - k(\hat\varepsilon) \ge k(\hat\varepsilon) \end{aligned}$$

if

$$Q(a') = \frac{1}{1+P^2(\hat a_{13})} \overset{\text{def}}{=} Q(a_{25}) \cong 0.99804. \tag{2}$$

Suppose that Player II does not fire. Then

$$K(\xi,\hat\eta) = 0.$$

On the other hand, suppose that Player I reaches the point a_{25} and does not fire before or at $\langle a_{25}\rangle$. For such a strategy $\hat\xi$ we have

$$\begin{aligned} K(\hat\xi,\eta) &\le -P(a_{25}) + Q(a_{25})v_{24}(2,a_{25}) + k(\hat\varepsilon) \\ &= -1 + (1+P^2(\hat a_{13}))Q(a_{25}) + k(\hat\varepsilon) = k(\hat\varepsilon). \end{aligned}$$

Suppose that Player I does not reach the point a_{25} and does not fire. We have

$$K(\hat\xi,\eta) = 0.$$

Suppose that Player I fires at $a' < a_{25}$. We obtain for such a strategy $\hat\xi$

$$\begin{aligned} K(\hat\xi,\eta) &\le P(a') + Q(a')v_{15}(1,a') + k(\hat\varepsilon) \\ &= 1 - 2Q(a') + Q^3(a') + k(\hat\varepsilon) \le k(\hat\varepsilon). \end{aligned}$$

The assertion is proved.

Duel $(2,5)$, $\langle a\rangle$

Case 4.2

$$\begin{aligned} 0.96894 \cong Q(\hat a_{25}) \le Q(a) \le Q(a_{25}) \cong 0.99804, \\ 0.95105 \cong Q(\hat a_{24}) \le Q(a) \le Q(\check a_{25}) \cong 0.95302, \\ 0.94671 \cong Q(\bar a_{25}) \le Q(a) \le Q(\hat a_{24}). \end{aligned}$$

We define strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the duel $(2,4)$, $\langle a', a' \wedge \hat\varepsilon\rangle$.

Strategy of Player II: Fire at $\langle a\rangle$ and play optimally the resulting duel.

For constants a satisfying the above conditions we assume that

$$\begin{aligned} v_{25}(a) &= -P(a) + Q(a)v_{24}(2,a) = \\ &= \begin{cases} -1 + (1+P^2(\hat a_{13}))Q(a) & \text{if } Q(a) \ge Q(\hat a_{24}), \\ -1 + (1+v_{23})Q^2(a) & \text{if } 0.93571 \cong Q(\bar a_{24}) \le Q(a) \le Q(\hat a_{24}) \end{cases} \end{aligned} \tag{3}$$

We prove that for constants a specified in Case 4.2 strategies ξ and η are optimal in the limit and the limit value of the game is given by (3).

Suppose that Player II fires at $a' < \bar{a}_{24}$. We obtain

$$\begin{aligned} K(\xi,\hat{\eta}) &\geq -P(a') + Q(a')v_{24}(2,a') - k(\hat{\varepsilon}) \\ &\geq -P(a) + Q(a)v_{24}(2,a) - k(\hat{\varepsilon}) = v_{25}(a) - k(\hat{\varepsilon}) \end{aligned}$$

by (3).

Suppose that Player II does not fire. For such a strategy $\hat{\eta}$

$$K(\xi,\hat{\eta}) = 0 \geq \begin{cases} 1 + (1 + P^2(\hat{a}_{13}))Q(a) & \text{if } Q(a) \geq Q(\hat{a}_{24}), \\ -1 + (1 + v_{23})Q^2(a) & \text{if } 0.93571 \cong \\ & \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{24}) \end{cases}$$

On the other hand, suppose that Player I fires at $\langle a \rangle$. For such a strategy $\hat{\xi}$ we have

$$\begin{aligned} K(\hat{\xi},\eta) &\leq Q^2(a)v_{14}(a) + k(\hat{\varepsilon}) \\ &= \begin{cases} -Q^2(a) + Q^3(a) + k(\hat{\varepsilon}) & \text{if } Q(a) \geq Q(\hat{a}_{13}), \\ -Q^2(a) + (1 + v_{12})Q^4(a) + k(\hat{\varepsilon}) & \text{if } Q(\hat{a}_{14}) \leq Q(a) \leq Q(\hat{a}_{13}), \end{cases} \end{aligned}$$

$Q(\hat{a}_{13}) \cong 0.95572$ (see Trybuła, 1993, part II), $Q(\hat{a}_{14}) \cong 0.89815$ (see Trybuła, 1993, part III).

i) Now, if $0.99804 \cong Q(a_{25}) \geq Q(a) \geq Q(\hat{a}_{23})$ we demand that

$$-Q^2(a) + Q^3(a) \leq -1 + (1 + P^2(\hat{a}_{13}))Q(a)$$

or

$$S(Q) = Q^3(a) - Q^2(a) - (1 + P^2(\hat{a}_{13}))Q(a) + 1 \leq 0. \tag{4}$$

This function is decreasing in Q and $S(Q(\hat{a}_{25})) = 0$, $Q(\hat{a}_{25}) \cong 0.96894$. Then the inequality is satisfied for

$$0.96894 \cong Q(\hat{a}_{25}) \leq Q(a) \leq Q(a_{25}) \cong 0.99804.$$

ii) If $0.91105 \cong Q(\hat{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{13}) \cong 0.95572$ we demand that

$$-Q^2(a) + (1 + v_{12})Q^4(a) \leq -1 + (1 + P^2(\hat{a}_{13}))Q(a)$$

or

$$S(Q) = (1 + v_{12})Q^4(a) - Q^2(a) - (1 + P^2(\hat{a}_{13}))Q(a) + 1 \leq 0. \tag{5}$$

This function has one minimum in [0.1], $S(Q(\hat{a}_{24})) \cong -0.00139$, $S(Q(\hat{a}_{13})) \cong 0.00196$. Then it has the root in the interval $[Q(\hat{a}_{24}), Q(\hat{a}_{13})]$ which is equal $Q \stackrel{\text{def}}{=} Q(\check{a}_{25}) \cong 0.95302$. Then the inequality holds for

$$0.95105 \cong Q(\hat{a}_{24}) \le Q(a) \le Q(\check{a}_{25}) \cong 0.95302.$$

iii) If $0.93571 \cong Q(\bar{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}) \cong 0.95105$ we demand that

$$-Q^2(a) + (1+v_{12})Q^4(a) \le -1 + (1+v_{23})Q^2(a)$$

or

$$S(Q) = (1+v_{12})Q^4(a) - (2+v_{23})Q^2(a) + 1 \le 0. \tag{6}$$

This function has the only root for $Q(a) = Q(\bar{a}_{25}) \cong 0.94671$ and is positive for $Q(a) < Q(\bar{a}_{25})$ and negative for $Q(a) > Q(\bar{a}_{25})$. Then the inequality holds for

$$0.94671 \cong Q(\bar{a}_{25}) \le Q(a) \le Q(\hat{a}_{24}) \cong 0.95105.$$

The assertion is proved.

