Control and Cybernetics

vol. **24** (1995) No. 2

Solutions of coupled Riccati equations arising in differential games

by

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In this paper, accurate solutions of coupled, variable coefficients Riccati equations arising in differential games are constructed. First, the existence interval is predeterminated, then a discrete numerical solution at a subnet of its points is obtained. Finally, a continuous numerical solution with a prefixed accuracy is constructed. Results are illustrated with an example.

Keywords: Matrix Riccati differential equation, differential games, initial value problems, numerical solutions, error bound.

1. Introduction

Let us consider a two-player linear quadratic differential game defined by

$$x' = Ax + B_1u_1 + B_2u_2;\ x(0) = x_0;\ 0 \le t \le t_f$$

where $x(t) \in \mathbf{R}^n$ and, for $i = 1, 2, u_i(t) \in \mathbf{R}^{r_i}$. The cost function associated with player i is

$$J_i = \frac{1}{2}\left\{x_f^T K_{if} x_f + \int_0^{t_f} \left(x^T Q_i x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2\right) dt\right\}$$

where $x_f \equiv x(t_f)$; $i = 1, 2$ and all matrices are symmetric and R_{ii}, $i = 1, 2$ are positive definite. The open-loop Nash controls must satisfy, Starr, Ho (1969)

$$u_i = -R_{ii}^{-1} B_i^T K_i(t)\Phi(t, 0)x_0;\ i = 1, 2$$

with $K_1(t)$ and $K_2(t)$ satisfying the coupled Riccati type differential equations

$$\begin{aligned} K_1' &= -Q_1 - A^T K_1 - K_1 A + K_1 S_1 K_1 + K_1 S_2 K_2;\ K_1(t_f) = K_{1f} \quad (1)\\ K_2' &= -Q_2 - A^T K_2 - K_2 A + K_2 S_2 K_2 + K_2 S_1 K_1;\ K_2(t_f) = K_{2f} \end{aligned}$$

wherein

$$S_i = B_i R_{ii}^{-1} B_i^T;\ i = 1, 2,$$

$$\Phi\prime(t, 0) = \left(A - \sum_{i=1}^{2} S_i K_i \right) \Phi(t, 0);\ \Phi(t, t) = I,$$

and I denotes the identity matrix.

The coupled Riccati equations are generally difficult to solve. Numerical approximate solutions are given in Cruz, Chen (1971), and an iterative algorithm has recently been proposed in Jódar, Abou-Kandil (1988). For a particular case, an analytic solution may be found as in Abou-Kandil, Bertrand (1986). The aim of this paper is to devise continuous numerical solutions and error bound of coupled Riccati matrix systems of the type (1), where $K_i(t)$, K_{if}, $S_i(t)$, $Q_i(t)$ for $i = 1, 2$, and $A(t)$ are twice continuously differentiable $\mathbf{C}^{n\times n}$ valued functions.

The system (1) may be written in the following rectangular Riccati type form

$$\begin{aligned} W\prime(t) &= C(t) - D(t)W(t) - W(t)A(t) - W(t)B(t)W(t) \\ W(t_f) &= W_f \end{aligned} \tag{2}$$

where

$$\begin{aligned} B(t) &= \begin{bmatrix} S_1(t) & \vdots & S_2(t) \end{bmatrix};\ D(t) = \begin{bmatrix} A^T(t) & 0_{n\times n} \\ 0_{n\times n} & A^T(t) \end{bmatrix}; \\ C(t) &= \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix};\ W(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \end{bmatrix};\ W_f = \begin{bmatrix} K_{1f} \\ K_{2f} \end{bmatrix} \in \mathbf{C}^{2n\times n} \end{aligned}$$

The nonautonomous Riccati differential equation (2) seems to have received little numerical treatment in the literature, especially for the study of accuracy and error bounds. Moreover, those studies which have been done are devoted almost exclusively to the autonomous case, in spite of the fact that many real systems are nonautonomous. Some exceptions can be found in Dieci (1992); Jódar, Abou-Kandil (1988); Kunkel, Mehrmann (1990); Kenney, Leipnik (1985); Oshman, Bar-Itzhack (1984), but in both cases no error bounds in terms of the data are given.

