Control and Cybernetics

vol. **24** (1995) No. 1

A normal possibility decision rule

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Decision-making in real problems occurs in a fuzzy environment. Thus, Fuzzy-Bayes decision rules are proposed to cope with a fuzzy state of nature. These decision rules are based on the probability of fuzzy events. Furthermore, a decision rule based on fuzzy utility functions are constructed. On the other hand, normal possibility theory is constructed by interpreting membership functions of fuzzy sets as normal possibility distributions which are most efficient among possibility distributions in the sense of both operations on possibility distributions and identification of their parameters. The purpose of our study is the application of normal possibility theory to Bayes decision problems. This normal possibility decision rule can be applied to the decision problems, in which both a normal possibility distribution of a state of nature and the membership functions of fuzzy events are given by our knowledge or our belief.

The theoretical main objective is to construct the intensive utility function by introducing the concept of a fuzzy event to a state of nature and to formulate a decision rule based on the intensive utility function named the fuzzy utility function.

1. Introduction

Decision-making in real problems is done in a fuzzy environment. Bayes decision rule is based on the statistical rule. When we have difficulties in obtaining utility functions in Bayes decision rule, Fuzzy-Bayes decision rule is constructed to cope with a fuzzy state of nature named fuzzy event. Okuda et al. (1978) and Tanaka et al. (1979) formulated Fuzzy-Bayes decision rule to facilitate determination of the utility function of Bayes decision rule in a fuzzy environment. In some situations, Uemura (1990a,b) proposed the decision rule on Fuzzy events introducing the concept of the indifference events and the reserved judgement into the above Fuzzy-Bayes decision rule. Further Uemura (1991a,b) showed how to obtain the fuzzy utility functions and proposed a decision rule based on these fuzzy utility functions. On the other hand, the normal possibility theory has been proposed as an evidence theory of expert's knowledge by Tanaka and Ishibuchi (1991;1992a;b). In this paper we discuss the application of normal possibility theory to Bayes decision problems and we construct a fuzzy decision theory using normal possibility distributions.

First, we obtain the possibility distributions of fuzzy utility via the extension principle for a mapping. Second, we obtain the possibility distributions of fuzzy expected utility via the extension principle for binary operations. Third, we discuss the application of this decision rule to the decision problems, in which the decision maker obtains both the normal possibility distribution of a state of nature and the membership functions of fuzzy events, on the basis of his knowledge and belief.

If we use the probability distribution of a state of nature, we calculate the probability of fuzzy events defined by Zadeh (1968). But there is the assumption in this calculation that the membership functions are orthogonal (the sum of membership functions is one). Without this orthogonal condition of membership functions, for the purpose of using the possibility measure of fuzzy events, we apply the possibility distribution of a state of nature rather than the probability distribution.

2. Normal possibility theory

By interpretation of membership functions as possibility distributions, possibility theory was proposed by Zadeh (1977) and many frameworks of possibility theory were developed Nguyen (1978); Histal (1978). Normal possibility distributions are the most efficient among possibility distributions in the sense of both operations on and identification of parameters of possibility distributions. Thereafter the framework of normal possibility theory has been developed, Tanaka, Ishibuchi (1991;1992a;b).

A multivariate normal possibility distribution of evidence A is defined as follows:

$$\prod_A(x) = \exp\{-(x-a)^t D_A(x-a)\} \tag{1}$$

where a is a central vector and D_A is a positive-definite matrix, and let us denote $\prod_A(x)$ by $(a, D_A)_e$.

The most important definition and theorems in normal possibility theory are as follows (see Tanaka, Ishibuchi 1991;1992a;b):

DEFINITION 2.1 *Possibility Measure of Fuzzy Event. Given a normal possibility distribution* $(a, D_A)_e$ *of evidence* A *and a normal possibility distribution* $(b, D_B)_e$ *of fuzzy event* B*, the possibility measure of fuzzy event* B *is defined as follows:*

$$\prod_A(B) = \max_x \prod_A(x) \cdot \mu_B(x) \tag{2}$$

THEOREM 2.1 *Possibility Distribution of a Linear System. If a normal possibility distribution* $(a, D_A)_e$ *of evidence* A *and a matrix* T *are given, the possibility distribution of the linear system* $y = Tx$ *is obtained as follows:*

$$\prod_{TA}(y) = \exp\{-(y - Ta)^t (T D_A^{-1} T^t)^{-1} (y - Ta)\} \tag{3}$$

where TA *is the fuzzy output induced by* $y = TA$ *and* $\prod_A(e)$

THEOREM 2.2 *Possibility Distribution of* $A + B$*. The possibility distribution of* $A + B$ *is obtained as follows*

$$\prod_{A+B}(z) = \exp\{-(z - a - b)^t (D_A^{-1} + D_B^{-1})^{-1} (z - a - b)\} \tag{4}$$

3. Possibility distribution of fuzzy expected utility

Bayes decision problem is denoted by $\langle S, D, U_1 \rangle$, where S is the set of states of nature ($S = \{S_1, \ldots, S_k\}$), D is the set of decisions ($D = \{D_1, \ldots, D_n\}$), U_1 is the utility function on $S \times D$. We denote normal possibility decision problem by $\langle B, D, U_2 \rangle$, where B is the set of fuzzy events ($B = \{B_1, \ldots, B_m\}$), and U_2 is the fuzzy utility function on $B \times D$.