Duel $(2,5)$, $\langle a\rangle$

CASE 4.3

$$\begin{aligned}
0.95572 &\cong Q(\hat{a}_{13}) \le Q(a) \le Q(\hat{a}_{25}) \cong 0.96894,\\
0.95302 &\cong Q(\check{a}_{25}) \le Q(a) \le Q(\hat{a}_{13}) \cong 0.95572,\\
0.93571 &\cong Q(\bar{a}_{24}) \le Q(a) \le Q(\bar{a}_{25}) \cong 0.94671,\\
0.91636 &\cong Q(\check{a}_{24}) \le Q(a) \le Q(\bar{a}_{24}).
\end{aligned}$$

We define strategies ξ and η of Players I and II.

Strategy of Player I: Fire at $\langle a\rangle$ and play optimally the resulting duel.

Strategy of Player II: Fire at $\langle a\rangle$ and play optimally the resulting duel.

We prove that for a defined in the above strategies ξ and η are optimal in the limit and that the value of the game is

$$\begin{aligned}
v_{25}(a) &= Q^2(a)v_{14}(a) \qquad (7)\\
&= \begin{cases} -Q^2(a) + Q^3(a) & \text{if } Q(a) \ge Q(\hat{a}_{13}),\\ -Q^2(a) + (1+v_{12})Q^4(a) & \text{if } 0.89815 \cong Q(\hat{a}_{14}) \le Q(a) \le Q(\hat{a}_{13}).\end{cases}
\end{aligned}$$

Suppose that Player II does not fire at $\langle a\rangle$. For such a strategy $\hat{\eta}$

$$\begin{aligned}
K(\xi,\hat{\eta}) &\ge P(a) + Q(a)v_{15}(a) - k(\hat{\varepsilon})\\
&= \begin{cases} 1 - 2Q(a) + Q^3(a) - k(\hat{\varepsilon}) \ge \\ \qquad -Q^2(a) + Q^3(a) - k(\hat{\varepsilon}) & \text{if } Q(a) \ge Q(\hat{a}_{13}),\\ 1 - 2Q(a) + (1+v_{12})Q^4(a) - k(\hat{\varepsilon}) \ge \\ \qquad -Q^2(a) + (1+v_{12})Q^4(a) - k(\hat{\varepsilon}) & \text{if } Q(\hat{a}_{14}) \le Q(a) \le Q(\hat{a}_{13}).\end{cases}
\end{aligned}$$

Suppose that Player I does not fire at $\langle a\rangle$. For such a strategy $\hat{\xi}$

$$K(\hat{\xi},\eta)\leq -P(a)+Q(a)v_{24}(2,a)+k(\hat{\varepsilon})$$
$$=\begin{cases}-1+(1+P^2(\hat{a}_{23}))Q(a)+k(\hat{\varepsilon}) \text{ if } Q(a)\geq Q(\hat{a}_{24})\cong 0.95105\\ -1+(1+v_{23})Q^2(a)+k(\hat{\varepsilon})\\ \qquad\qquad \text{if } 0.93571\cong Q(\bar{a}_{24})\leq Q(a)\leq Q(\hat{a}_{24}),\\ -1+2Q(a)-2Q^2(a)+(1+v_{12})Q^4(a)+k(\hat{\varepsilon})\\ \qquad\qquad \text{if } 0.91636\cong Q(\check{a}_{24})\leq Q(a)\leq Q(\bar{a}_{24}).\end{cases}$$

i) Let

$$Q(a)\geq Q(\hat{a}_{13})\cong 0.95572. \tag{8}$$

We demand that

$$-1+(1+P^2(\hat{a}_{13})Q(a)\leq -Q^2(a)+Q^3(a)$$

or

$$Q^3(a)-Q^2(a)-(1+P^2(\hat{a}_{13}))Q(a)+1\geq 0.$$

The inequality is opposite (leaving out of account the case of equality) to the inequality (4) and is satisfied for

$$Q(a)\leq Q(\hat{a}_{25})\cong 0.96894. \tag{9}$$

The conditions (8) and (9) give

(a) $0.95572\cong Q(\hat{a}_{13})\leq Q(\hat{a}_{25})\cong 0.96894.$

ii) Let

$$0.95105\cong Q(\hat{a}_{24})\leq Q(a)\leq Q(\hat{a}_{13})\cong 0.95572. \tag{10}$$

We demant that

$$-1+(1+P^2(\hat{a}_{13}))Q(a)\leq -Q^2(a)+(1+v_{12})Q^4(a)$$

or

$$(1+v_{12})Q^4(a)-Q^2(a)-(1+P^2(\hat{a}_{13}))Q(a)+1\geq 0.$$

From (5) it follows that this inequality is satisfied for

$$Q(a)\geq Q(\check{a}_{25})\cong 0.905302. \tag{11}$$

The conditions (10) and (11) give

$$0.95302\cong Q(\check{a}_{25})\leq Q(a)\leq Q(\hat{a}_{13})\cong 0.95572.$$

iii) Let

$$0.93571 \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{24}) \cong 0.95105. \qquad (12)$$

We demand that

$$-1 + (1 + v_{23})Q^2(a) \leq -Q^2(a) + (1 + v_{12})Q^4(a)$$

From (6) it follows that this inequality is satisfied for

$$Q(a) \leq Q(\bar{a}_{25}) \cong 0.94671. \qquad (13)$$

The conditions (12) and (13) give

$$\text{(c) } 0.93571 \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(\bar{a}_{25}) \cong 0.94671.$$

iv) Let, at the end,

$$\text{(d) } 0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(\bar{a}_{24}) \cong 0.93571.$$

We ask for

$$-1 + 2Q(a) - 2Q^2(a) + (1 + v_{12})Q^4(a) \leq -Q^2(a) + (1 + v_{12})Q^2(a)$$

which is always satisfied.

Then, for constants specified in the cases (a)-(d) the strategies ξ and η are optimal in the limit. The limit values of the game for particular cases are given in Section 5.

Duel $(2,5), \langle a \wedge c, a \rangle$

Case 4.4 $Q(a) \geq Q(a_{25}) \cong 0.99804$

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the resulting duel $(2,4), \langle a', a' \wedge c_1 \rangle$.

Strategy of Player II: If Player I did not reach the point a_{25} and did not fire, do not fire, neither. If he had reached this point and did not fire, fire at $\langle a_{25} \rangle$ and play optimally the resulting duel. If Player I has fired at $a' < a_{25}$ play optimally the duel $(1,5), \langle a' \wedge \hat{\varepsilon}, a' \rangle$.

Proof of limit optimality of above strategies is omitted. It is easy to see that in this case

$$v_{25}(1, a) = 0.$$

Case 4.5 $0.93571 \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(a_{25}) \cong 0.99804.$

Strategy of Player I: Escape. If Player II fires (say at a') play optimally the resulting duel $(2,4), \langle a', a' \wedge c_1 \rangle$.

Strategy of Player II: Fire before $\langle a \rangle + c$ and play optimally afterwards.

The proof of limit optimality of these strategies is omitted. Now

$$v_{25}(1,a) = \begin{cases} -1 + (1 + P^2(\hat{a}_{13}))Q(a) \\ \qquad \text{if } 0.95105 \cong Q(\hat{a}_{24}) \le Q(a) \le Q(a_{25}) \cong 0.99804, \\ -1 + (1 + v_{23})Q^2(a) \\ \qquad \text{if } 0.93571 \cong Q(\bar{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}). \end{cases} \tag{14}$$

CASE 4.6 $0.91636 \cong Q(\breve{a}_{24}) \le Q(a) \le Q(\bar{a}_{24}) \cong 0.93571$.