From Reid (1972), the solution of (2) is given by

$$W(t) = \hat{V}(t)\hat{U}^{-1}(t) = [0_{2n\times n} I_{2n\times 2n}] Z(t) \{[I_{n\times n} 0_{n\times 2n}] Z(t)\}^{-1} \tag{3}$$

where

$$\begin{aligned} Z\prime(t) &= \begin{bmatrix} \hat{U}\prime(t) \\ \hat{V}\prime(t) \end{bmatrix} = \hat{S}(t) \begin{bmatrix} \hat{U}(t) \\ \hat{V}(t) \end{bmatrix} = \hat{S}(t) Z(t) \\ Z(t_f) &= \begin{bmatrix} I \\ W_f \end{bmatrix} \in \mathbf{C}^{3n\times n};\ \hat{S}(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & -D(t) \end{bmatrix} \in \mathbf{C}^{3nx3n}; \end{aligned} \tag{4}$$

with $W(t)$ given by (3) defined in an the interval where $U(t)$ is invertible. If we write

$$t = t(s) = t_f - s;\ 0 = t_0 \leq s \leq t_f$$

and

$$\begin{aligned} Y(s) &= Z(t_f - s) = Z(t);\ S(s) = \hat{S}(t_f - s) = \hat{S}(t) \\ U(s) &= \hat{U}(t_f - s) = \hat{U}(t);\ V(s) = \hat{V}(t_f - s) = \hat{V}(t) \end{aligned}$$

then, equation (4) takes the form

$$Y\prime(s) = -S(s)Y(s);\ Y(0) = Y_0 = \begin{bmatrix} I_{n\times n} \\ W_f \end{bmatrix} \in \mathbf{C}^{3n\times n} \tag{5}$$

The paper is organized as follows. In the next section, an existence interval for problem (2) is determined in terms of the data. Then, we construct a discrete numerical solution of equation (5) in a net of points where the solution is defined, and error bounds are obtained. Given an admissible error $\epsilon > 0$, and interpolating the discrete numerical solution using linear B-spline matrix functions, we determine a continuous numerical solution, in the predetermined interval, whose error is smaller than ϵ, uniformly in the existence domain. Finally, an illustrative example is presented.

If C is a matrix in $\mathbf{C}^{p\times q}$, we denote by $\|C\|$ the 2-norm, defined as the square root of the maximum eigenvalue of $C^H C$, where C^H denotes the conjugate transpose of C. From Golub, van Loan (1983) it follows that

$$\max|c_{ij}| \leq \|C\| \leq \sqrt{pq}\max|c_{ij}| \tag{6}$$

If P and Q are matrices in $\mathbf{C}^{r\times r}$ and P is invertible, then from the Banach lemma, Golub, van Loan (1983), if $\|P - Q\| < \|P^{-1}\|^{-1}$, one gets the invertibility of Q and $\|P^{-1} - Q^{-1}\| < \|P^{-1}\|\|Q^{-1}\|\|P - Q\|$. Taking into account the inequality $\|Q^{-1}\| \leq \|Q^{-1} - P^{-1}\| + \|P^{-1}\|$, we can write

$$\|P^{-1} - Q^{-1}\| \leq (1 - \|P^{-1}\|\|P - Q\|)^{-1}\|P^{-1}\|^2\|P - Q\| \tag{7}$$

2. Discrete numerical solution using one-step matrix methods

We begin this section with a lemma which extends Theorem 1 of Jódar (1992), and determine an existence interval for the solution $W(t)$ of (2) as well as an upper bound of $U^{-1}(t)$ in such interval, where $U(t) = \begin{bmatrix} I & 0 \end{bmatrix} Y(t)$, and $Y(t)$ is the solution of (5).