First, we assume that normal possibility distribution $(a, D_A)_e$ of evidence A and normal possibility distribution $(b_i, D_{Bi})_e$ of fuzzy event B_i are given by a decision maker, on the basis of his knowledge and belief, as follows:

$$\begin{aligned} \prod_A(s) &= \exp\{-(s - a)^t D_A (s - a)\} \\ \mu_{Bi}(s) &= \exp\{-(s - b_i)^t D_{Bi} (s - b_i)\} \end{aligned} \tag{5}$$

where s is the variable on a set of states of nature S.

By Definition 2.1, the possibility measure of fuzzy event B_i is obtained as follows:

$$\prod_A(B_i) = \max_s \prod_A(s) \cdot \mu_{Bi}(s) \tag{6}$$

Second, we assume that fuzzy utility function $U_2(B_i, D_j)(z)$ is defined by a normal possibility distribution, on the basis of our knowledge and belief, as follows:

$$\prod_{U_2(B_i, D_j)}(z) = \exp\{-(z - U_i j)^t D_{U_i j} (z - U_{ij})\} \tag{7}$$

whose parameter representation is $(U_{ij}, D_{U_{ij}})_e$.

Let us denote the value of possibility measure of fuzzy event B_i by $k_i (i = 1, \ldots, n)$ which is constant. By the extension principle, the possibility distribution of fuzzy utility multiplied by possibility measure of fuzzy event is obtained as the following theorem.

THEOREM 3.1 *Possibility Distribution of Fuzzy Utility Multiplied by Scalar Numbers*

$$\prod_{k_i \cdot U_2(B_i,D_j)}(z) = \exp\{-(z - k_i U_{ij})^t k_i^t D_{U_{ij}} k_i (z - k_i U_{ij})\} \tag{8}$$

Now, fuzzy expected utility $E(D_j)$ is given as the sum of the possibility distributions of a state of nature multiplied by the possibility measures of fuzzy events. Therefore, fuzzy expected utility is defined as follows:

DEFINITION 3.1 *Fuzzy Expected Utility*

$$E(D_j) = k_1 \cdot U(B_i, D_j) + \cdots + k_n \cdot U(B_n, D_j) \tag{9}$$

By Theorem 2.2, the possibility distribution of fuzzy expected utility $E(D_j)$ is obtained as follows:

THEOREM 3.2 *Possibility Distribution of Fuzzy Expected Utility*

$$\prod_{E(D_j)}(z) = (\sum_i k_i U_{ij} \; (\sum_i (k_i^t D_{U_{ij}} k_i)^{-1})^{-1})_e \tag{10}$$

4. Identification of possibility distribution of fuzzy utility

The fuzzy utility function $U_2(D_j, F_i)$ is obtained by the following mapping of the strict utility function $U_1(D_j, s)$ in Uemura (1991b), Tanaka and Ishibuchi (1991):

$$U_2(D_j, F_i) = \int \mu_{F_i}(s)/U_1(D_j, s) \tag{11}$$

where $U_1(D_j, s) : D_j \times s \longrightarrow [0,1]$, $\mu_{F_i}(s) : s \longrightarrow [0,1]$, and $U_2(D_j, F_i) : D_j \times F_i \longrightarrow [0,1]$.

Let us assume that $U_1(s, D_j)$ is a monotone and continuous function, then possibility distribution $\prod_{U_2(B_i,D_j)}(z)$ of fuzzy utility is obtained by the following theorem:

THEOREM 4.1 *Possibility Distribution of Fuzzy Utility*

$$\begin{aligned} \prod_{U_2(B_i,D_j)}(z) &= \int \mu_{B_i}(s)/U_1(s, D_j) \\ &= \sup_{\{z \mid z = U_1(s,D_j)\}} \mu_{B_i}(s) \\ &= \mu_{B_i}(U_1^{-1}(s, D_j)) \end{aligned} \tag{12}$$

where $U_1^{-1}(U_1(s, D_j)) = s$.