Strategy of Player I: If Player II did not fire before, fire at $\langle a \rangle + c$ and play optimally the resulting duel. If he has fired (say at a') play optimally the resulting duel $(2,4), \langle a', a' \wedge c_1 \rangle$.

Strategy of Player II: Fire before $\langle a \rangle + c$ and play optimally afterwards.

We prove that for given constants a above strategies are optimal in the limit and that the limit value of the game is

$$v_{25}(1,a) = -1 + 2Q(a) - 2Q^2(a) + (1 + v_{12})Q^4(a). \tag{15}$$

Suppose that Player II applies above strategy. Then for any strategy $\hat{\xi}$ of Player I

$$\begin{aligned} K(\hat{\xi}, \eta) &\le -P(a) + Q(a)v_{24}(2,a) + k(\hat{\varepsilon}) \\ &= \quad -1 + 2Q(a) - 2Q^2(a) + (1 + v_{12})Q^4(a) + k(\hat{\varepsilon}) = v_{25}(1,a) - k(\hat{\varepsilon}). \end{aligned}$$

$v_{12} \cong 0.04633$ (see Trybuła, 1993, part II).

On other hand, suppose that Player II fires before $\langle a \rangle + c$. For such a strategy $\hat{\eta}$

$$K(\xi, \hat{\eta}) \ge -P(a) + Q(a)v_{24}(2,a) - k(\hat{\varepsilon}) = v_{25}(1,a) - k(\hat{\varepsilon}).$$

If Player II fires after $\langle a \rangle + c$ or does not fire at all

$$\begin{aligned} K(\xi, \hat{\eta}) &\ge P(a) + Q(a)v_{15}(1,a) - k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + (1 + v_{12})Q^4(a) - k(\hat{\varepsilon}) \\ &\ge -1 + 2Q(a) - 2Q^2(a) + (1 + v_{12})Q^4(a) - k(\hat{\varepsilon}) \end{aligned}$$

for any a.

If, at the end, Player II fires at $\langle a \rangle + c$

$$\begin{aligned} K(\xi, \hat{\eta}) &\ge Q^2(a)v_{14}(a) - k(\hat{\varepsilon}) = -Q^2(a) + (1 + v_{12})Q^4(a) - k(\hat{\varepsilon}) \\ &\ge -1 + 2Q(a) - 2Q^2(a) + (1 + v_{12})Q^4(a) - k(\hat{\varepsilon}). \end{aligned}$$

The assertion is proved.

Duel $(2,5), \langle a, a \wedge c \rangle$.

CASE 4.7 $Q(a) \geq Q(a_{25}) \cong 0.99804$

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the resulting duel $(2,4), \langle a', a' \wedge \hat{\varepsilon} \rangle$.

Strategy of Player II: If Player I did not reach the point a_{25} and did not fire do not fire, neither. If he had reached this point and did not fire, fire at $\langle a_{25} \rangle$ and play optimally the resulting duel. If Player I has fired at $a' < a_{25}$ play optimally the resulting duel $(1,5), \langle a' \wedge c_1, a' \rangle$.

The proof that above strategies are optimal in the limit is omitted. We have, similarly as for duel $(2,5), \langle a \rangle$

$$v_{25}(2, a) = 0.$$

CASE 4.8 $0.97465 \cong Q(a_{25}^{(1)}) \leq Q(a) \leq Q(a_{25}) \cong 0.99804$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: Escape. If Player II has fired (say at a') play optimally the resulting duel $(2,4), \langle a', a' \wedge \hat{\varepsilon} \rangle$.

Strategy of Player II: Fire at $\langle a \rangle + c$ and play optimally afterwards.

Suppose that Player I applies the strategy ξ and Player II applies a strategy $\hat{\eta}$. It is easy to prove that

$$\begin{aligned} K(\xi, \hat{\eta}) \quad &\geq \quad -1 + (1 + P^2(\hat{a}_{13}))Q(a) - k(\hat{\varepsilon}) \\ &\text{if} \quad 0.95105 \cong Q(\hat{a}_{24}) \leq Q(a) \leq Q(a_{25}) \cong 0.99804 \end{aligned}$$

(compare with the formulae (14) for the duel $(2,5), \langle a \rangle$).

On the other hand, also comparing with the duel $(2,5), \langle a \rangle$ we obtain that here comes in addition the case in which Player I fires before $\langle a \rangle + c$. For such a strategy $\hat{\xi}$ we have

$$\begin{aligned} K(\hat{\xi}, \eta) \quad &\leq \quad P(a) + Q(a)v_{15}(1, a) + k(\hat{\varepsilon}) \\ &= \quad 1 - 2Q(a) + Q^3(a) + k(\hat{\varepsilon}) \\ &\leq \quad -1 + (1 + P^2(\hat{a}_{13}))Q(a) + k(\hat{\varepsilon}) \end{aligned}$$

if

$$S(Q) = Q^3(a) - (3 + P^2(\hat{a}_{13}))Q(a) + 2 \leq 0.$$

This function is decreasing in Q and $S(Q(a^{(1)})) \cong 0$ for $Q(a^{(1)}) \cong 0.97465$. Then the inequality holds for $Q \geq Q(a^{(1)})$ and strategies ξ and η are optimal in the limit for a specified in Case 4.8. We also have that

$$v_{25}(2, a) = -1 + (1 + P^2(\hat{a}_{13}))Q(a).$$

CASE 4.9 $0.91636 \cong Q(\check{a}_{24}) \leq Q(a) \leq Q(a_{25}^{(1)}) \cong 0.97645$.

Strategy of Player I: Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II: If Player I did not fire before, fire at $\langle a\rangle + c$ and play optimally the resulting duel.

The above strategies are optimal in the limit for given a and the limit value of the game is

$$v_{25}(2,a) = \begin{cases} 1 - 2Q(a) + Q^3(a) & \text{if } Q(\hat{a}_{13}) \le Q(a) \le Q(a_{25}^{(1)}), \\ 1 - 2Q(a) + (1+v_{12})Q^3(a) & \text{if } Q(\breve{a}_{24}) \le Q(a) \le Q(\hat{a}_{13}). \end{cases}$$

The proof is omitted.