LEMMA 2.1 *Let* $q_0 = \max\{\| \begin{bmatrix} A(t) & B(t) \end{bmatrix} \|;\ 0 \le t \le t_f\}$, $k_0 = \max\{\|S(t)\|;\ 0 \le t \le t_f\}$, *and* $\delta \le t_f$ *be a positive number satisfying*

$$\delta k_0 + \ln(\delta) < -\ln\left(q_0 \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\|\right) \tag{8}$$

Then the local solution $W(t)$ *of problem (2) is defined by (3) in* $[0,\delta]$. *Furthermore, if* $Y(t)$ *is the solution of (5) then* $U(t) = \begin{bmatrix} I & 0 \end{bmatrix} Y(t)$ *is invertible in* $[0,\delta]$, *and*

$$\|U^{-1}(t)\| < (1 - M\delta)^{-1};\ M = q_0 \exp(\delta k_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\|;\ 0 \le t \le \delta \tag{9}$$

Proof. Since $U(0) = I$, from the Perturbation lemma Ortega, Rheinboldt (1972), $U(t)$ is invertible if $\|U(t) - I\| < 1$. Note that

$$Y(t) - Y(0) = \int_0^t Y\prime(r)dr = -\int_0^t S(r)Y(r)dr,$$

and

$$\begin{aligned} U(t) - I &= U(t) - U(0) = \begin{bmatrix} I & 0 \end{bmatrix} (Y(t) - Y(0)) = \\ &= -\int_0^t \begin{bmatrix} A(r) & B(r) \end{bmatrix} Y(r)dr \end{aligned}$$

Taking norms, and from the definition of q_0, it follows that

$$\|U(t) - I\| \le t q_0 \max\{\|Y(r)\|;\ 0 \le r \le t\} \tag{10}$$

From Flett (1980), the solution $Y(t)$ of (5) satisfies

$$\|Y(t)\| \le \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| \exp(\delta k_0);\ 0 \le t \le t_r \tag{11}$$

Using (10) and (11), if we select a positive number $\delta \le t_r$ such that

$$\delta q_0 \exp(\delta k_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| < 1, \tag{12}$$

then $U(t)$ is invertible in $0 \le t \le \delta$. Taking logarithms in (12) one gets the inequality (8) and the invertibility of $U(t)$ in $0 \le t \le \delta$. Note that

$$U\prime(t) = \begin{bmatrix} I & 0 \end{bmatrix} Y\prime(t) = -\begin{bmatrix} I & 0 \end{bmatrix} S(t)Y(t) = -\begin{bmatrix} A(t) & B(t) \end{bmatrix} Y(t)$$

and using norms for $0 \le t \le \delta$, (11) implies that

$$\|U\prime(t)\| \le q_0 \exp(\delta q_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\|;\ 0 \le t \le \delta$$

Finally, from Freedman (1969) and from (12) one concludes (9). ■

Let us consider the one-step matrix method defined by

$$Y_{n+1} - Y_n = h\{B_1 f_{n+1} + B_0 f_n\};\ B_0 + B_1 = I \tag{13}$$

where B_0, B_1 are matrices in $\mathbf{C}^{r\times r}$ and Y_n and $f_n = f(t_n, Y_n)$ belong to $\mathbf{C}^{r\times r}$ and $t_n = nh \in [0,\delta]$, $h > 0$, $0 \le n \le N-1$, $N = \delta/h$. Definition (13) is associated to the problem

$$Y\prime(t) = f(t, Y(t));\ Y(0) = Y_0 \in \mathbf{C}^{r\times q};\ 0 \le t \le \delta \tag{14}$$

where $f : [0,\delta] \times \mathbf{C}^{r\times q} \to \mathbf{C}^{r\times q}$ is bounded, continuous, and satisfies the Lipschitz condition

$$\|f(t,P) - f(t,Q)\| < L\|P - Q\| \tag{15}$$

to guarantee the existence of a unique continuously differentiable matrix function $Y(t)$, a solution of (14), Flett (1980). By Jódar, Ponsoda (1993), this method is zero-stable and consistent, and the global discretization error at t_n