The identification of $U_1(s, D_j)$ is described in Keeney and Raiffa (1976). If the decision maker is a risk neutral person, the possibility distribution of fuzzy utility is obtained by Theorem 4.1 and the extension principle as follows:

THEOREM 4.2 *Possibility Distribution of Fuzzy Utility. Possibility distribution of fuzzy utility is a normal possibility distribution as follows:*

$$\prod_{U_2(B_i,D_j)} = (d_j + c_j b_i \, (c_j D_{B_i}{}^{-1} c_j^t)^{-1})_e \tag{13}$$

where $U_1(s, D_j) = c_j s + d_j$ *which is obtained by certainty equivalent, on the basis of a risk neutral decision maker's knowledge and belief.*

The parameter representation $(U_{ij}, D_{U_{ij}})_e$ can be written as:

$$U_{ij} = d_j + c_j b_i \tag{14}$$

$$D_{U_{ij}} = (c_j D_{B_i}^{-1} c_j^t)^{-1} \tag{15}$$

If a decision maker is a risk neutral person, the utility function is a linear function. Using the extension principle for a mapping in formula (11), we obtain the fuzzy utility function. Therefore, if a decision maker is a risk neutral person, the fuzzy utility function is normal possibility distributed. However, in this, as risk aversion and proneness cases, the fuzzy utility function is obtained by formula (11) (see Uemura 1991a;b). But in the latter cases, the fuzzy utility function is not normally distributed but is a fuzzy number. In this paper, because of the wish to construct a normal possibility decision rule, the decision maker is only a risk neutral person. And except for a risk neutral person, a decision maker makes a decision on the basis of the before proposed decision rule Uemura (1991a;b) by using the possibility measure of a fuzzy event in (2) instead of the probability of a fuzzy event.

5. Ordering of fuzzy expected utility

For decision-making, ordering of fuzzy expected utility is indispensable. The weight of a state of nature S_i in a decision D_j is denoted by $w_{ij}(i = l, \ldots, n)$, a factor of importance given by the decision maker. We consider linear system $E(D_j) = w_j^t z$, where w_j is an important weight vector whose elements are w_{ij}. By Theorem 2.1, the possibility distribution of this linear system $E(D_j)$ is obtained as follows:

$$\prod_{E(D_j)}(y) = (w_j^t \sum_i k_i U_{ij} \; ((w_j^t \sum_i (k_i^t D_{U_{ij}} k_i)^{-1} w_j)^{-1})_e \tag{16}$$

where the possibility distribution $\prod_{E(D_j)}(y)$ of this linear system $E(D_j)$ is obtained in one dimensional space.

Dubois and Prade (1988) defined the ordering of fuzzy numbers as follows:

$$Pos(A > B) = \sup_y \inf_{x \geq y} \min(\mu_A(x), 1 - \mu_B(y)) \tag{17}$$

where A and B are fuzzy numbers, x and y are elements, and $\mu_A(x)$ and $\mu_B(y)$ are possibility distributions of A and B.

This indicator has the following property (see Dubois and Prade, 1988):

$$Pos(A > B) + Pos(B > A) = 1 \tag{18}$$

The ordering of fuzzy expected utilities and the selection of the optimal decision can be obtained by the following rule:

If $Pos(E(D_j) > E(D_i)) > 0.5$ for all i $(i \neq j)$, then the optimal decision D^* is D_j.

In particular, when the possibility distributions of fuzzy expected utilities are normal possibility distributions, we obtain the ordering of fuzzy expected utilities through the following theorem:

THEOREM 5.1 *Ordering of fuzzy expected utilities*

If $a_{E(D_j)} \geq a_{E(D_i)}$ *then* $Pos(E(D_j) > E(D_i)) = 1$.

where $a_{E(D_j)}$ *is the central parameter of* $E(D_j)$ *and* $a_{E(D_i)}$ *is the central parameter of* $E(D_i)$

By Theorem 5.1, we obtain the following simple decision rule:

If $a_{E(D_j)} > a_{E(D_i)}$ for all i $(i \neq j)$, then the optimal decision D^* is D_j.

6. Example

Now, let us consider the judgement problem of the recommendation entrance on the basis of the letter of recommendation. Let us set a state of nature as the continuous score $[0, 100]$ of the evaluation of result. We consider two decisions, $D_1 = \{success\}$, $D_2 = \{failure\}$.

We have the belief that the degree of difficulty in the entrance examination of this year is almost equal to the one of the last year. And we have the belief that the possibility distribution of a state of nature is a normal possibility distribution. Furthermore, we obtain the data of the score of evaluation in last year. By our information about a state of nature, we set the possibility distribution of a state of nature as.

$$\prod_A(s) = \exp\{-(s-a)^2/2c^2\} \tag{19}$$

where a and c are identified on the basis of data (see Tanaka, Ishibuchi, 1991; 1992a;b).