5. Results for the duels $(2,5)$.

$$v_{25}(1,a) = \begin{cases} 0 \quad \text{if } Q(a) \ge Q(a_{25}) \cong 0.99804, \\ -1 + (1 + P^2(\hat{a}_{13}))Q(a) \\ \qquad \text{if } 0.95105 \cong Q(\hat{a}_{24}) \le Q(a) \le Q(a_{25}), \\ -1 + (1+v_{23})Q^2(a) \\ \qquad \text{if } 0.93571 \cong Q(\bar{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}), \\ -1 + 2Q(a) - 2Q^2(a) + (1+v_{12})Q^4(a) \\ \qquad \text{if } 0.91636 \cong Q(\breve{a}_{24}) \le Q(a) \le Q(\bar{a}_{24}). \end{cases}$$

$$v_{25}(a) = \begin{cases} 0 \quad \text{if } Q(a) \ge Q(a_{25}), \\ -1 + (1 + P^2(\hat{a}_{13}))Q(a) \\ \qquad \text{if } 0.96894 \cong Q(\hat{a}_{25}) \le Q(a) \le Q(a_{25}), \\ -Q^2(a) + Q^3(a) \\ \qquad \text{if } 0.95572 \cong Q(\hat{a}_{13}) \le Q(a) \le Q(\hat{a}_{25}), \\ -Q^2(a) + (1+v_{12})Q^4(a) \\ \qquad \text{if } 0.95302 \cong Q(\breve{a}_{25}) \le Q(a) \le Q(\hat{a}_{13}), \\ -1 + (1 + P^2(\hat{a}_{13}))Q(a) \\ \qquad \text{if } Q(\hat{a}_{24}) \le Q(a) \le Q(\breve{a}_{25}), \\ -1 + (1+v_{23})Q^2(a) \\ \qquad \text{if } 0.94671 \cong Q(\bar{a}_{25}) \le Q(a) \le Q(\hat{a}_{24}), \\ -Q^2(a) + (1+v_{12})Q^4(a) \\ \qquad \text{if } Q(\breve{a}_{24}) \le Q(a) \le Q(\bar{a}_{25}). \end{cases}$$

$$v_{25}(2,a) = \begin{cases} 0 \quad \text{if } Q(a) \ge Q(a_{25}), \\ -1 + (1 + P^2(\hat{a}_{13}))Q(a) \\ \qquad \text{if } 0.97465 \cong Q(\hat{a}_{25}^{(1)}) \le Q(a) \le Q(a_{25}), \\ 1 - 2Q(a) + Q^3(a) \\ \qquad \text{if } Q(\hat{a}_{13}) \le Q(a) \le Q(a_{25}^{(1)}), \\ 1 - 2Q(a) + (1+v_{12})Q^4(a) \\ \qquad \text{if } Q(a_{24}) \le Q(a) \le Q(\hat{a}_{13}). \end{cases}$$

For the duels with arbitrary movements see Trybuła (1990-1991, 1993a).

For noisy duels see Fox and Kimeldorf (1969), Karlin (1959), Trybuła (1992, 1990-1991, 1993b).

For other duels see Cegielski (1986a;b), Kimeldorf (1983), Orłowski and Radzik (1985a;b), Restrepo (1957), Styszyński (1974), Teraoka (1979).

Part V

1. Reference

The present part V constitutes the sequel to parts I, II and III published in *Control and Cybernetics*, vol. **22**, 1993 No 3, and part IV, in this issue. The definitions and assumptions necessary for the consideration herein contained can be found in these previous parts. For more basic definitions and notions the reader is refered to Karlin (1959).

2. Duels $(3, 5)$

Duel $(3, 5), \langle a \rangle$

CASE 2.1 $Q(a) \geq Q(a_{35}) \cong 0.95288$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: Go ahead and if Player II did not fire before, fire with an absolutely continuous probability distribution (ACPD) in the interval $(\langle a_{35} \rangle, \langle a_{35} \rangle + \alpha(\varepsilon))$ and play optimally the resulting duel. If Player II has fired (say at a') play optimally the duel $(2, 5), \langle a' \wedge \hat{\varepsilon}, a \rangle$.

Strategy of Player II: If Player I had not reached the point a_{35} and did not fire, do not fire, neither. If he reached the point a_{35} and did not fire, fire at $\langle a_{35} \rangle$ and play optimally the resulting duel. If he had fired before he reached a_{35} (say at a') play optimally the duel $(2, 5), \langle a' \wedge \hat{\varepsilon}, a' \rangle$.

The sign $\langle s \rangle$ denotes the first time when Player I reached the point s.

"Play optimally" means apply a strategy optimal in the limit.

The ACPD is chosen to make strategy ξ ε–optimal in limit–the values of ε are chosen for particular values of $\hat{\varepsilon}$.

Let v_{35} and a_{35} be the numbers satisfying the equations

$$\begin{aligned} v_{35} &= P(a_{35}) + Q(a_{35}) v_{25}(1, a_{35}) \\ &= -P(a_{35}) + Q(a_{35}) v_{34} \end{aligned} \tag{16}$$

where $Q(a) = 1 - P(a)$; v_{34}, $v_{25}(1, a_{35})$ are the limit values of the games $(3, 4)$, $(2, 5)$, $\langle a_{35} \wedge c, a_{35} \rangle$, respectively.

Let

$$0.95105 \cong Q(\hat{a}_{24}) \leq Q(a_{35}) \leq Q(a_{25}) \cong 0.99804.$$

For these a_{35}

$$v_{25}(1, a_{35}) = -1 + (1 + P^2(\hat{a}_{13}))Q(a_{35}),$$

where $P(\hat{a}_{13}) \cong 0.04428$. Then from (16) we obtain

$$(1 + P^2(\hat{a}_{13}))Q^3(a_{35}) - (3 + v_{34})Q(a_{35}) + 2 = 0$$

which gives

$$Q(a_{35}) \cong 0.95288, \; v_{35} \cong 0.00400. \tag{17}$$

We prove that strategies ξ and η are optimal in the limit and that the limit value of the game is v_{35} given in the above.

Suppose that Player II fires at $a' < a_{35}$. For such a strategy (denote it by $\hat{\eta}$) we have

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + Q(a')v_{34} - k(\hat{\varepsilon}) \\ &\geq -P(a_{35}) + Q(a_{35})v_{34} - k(\hat{\varepsilon}) = v_{35} - k(\hat{\varepsilon}) \end{aligned}$$

where $K(\cdot, \cdot)$ is the expected gain of Player I and $k(\hat{\varepsilon}) \to 0$ if $\hat{\varepsilon} \to 0$.

Suppose that Player II does not fire before $\langle a_{35} \rangle$. For such a strategy $\hat{\eta}$

$$K(\xi, \hat{\eta}) \geq P(a_{35}) + Q(a_{35})v_{25}(1, a_{35}) - k(\hat{\varepsilon}) = v_{35} - k(\hat{\varepsilon}).$$

Then

$$K(\xi, \breve{\eta}) \geq v_{35} - k(\hat{\varepsilon})$$

for any strategy $\breve{\eta}$ of Player II.