$$e_n = Y(t_n) - Y_n, \tag{16}$$

where $Y(t_n)$, is the theoretical exact solution at the net of points t_n. Furthermore Y_n the numerical solution obtained by application of the method (13), satisfies

$$\|e_n\| \le \Gamma^* t_n h^p G D \exp(t_n L \Gamma^* B^*);\ n \ge 0 \tag{17}$$

where L is the Lipschitz constant defined by (15), $t_n nh$, $N = \delta/h$, and

$$\Gamma^* = (1 - hL\|B_1\|)^{-1};\ h < (L\|B_1\|)^{-1} \tag{18}$$

where

$$B^* = \|B_0\| + \|B_1\|$$

In addition

$$D \ge \max\{\|Y^{(p+1)}(t)\|;\ 0 \le t \le \delta\};\ G = \|C_{p+1}\|; \tag{19}$$

if we assume that the method (13) is of order p, i.e.

$$C_0 = C_1 = 0;\ C_s = \frac{1}{s!}I - \frac{1}{(s-1)!}B_1;\ s = 2,3,\ldots$$
$$C_p = 0;\ C_{p+1} = 0;$$

see Jódar, Ponsoda (1993).

Now, applying method (13), with $B_0 = B_1 = I/2$, to the problem (5), yields

$$Y_{n+1} - Y_n = \frac{h}{2}\{-S(t_{n+1})Y_{n+1} - S(t_n)Y_n\}; \tag{20}$$

where $Y_0 = \begin{bmatrix} I \\ W_f \end{bmatrix}$; $t_n = nh$, $0 \le n \le N-1$, $N = \delta/h$, δ is defined by Lemma 2.1 and $S(t)$ is given by (4)-(5). Note that in this case we have $C_1 = C_2 = 0$ and $C_3 = -I/12$. Thus, (20) defines a method of order $p = 2$. The constants appearing in (18)-(19) take the values

$$G = \|C_3\| = 1/12,\ B^* = \|B_0\| + \|B_1\| = 1, \qquad \Gamma^* = (1 - hL/2)^{-1} \tag{21}$$

where k_0, defined by Lemma 2.1, is the Lipschitz constant provided in the initial value problem (5), and $D \ge \max\{\|Y^{(3)}(t)\|;\ 0 \le t \le \delta\}$. Note that the theoretical solution $Y(t)$ of (5) satisfies

$$\begin{aligned} Y^{(2)}(t) &= -S\prime(t)Y(t) + S^2(t)Y(t), \\ Y^{(3)}(t) &= -S^{(2)}(t)Y(t) + 2S\prime(t)S(t)Y(t) + S(t)S\prime(t)Y(t) - S^3(t)Y(t) \end{aligned} \tag{22}$$

Let us denote by k_1 and k_2 the positive constants defined by

$$k_i = \max\{\|S^{(i)}(t)\|;\ 0 \le t \le \delta\};\ i = 1, 2, \tag{23}$$

then, from (11), (22) and (23), it follows that

$$\max\{\|Y^{(3)}(t)\|;\ 0 \le t \le \delta\} \le \exp(\delta k_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| \{k_0^3 + 3k_1k_0 + k_2\} \tag{24}$$

Note that, selecting $h < 1/k_0$, the constant Γ^* defined by (21) satisfies $\Gamma^* < 2$, and taking into account (17), (21), (24), the discretization error e_n defined by (16) takes the form

$$\|e_n\| \le \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| \frac{h^2 t_n}{6} \exp(\delta k_0)\{k_0^3 + 3k_1k_0 + k_2\} \exp(2t_nk_0) \tag{25}$$
$$0 \le n \le N;\ N = \delta/h$$