We consider two fuzzy events on the state of nature, $B_1 = \{bad\}$ and $B_2 = \{good\}$. The membership functions of two fuzzy events are shown in (20).

$$\begin{aligned} \mu_{B_1}(s) &= \exp\{-(s-b_1)^2/2d^2\} \\ \mu_{B_2}(s) &= \exp\{-(s-b_2)^2/2e^2\} \end{aligned} \tag{20}$$

where $b_i (i = 1, 2)$, d and e are given by the decision maker, on the basis of his knowledge and belief.

Further, we assume that the decision maker has the belief that he is a risk neutral person and he would like to make a rough decision in above fuzzy situation.

By (6), the possibility measures of fuzzy events are obtained as follows:

$$\begin{aligned} \prod_A(B_1) &= \exp\{(ac + b_1d)^2/2(c+d) - a^2c - b_1^2d\} \\ \prod_A(B_2) &= \exp\{(ac + b_2e)^2/2(c+e) - a^2c - b_2^2e\} \end{aligned} \qquad (21)$$

For simplicity, set $\prod_A(B_i) = k_i$ $(i = 1, 2)$. The utility functions with respect to D_1 and D_2 are obtained as follows:

$$\begin{aligned} U_1(s, D_1) &= (p_{11} - p_{21})/(s_l - s_2)s + (s_1p_{21} + s_2p_{11})/(s_1 - s_2) \\ U_1(s, D_2) &= (p_{12} - p_{22})/(s_l - s_2)s + (s_1p_{22} + s_2p_{12})/(s_1 - s_2) \end{aligned} \qquad (22)$$

where $p_{ij} = U_1(s_i, D_j)(i = 1, 2; j = 1, 2)$ which is obtained by certainty equivalent, on the basis of a risk neutral decision maker's knowledge and belief.

Setting $U_1(s, D_1) = g_1s + r_1$ and $U_1(s, D_2) = q_2s + r_2$, we have the following possibility distributions of fuzzy utility functions $U_2(B_i, D_j)$:

$$\begin{aligned} \prod_{U_2(B_1,D_1)}(z) &= \exp\{-(z - r_1 - q_1b_1)^2/2q_1^2d^2\} \\ \prod_{U_2(B_2,D_1)}(z) &= \exp\{-(z - r_1 - q_1b_2)^2/2q_1^2e^2\} \\ \prod_{U_2(B_1,D_2)}(z) &= \exp\{-(z - r_1 - q_2b_1)^2/2q_2^2d^2\} \\ \prod_{U_2(B_2,D_2)}(z) &= \exp\{-(z - r_1 - q_2b_2)^2/2q_2^2e^2\} \end{aligned} \qquad (23)$$

In what follows, $\prod_{U_2(B_i,D_j)}$ is represented as $(g_{ij}, h_{ij})_e$ $(i = 1, 2; j = 1, 2)$

By Definition 3.1, fuzzy expected utilities are defined as follows:

$$\begin{aligned} E(D_1) &= k_1 \cdot U_2(B_1, D_1) + k_2 \cdot U_2(B_2, D_1) \\ E(D_2) &= k_1 \cdot U_2(B_1, D_2) + k_2 \cdot U_2(B_2, D_2) \end{aligned} \qquad (24)$$

By Theorem 3.2, possibility distributions of fuzzy expected utility are obtained as follows:

$$\begin{aligned} \prod_{E(D_1)}(z) &= \ exp\{-(z - k_1g_{11}k_2g_{21})^2/2(k_1h_{11} + k_2h_{21})^2\} \\ \prod_{E(D_2)}(z) &= \ exp\{-(z - k_1g_{12}k_2g_{22})^2/2(k_1h_{12} + k_2h_{22})^2\} \end{aligned} \qquad (25)$$

Finally by Theorem 5.1, the ordering of fuzzy expected utility and the optimal decision are obtained as follows:

1. $k_1g_{11} + k_2g_{21} > k_1g_{12} + k_2g_{22}$: the optimal decision is D_1.
2. $k_1g_{11} + k_2g_{21} \leq k_1g_{12} + k_2g_{22}$: the optimal decision is D_2.

7. Conclusions

In this paper we discussed the application of normal possibility theory to Bayes decision problems and constructed a normal possibility decision rule.

There are two suppositions in this decision rule, namely:

1. The possibility distributions of evidence and fuzzy events are normal possibility distributions given by the decision-maker.

2. The decision maker is a risk neutral person.

There are four conclusions for the construction of this normal possibility decision rule:

1. The possibility distribution of fuzzy utility is obtained as a normal possibility distribution whose parameters are indicated by the above two suppositions.

2. Fuzzy expected utility is defined as the sum of fuzzy utility functions multiplied by possibility measures of fuzzy events.

3. The possibility distribution of fuzzy expected utility is obtained as a normal possibility distribution whose parameters are indicated by the above two suppositions and the conclusions 1.

4. The rule of ordering fuzzy expected utilities and selecting the optimal decision is constructed.

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