On the other hand, if Player I fires at $a' < a_{35}$ we obtain for such a strategy $\hat{\xi}$

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq P(a') + Q(a')v_{25}(1, a') + k(\hat{\varepsilon}) \\ &= \begin{cases} 1 - Q(a') + k(\hat{\varepsilon}) & \text{if} \quad Q(a) \geq Q(a_{25}) \cong 0.99804, \\ 1 - 2Q(a') + (1 + P^2(\hat{a}_{13}))Q^2(a') + k(\hat{\varepsilon}) & \\ & \text{if} \quad 0.95105 \cong Q(\hat{a}_{24}) \leq Q(a) \leq Q(a_{25}). \end{cases} \end{aligned}$$

It is easy to prove that in both cases

$$K(\hat{\xi}, \eta) \leq v_{35} + k(\hat{\varepsilon}).$$

Suppose that Player I does not fire before or at $\langle a_{35} \rangle$. For such a strategy $\hat{\xi}$

$$K(\hat{\xi}, \eta) \leq -P(a_{35}) + Q(a_{35})v_{34} + k(\hat{\varepsilon}) = v_{35} + k(\hat{\varepsilon}).$$

Suppose that Player I fires at $\langle a_{35}\rangle$. In this case

$$\begin{aligned} K(\hat{\xi},\eta) &\le Q^2(a_{35})v_{24}(a_{35}) + k(\hat{\varepsilon}) \\ &= Q^2(a_{35})P^2(\hat{a}_{13}) + k(\hat{\varepsilon}) < v_{35} + k(\hat{\varepsilon}). \end{aligned}$$

At the end, suppose that Player I does not reach the point a_{35} and does not fire. We have

$$K(\hat{\varepsilon},\eta) = 0 < v_{35}.$$

The assertion is proved.

CASE 2.2 $0.94812 \cong Q(\hat{a}_{35}) \le Q(a) \le Q(a_{35}) \cong 0.95288$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: If Player II did not fire before, fire with an ACPD in the interval $(\langle a\rangle, \langle a\rangle + \alpha(\varepsilon))$ and play optimally the resulting duel. If he has fired (say at a') play optimally the duel $(2,4), \langle a', a' \wedge \hat{\varepsilon}\rangle$.

Strategy of Player II: Fire at $\langle a\rangle$ and play optimally the resulting duel.

We prove that strategies ξ and η are optimal in the limit and that the value of the game is

$$v_{35}(a) = -1 + (1 + v_{34})Q(a)$$

Suppose that Player II fires after $\langle a\rangle + \alpha(\varepsilon)$. For such a strategy $\hat{\eta}$ we have

$$\begin{aligned} K(\xi,\hat{\eta}) &\ge P(a) + Q(a)v_{25}(1,a) - k(\hat{\varepsilon}) \\ &= \begin{cases} 1 - 2Q(a) + (1 + P^2(\hat{a}_{13}))Q^2(a) - k(\hat{\varepsilon}) \\ \qquad \text{if} \quad 0.95105 \cong Q(\hat{a}_{24}) \le Q(a) \le Q(a_{35}), \\ 1 - 2Q(a) + (1 + v_{12})Q^3(a) - k(\hat{\varepsilon}) \\ \qquad \text{if} \quad 0.93571 \cong Q(\bar{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}). \end{cases} \end{aligned}$$

Consider the subcases:

i) $1 - 2Q(a) + (1 + P^2(\hat{a}_{13}))Q^2(a) \ge -1 + (1 + v_{34})Q(a)$.

From Case 2.1 it follows that it is satisfied for

$$Q(a) \le Q(a_{35}) \cong 0.95288.$$

ii) $1 - 2Q(a) + (1 + v_{23})Q^3(a) \ge -1 + (1 + v_{34})Q(a)$

or

$$S(Q) = (1 + v_{23})Q^3(a) - (3 + v_{34})Q(a) + 2 \ge 0.$$

This function is positive for $Q < Q(\hat{a}_{24})$. Then the inequality always holds for $Q(\bar{a}_{24}) \le Q(a) \le Q(\hat{a}_{24})$.

On the other hand, suppose that Player I also fires at $\langle a \rangle$. For such a strategy $\hat{\xi}$ we have

$$K(\hat{\xi},\eta) \le Q^2(a)v_{24}(a) + k(\hat{\varepsilon})$$
$$= \begin{cases} P^2(\hat{a}_{13})Q^2(a) + k(\hat{\varepsilon}) \text{ if } Q(a) \ge Q(\hat{a}_{24}), \\ -Q^2(a) + (1+v_{23})Q^3(a) + k(\hat{\varepsilon}) \\ \qquad \text{if } 0.91636 \cong Q(\check{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}). \end{cases}$$

Consider the subcases

i) $P^2(\hat{a}_{13})Q^2(a) \le -1 + (1+v_{34})Q(a)$.

This inequality always holds for $Q(a) \ge Q(\hat{a}_{24})$.

ii) $-Q^2(a) + (1+v_{23})Q^3(a) \le -1 + (1+v_{34})Q(a)$

or

$$S(Q) = (1+v_{23})Q^3(a) - Q^2(a) + (1+v_{34})Q(a) + 1 \le 0.$$

This function is negative for $Q > Q(\hat{a}_{35}) \cong 0.94812$.

This ends the proof of the assertion.

CASE 2.3 $0.91636 \cong Q(\check{a}_{24}) \le Q(a) \le Q(\hat{a}_{35}) \cong 0.94812$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: Fire at $\langle a \rangle$ and play optimally the resulting duel.

Strategy of Player II: Fire at $\langle a \rangle$ and play optimally the resulting duel.

We prove that now

$$v_{35}(a) = Q^2(a)v_{24}(a) = -Q^2(a)+(1+v_{23})Q^3(a) \text{ if } Q(\check{a}_{24}) \le Q(a) \le Q(\hat{a}_{35}). \quad (18)$$

Suppose that Player II does not fire at $\langle a \rangle$. For such a strategy $\hat{\eta}$ we have

$$K(\xi,\hat{\eta}) \ge P(a) + q(a)v_{25}(1,a) - k(\hat{\varepsilon})$$
$$= \begin{cases} 1 - 2Q(a) + (1+v_{23})Q^3(a) - k(\hat{\varepsilon}) \\ \qquad \text{if } 0.93571 \cong Q(\bar{a}_{24}) \le Q(a) \le Q(\hat{a}_{24}), \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + (1+v_{12})Q^5(a) - k(\hat{\varepsilon}) \\ \qquad \text{if } 0.91636 \cong Q(\check{a}_{24}) \le Q(a) \le Q(\bar{a}_{24}). \end{cases}$$

Consider the subcases:

i) $1 - 2Q(a) + (1+v_{23})Q^3(a) \ge -Q^2(a) + (1+v_{23})Q^3(a)$.

This inequality is satisfied for any a.

ii) $1 - 2Q(a) + 2Q^2(a) + (1+v_{12})Q^5(a) \ge -Q^3(a) + (1+v_{23})Q^3(a)$

or

$$S(Q) = (1+v_{12})Q^5(a) - (3+v_{23})Q^3(a) + 3Q^2(a) - 2Q(a) + 1 \ge 0.$$

This multinomial is decreasing for $Q \in (Q(\check{a}_{24}), Q(\bar{a}_{24}))$ and $S(Q(\bar{a}_{24})) \cong S(0.93571) \cong 0.00413$. Then the inequality holds for Q belonging to this interval.

On the other hand, suppose that Player I does not fire at $\langle a\rangle$. We have for such a strategy $\hat{\xi}$

$$\begin{aligned} K(\hat{\xi},\eta) &\leq -P(a)+Q(a)v_{34}+k(\hat{\varepsilon}) \\ &\leq -Q^2(a)+(1+v_{23})Q^3(a)+k(\hat{\varepsilon}). \end{aligned}$$

From 2.2 it follows that it holds for $Q(a)\leq Q(\hat{a}_{35})\cong 0.94812$.

Then for given constants a strategies ξ and η are optimal in the limit and the limit value of the game is given by (18).

Duel $(3,5)$, $\langle a\wedge c, a\rangle$

We define optimal in the limit strategies for particular cases.

CASE 2.4 $Q(a)\geq Q(a_{35})\cong 0.95288$.

Strategy of Player I: Go ahead and if Player II did not fire before, fire with an ACPD in the interval $(\langle a_{35}\rangle,\langle a_{35}\rangle+\alpha(\varepsilon))$ and play optimally the resulting duel. If Player II has fired (say at a') play optimally the resulting duel $(2,5)$, $\langle a'\wedge c_1, a'\rangle$.

Strategy of Player II: If Player I did not reach the point a_{35} and did not fire, do not fire also. If he reached the point a_{35} and did not fire, fire at $\langle a_{35}\rangle$ and play optimally the resulting duel. If he fired before he reached a_{35} (say at a') play optimally the duel $(2,5)$, $\langle a'\wedge\hat{\varepsilon}, a'\rangle$.

Now

$$v_{35}(1,a)=-1+(1+v_{34})Q(a_{35})\cong 0.00400.$$

The proof is the same as for the duel $(3,5)$, $\langle a\rangle$.

CASE 2.5 $0.93571\cong Q(\bar{a}_{24})\leq Q(a)\leq Q(a_{35})\cong 0.95288$.

Strategy of Player I: If Player II did not fire before, fire with an ACPD in the interval $(\langle a\rangle+c,\langle a\rangle+c+\alpha(\varepsilon))$ and play optimally the resulting duel. If he has fired (say at a') play optimally the obtained duel $(2,4)$, $\langle a', a'\wedge c_1\rangle$.

Strategy of Player II: Fire before $\langle a\rangle+c$ and play optimally the duel $(3,4)$.

We have now

$$v_{35}(1,a)=-1+(1+v_{34})Q(a).$$

The proof of limit optimality of above strategies is omitted.

CASE 2.6 $0.91636\cong Q(\breve{a}_{24})\leq Q(a)\leq Q(\bar{a}_{24})\cong 0.93571$.

Strategy of Player I: Fire at $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II: Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

Now also

$$v_{35}(1,a) = -1 + (1 + v_{34})Q(a)$$

and the proof of limit optimality is omitted as well.

Duel $(3,5), \langle a, a \wedge c\rangle$

Case 2.7 $Q(a) \geq Q(a_{35}) \cong 0.95288$.

Optimal in the limit strategies of Player I and II are the same as for the duel $(3,5), \langle a\rangle$ and the limit value of the game is the same.

Case 2.8 $0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(a_{35}) \cong 0.95288$.

Optimal in the limit strategies of Player I and II are:

Strategy of Player I: Fire before $\langle a\rangle + c$ and play optimally the resulting duel.

Strategy of Player II: If Player I did not fire before fire at $\langle a\rangle + c$ and play optimally the resulting duel. If he has fired (say at a') play optimally the duel $(2,5), \langle a' \wedge c_1, a'\rangle$.

We have

$$v_{35}(2,a) = P(a) + Q(a)v_{25}(1,a)$$

$$= \begin{cases} 1 - 2Q(a) + (1 + P^2(\hat{a}_{13}))Q^2(a) \\ \qquad \text{if} \quad Q(\hat{a}_{24}) \leq Q(a) \leq Q(a_{35}), \\ 1 - 2Q(a) + (1 + v_{23})Q^3(a) \\ \qquad \text{if} \quad 0.93571 \cong Q(\bar{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{24}), \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + (1 + v_{12})Q^5(a) \\ \qquad \text{if} \quad 0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(\bar{a}_{24}). \end{cases}$$

The proof is omitted.

3. Results for the duels $(3,5)$

$$v_{35}(1,a) = \begin{cases} -1 + (1 + v_{34})Q(a_{35}) \cong 0.00400 \\ \qquad \qquad \text{if } Q(a) \geq Q(a_{35}) \cong 0.95288, \\ -1 + (1 + v_{34})Q(a) \ \text{if } 0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(a_{35}), \end{cases}$$

$$v_{35}(a) = \begin{cases} -1 + (1 + v_{34})Q(a_{35}) & \text{if } Q(a) \geq Q(a_{35}), \\ -1 + (1 + v_{34})Q(a) & \text{if } 0.94812 \cong Q(\hat{a}_{35}) \leq Q(a) \leq Q(a_{35}), \\ -Q^2(a) + (1 + v_{23})Q^3(a) & \text{if } 0.91636 \cong Q(\breve{a}_{24}) \leq Q(a) \leq Q(\hat{a}_{35}), \end{cases}$$

$$v_{35}(2,a)=\begin{cases}-1+(1+v_{34})Q(a_{35}) \text{ if } Q(a)\geq Q(a_{35}),\\ 1-2Q(a)+(1+P^2(\hat{a}_{13}))Q^2(a)\\ \qquad\qquad \text{if } 0.95105\cong Q(\hat{a}_{24})\leq Q(a)\leq Q(a_{35}),\\ 1-2Q(a)+(1+v_{23})Q^3(a)\\ \qquad\qquad \text{if } 0.93571\cong Q(\bar{a}_{24})\leq Q(a)\leq Q(\hat{a}_{24}),\\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+(1+v_{12})Q^5(a)\\ \qquad\qquad \text{if } 0.91636\cong Q(\breve{a}_{24})\leq Q(a)\leq Q(\bar{a}_{24}),\end{cases}$$

$v_{12}\cong 0.04633$, $v_{23}\cong 0.05354$, $v_{34}\cong 0.05365$, $P(\hat{a}_{13})\cong 0.04428$.

4. Duels $(4,5)$ and $(5,5)$

Duel $(4,5)$, $\langle a\rangle$. $Q(a)\geq Q(a_{45})\cong 0.91791$.

We define strategies ξ and η of Players I and II.

Strategy of Player I: Go ahead and if Player II did not fire, fire with an ACPD in the interval $(\langle a_{45}\rangle, \langle a_{45}\rangle+\alpha(\varepsilon))$ and play optimally the resulting duel. If he has fired play optimally the duel $(4,4)$.

Strategy of Player II: If Player I did not reach the point a_{45} and did not fire, do not fire, neither. If he has reached a_{45} and did not fire, fire at $\langle a_{45}\rangle$ and play optimally the duel $(4,4)$. If Player I had fired (say at a') before he reached the point a_{45} play optimally the duel $(3,5)$, $\langle a'\wedge\hat{\varepsilon}, a'\rangle$.

We determine the constant v_{45} and a_{45} from the equations

$$\begin{aligned} v_{45} &= P(a_{45})+Q(a_{45})v_{35}(1,a_{45}) \\ &= -P(a_{45})+Q(a_{45})v_{44}. \end{aligned} \tag{19}$$

Let

$$0.91636\cong Q(\breve{a}_{24})\leq Q(a_{45})\leq Q(a_{35})\cong 0.95288.$$

For this a_{45}

$$v_{35}(1,a_{45})=-1+(1+v_{34})Q(a_{45})$$

and we obtain from (19)

$$(1+v_{34})Q^2(a_{45})-(3+v_{44})Q(a_{45})=0$$

what gives

$$Q(a_{45})\cong 0.91791,\ v_{45}\cong 0.05195. \tag{20}$$

We prove that for this a_{45} strategies ξ and η are optimal in the limit if $a\leq a_{45}$ and the limit value of the game is v_{45}.

Suppose that Player II fires at $a' < a_{45}$. For this strategy (denote it by $\hat{\eta}$) we have

$$\begin{aligned} K(\xi,\hat{\eta}) &\geq -P(a') + Q(a')v_{44} - k(\hat{\varepsilon}) \\ &\geq -P(a_{45}) + Q(a_{45})v_{44} - k(\hat{\varepsilon}) = v_{45} - k(\hat{\varepsilon}). \end{aligned}$$

Suppose that Player II does not fire before $\langle a_{45}\rangle + \alpha(\varepsilon)$. We obtain for such a strategy $\hat{\eta}$

$$K(\xi,\hat{\eta}) \geq P(a_{45}) + Q(a_{45})v_{35}(1,a_{45}) - k(\hat{\varepsilon}) = v_{45} - k(\hat{\varepsilon})$$

for properly chosen $\alpha(\varepsilon)$.

On the other hand, suppose that Player I fires at $a' < a_{45}$ (strategy $\hat{\xi}$). We have

$$K(\hat{\xi},\eta) \leq P(a') + Q(a')v_{35}(1,a') + k(\hat{\varepsilon})$$

$$= \begin{cases} 1-(1-v_{35})Q(a') & \text{if} \quad Q(a) \geq Q(a_{35}) \cong 0.95288, \\ 1-2Q(a') + (1+v_{34})Q^2(a') & \\ & \text{if} \quad 0.91791 \cong Q(a_{45}) \leq Q(a') \leq Q(a_{35}). \end{cases}$$

It is easy to prove that in both cases

$$K(\hat{\xi},\eta) \leq v_{45} \cong 0.051195.$$

Suppose that Player I does not fire before or at $\langle a_{45}\rangle$ but reaches the point a_{45}. For such a strategy $\hat{\xi}$

$$K(\hat{\xi},\eta) \leq -P(a_{45}) + Q(a_{45})v_{44} + k(\hat{\varepsilon}) = v_{45} + k(\hat{\varepsilon}).$$

Suppose that Player I fires at a_{45}. In this case

$$\begin{aligned} K(\hat{\xi},\eta) &\leq Q^2(a_{45}) + k(\hat{\varepsilon}) \\ &\cong 0.04521 + k(\hat{\varepsilon}) < v_{45} + k(\hat{\varepsilon}). \end{aligned}$$

The assertion is proved.

It is easy to see that if $a \leq a_{45}$ the same strategies ξ and η are optimal in the limit for the duels $(4,5), \langle a \wedge c, a\rangle$ and $(4,5), \langle a, a \wedge c\rangle$. Then we shall denote these duels together with $(4,5), \langle a\rangle$ simply by $(4,5)$.

Duel $(5,5), \langle a\rangle$. $Q(a) \geq Q(a_{55}) \cong 0.92082$.

Strategy of Player I: Go ahead and if Player II did not fire before, fire with an ACPD in the interval $(\langle a_{55}\rangle, \langle a_{55}\rangle + \alpha(\varepsilon))$ and play optimally the duel $(4,5)$. If he has fired play optimally the duel $(5,4)$.

Strategy of Player II: If Player I did not fire before, fire at $\langle a_{55}\rangle$ and play optimally the resulting duel $(5,4)$ or $(4,4)$. If he has fired play optimally the duel $(4,5)$.

Now we have

$$Q(a_{55}) = \frac{2}{2+v_{54}+v_{45}} \cong 0.92082, \quad (21)$$
$$v_{55} = -1+(1+v_{54})Q(a_{55}) \cong 0.12701$$

since $v_{54} \cong 0.22392$ (see Section 5.).

The proof of limit optimality of above strategies and the formulae (21) is omitted.

5. Duels (m,n), $m < n$.

Duel $(m,n), \langle a\rangle$. $Q(a) \geq Q(a_{mn})$.

At the end, let us consider the duel in which Player I has m bullets, Player II has n bullets, $m > n \geq 1$, and the game is beginning when $a \leq a_{mn}$. Let us define strategies ξ and η of Players I and II.

CASE 5.1

Strategy of Player I: Go ahead and if Player II did not fire before, fire with an ACPD in the interval $(\langle a_{mn}\rangle, \langle a_{mn}\rangle + \alpha(\varepsilon))$ and play optimally the duel $(m-1,n)$. If Player II has fired, play optimally the duel $(m,n-1)$.

Strategy of Player II: If Player I did not fire before, fire at $\langle a_{mn}\rangle$ (i.e. when Player I reaches the point a_{mn}) and play optimally the resulting duel $(m,n-1)$ or $(m-1,n-1)$. If Player I has fired play otimally the duel $(m-1,n)$. If Player I did not reach the point a_{mn} and did not fire do not fire, neither.

Let, for $a < a_{mn}$

$$v_{mn}(a) \stackrel{\text{def}}{=} v_{mn} = P(a_{mn}) + Q(a_{mn})v_{m-1,n} = -P(a_{mn}) + Q(a_{mn})v_{m,n-1}, \quad (22)$$

for $v_{\hat{m}1}$, $v_{\hat{n}\hat{n}}$, $\hat{m} \leq m$, $\hat{n} \leq n$, given.

From (22) we obtain

$$Q(a_{mn}) = \frac{2}{2+v_{m,n-1}-v_{m-1,n}},$$

$$v_{mn} = -1+(1+v_{m,n-1})Q(a_{mn}), \quad (23)$$

for $v_{\hat{m}1}$, $v_{\hat{n}\hat{n}}$, $\hat{m} \leq m$, $\hat{n} \leq n$, given.

We prove that if

$$n \leq 5, m > n > 1, \quad (24)$$

$$Q^2(a_{mn})v_{m-1,n-1} \leq v_{mn}, \quad (25)$$

$$Q(a_{\hat{m}\hat{n}}) > Q(a_{\hat{m},\hat{n}-1}),\ Q(a_{\hat{m}\hat{n}}) > Q(a_{\hat{m}-1,\hat{n}}) \tag{26}$$

for all $\hat{m} > \hat{n} > 1$, $\hat{m} \leq m$, $\hat{n} \leq n$ then strategies ξ and η are optimal in the limit and v_{mn} given by (23) is the limit value of the game.

Suppose that Player II fires before $\langle a_{mn} \rangle$. For such strategy $\hat{\eta}$ we have

$$\begin{aligned} K(\xi, \hat{\eta}) &\geq -P(a') + Q(a')v_{m,n-1} - k(\hat{\varepsilon}) \\ &\geq -P(a_{mn}) + Q(a_{mn})v_{m,n-1} - k(\hat{\varepsilon}) = v_{mn} - k(\hat{\varepsilon}) \end{aligned}$$

if $a_{mn} < a_{m,n-1}$.

Suppose that Player II fires after $\langle a_{mn} \rangle + \alpha(\varepsilon)$. With properly chosen $\alpha(\varepsilon)$ for given $\hat{\varepsilon}$ we have

$$K(\xi, \hat{\eta}) \geq P(a_{mn}) + Q(a_{mn})v_{m-1,n} - k(\hat{\varepsilon}) = v_{mn} - k(\hat{\varepsilon})$$

if $a_{mn} < a_{m-1,n}$.

On the other hand, if Player I fires at a' before $\langle a_{mn} \rangle$ (strategy $\hat{\xi}$) we obtain

$$\begin{aligned} K(\hat{\xi}, \eta) &\leq P(a') + Q(a')v_{m-1,n} + k(\hat{\varepsilon}) \\ &\leq P(a_{mn}) + Q(a_{mn})v_{m-1,n} + k(\hat{\varepsilon}) = v_{mn} + k(\hat{\varepsilon}) \end{aligned}$$

if $a_{mn} < a_{m-1,n}$.

If Player I does not fire at $\langle a_{mn} \rangle$ or before we have

$$K(\hat{\xi}, \eta) \leq -P(a_{mn}) + Q(a_{mn})v_{m,n-1} + k(\hat{\varepsilon}) = v_{mn} + k(\hat{\varepsilon})$$

if $a_{mn} < a_{m-1,n}$.

If Player I fires at $\langle a_{mn} \rangle$ then

$$K(\hat{\xi}, \eta) \leq Q^2(a_{mn})v_{m-1,n-1} + k(\hat{\varepsilon}) \leq v_{mn} + k(\hat{\varepsilon})$$

by assumptions (25) and (26), since also

$$a_{mn} < a_{m,n-1} < a_{m-1,n-1}.$$

At the end, suppose that Player I does not reach the point a_{mn} and does not fire. For such a strategy $\hat{\xi}$ we obtain

$$K(\hat{\xi}, \eta) = 0 \leq v_{mn}$$

since by (22)

$$v_{mn} = 1 - (1 - v_{m-1,n})Q(a_{mn}) \geq 0$$

for $m > n$, $n \leq 5$ because $v_{nn} \geq 0$ for $n \leq 5$, $v_{m1} \geq 0$ for $m \leq 5$ and then $v_{m-1,n} \geq 0$ for $m > n$, $n \leq 5$ by inductive argument with respect to m.

Moreover v_{m1} can be determined for any m (see Trybuła, 1993b, part I) and v_{nn} are determined for $n \leq 5$ (see Trybuła, 1993b, parts I-III) then v_{mn} can be determined for all natural m, n satisfying the condition (24).

This ends the proof of the assertion.

CASE 5.2

Strategy of Player I: Go ahead and if Player II did not fire before, fire at $\langle a_{mn}\rangle$ and play optimally the resulting duel $(m-1, n)$ or $(m-1, n-1)$. If he fired play optimally the duel $(m, n-1)$.

Strategy of Player II: If Player I did not fire before, fire with an ACPD in the interval $(\langle a_{mn}\rangle, \langle a_{mn}\rangle + \alpha(\varepsilon))$ and play optimally the duel $(m, n-1)$. If he has fired play optimally the duel $(m-1, n)$. If Player I did not reach the point a_{mn} and did not fire do not fire, neither.

These strategies are optimal in the limit if, besides (24) and (25), the condition

$$Q^2(a_{mn})v_{m-1,n-1} \geq v_{mn} \tag{27}$$

holds and for these (m, n) the formulae (23) hold as well. The proof is omitted.

It is easy to see that if $a \leq a_{mn}$ then the same strategies are optimal in the limit for the duels $(m, n), \langle a \wedge c, a\rangle$ and $(m, n), \langle a, a \wedge c\rangle$. We denote these duels together with the duel $(m, n), \langle a\rangle$ simply by (m, n).

Now we present the tables of the values v_{mn} and $Q(a_{mn})$, $m > n$, $m \leq 20$, $n \leq 5$, computed on the basis of Trybuła (1993b), part I, obtained values v_{mm}, $Q(a_{mm})$, $m \leq 5$ and formulae (23). By asterisks we denote these (m, n) for which inequality (25) holds.

m	v_{m1}	$Q(a_{m1})$	v_{m2}	$Q(a_{m2})$	v_{m3}	$Q(a_{m3})$
2	0.50000	0.75000				
3	0.60000	0.80000	0.34604	0.84128		
4	0.66667	0.83333	0.43639	0.86184	0.26997	0.88414
5	0.71429	0.85714	0.50515	0.87801	0.34678	0.89478
6	0.75000	0.87500	0.55913	0.89093	0.40948	0.90402
7	0.77778	0.88889	0.60257	0.90145	0.46147	0.91195
8	0.80000	0.90000	0.63828	0.91016	0.50521	0.91878
9	0.81818	0.90909	0.66813	0.91747	0.54248	0.92468
10	0.83333	0.91667	0.69345	0.92370	0.57459	0.92981
11	0.84615	0.92308	0.71520	0.92907	0.60254	0.93432
12	0.85714	0.92857	0.73407	0.93373	0.62706	0.93829
13	0.86667	0.93333	0.75061	0.93782	0.64876	0.94182
14	0.87500	0.93750	0.76521	0.94144	0.66808	0.94498
15	0.88235	0.94118	0.77820	0.94467	0.68541	0.94782
16	0.88889	0.94444	0.78983	0.94756	0.70102	0.95038
17	0.89474	0.94737	0.80031	0.95016	0.71516	0.95270
18	0.90000	0.95000	0.80979	0.95252	0.72803	0.95482
19	0.90476	0.95238	0.81841	0.95467	0.73978	0.95676
20	0.90909	0.95455	0.82629	0.95663	0.75057	0.95854

m	v_{m4}	$Q(a_{m4})$	v_{m5}	$Q(a_{m5})$
5	0.22392*	0.90875		
6	0.28981	0.91510	0.19272*	0.92473
7	0.34595	0.92095	0.25017*	0.92884
8	0.39419	0.92624	0.30054	0.93283
9	0.43601	0.93097	0.34491	0.93656
10	0.47256	0.93520	0.38421	0.94001
11	0.50474	0.93898	0.41921	0.94316
12	0.53329	0.94236	0.45055	0.94604
13	0.55876	0.94542	0.47875	0.94867
14	0.58163	0.94817	0.50425	0.95108
15	0.60227	0.95067	0.52741	0.95328
16	0.62098	0.95295	0.54853	0.95531
17	0.63803	0.95503	0.56787	0.95717
18	0.65361	0.95694	0.58563	0.95889
19	0.66792	0.95869	0.60201	0.96048
20	0.68110	0.96031	0.61715	0.96196

The duels considered in Trybuła (1993b), in the preceding part, in this volume, and in the present part may be treated as mathematical idealization of real duels. For example, the duel (m, n) can be a model for the duel in which Player I has m missiles and some machine guns or hands-grenades (short distance weapon) and Player II has only n missiles.

Noisy duels are considered in Berzin (1983), Fox and Kimeldorf (1969), Karlin (1959), Kimeldorf (1983), Teraoka (1976), Trybuła (1992, 1990-1991), Trybuła (1993b).

For other duels see Cegielski (1986), Cegielski (1986), Orłowski and Radzik (1985a;b), Radzik (1988), Restrepo (1957), Styszyński (1974), Teraoka (1979), Trybuła (1990), Yanovskaya (1969).

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