Solving (20), it is easy to show that

$$Y_0 = \begin{bmatrix} I \\ W_f \end{bmatrix};\ Y_n = \prod_{j=0}^{n-1} \left\{ \left(I + \frac{h}{2}S(t_{n-j})\right)^{-1} \left(I - \frac{h}{2}S(t_{n-j-1})\right) \right\} Y_0 \tag{26}$$
$$1 \le n \le N$$

The matrices Y_n given by (26) are well defined because, for $h < 2/k_0$, and from the perturbation lemma, $\left(I + \frac{h}{2}S(t_{n-j})\right)$ are invertible for $0 \leq j \leq n-1$. Taking the block componets of Y_n, we can write

$$U_n = \begin{bmatrix} I & 0 \end{bmatrix} Y_n;\ V_n = \begin{bmatrix} 0 & I \end{bmatrix} Y_n;\ n \geq 1;\ U_0 = I;\ V_0 = W_f \tag{27}$$

Let us denote by $\{u_n\}$ and $\{v_n\}$ the matrix sequences defined by

$$\begin{aligned} u_n &= U(t_n) - U_n = \begin{bmatrix} I & 0 \end{bmatrix} \{Y(t_n) - Y_n\} \\ v_n &= V(t_n) - V_n = \begin{bmatrix} 0 & I \end{bmatrix} \{Y(t_n) - Y_n\} \end{aligned} \tag{28}$$

where Y_n is given by (26). Taking norms in (28), it follows that

$$\|u_n\| \leq \|e_n\|;\ \|v_n\| \leq \|e_n\|;\ 0 \leq n \leq N \tag{29}$$

Select a step-size h for which

$$h < \left\{ 6(1 - M\delta)\exp(-3\delta k_0) \left[\left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| (k_0^3 + 3k_1k_0 + k_2)\delta \right]^{-1} \right\}^{1/2} \tag{30}$$

where M is defined by (9). From (25), (29) and (9), if h satisfies (30), then

$$\|u_n\| = \|U(t_n) - U_n\| \leq 1 - M\delta \leq \|U^{-1}(t)\|^{-1};\ 0 \leq n \leq N, \tag{31}$$

From (31) and the perturbation lemma, invertibility of U_n follows. From (27) we therefore have that

$$W_n = V_n U_n^{-1} = \left\{ \begin{bmatrix} 0 & I \end{bmatrix} Y_n \right\} \left\{ \begin{bmatrix} I & 0 \end{bmatrix} Y_n \right\}^{-1};\ 0 \leq n \leq N \tag{32}$$

is a numerical approximation of the theoretical value $W(t_n)$ of the exact solution of (2) at $t_n = nh$, $0 \leq n \leq N$, $N = \delta/h$.

3. Construction of continuous numerical solutions of a prescribed accuracy

The corresponding linear B-spline matrix functions which interpolate the sequences U_n, V_n defined by (27)-(28) in the interval $[0, \delta]$ take the form, Jódar, Ponsoda (1993)

$$\begin{aligned} S_U(t) &= h^{-1}\{(t_{n+1} - t_n)U_n + (t - t_n)U_{n+1}\} = \begin{bmatrix} I & 0 \end{bmatrix} S_Y(t); \\ & t_n \leq t \leq t_{n+1} \end{aligned} \tag{33}$$

$$\begin{aligned} S_V(t) &= h^{-1}\{(t_{n+1} - t_n)V_n + (t - t_n)V_{n+1}\} = \begin{bmatrix} 0 & I \end{bmatrix} S_Y(t); \\ & t_n \leq t \leq t_{n+1} \end{aligned} \tag{34}$$

where

$$T_Y(t) = \sum_{n=-1}^{N-1} Y(t_{n+1})B_{1n}(t);\ S_Y(t) = \sum_{n=-1}^{N-1} Y_{n+1}B_{1n}(t); \qquad (35)$$
$$0 = t_0 \le t \le t_N = \delta$$

with

$$B_{1n}(t) = h^{-1}\begin{cases} t - t_n; & \text{for} \quad t_n \le t \le t_{n+1} \\ t_{n+2} - t; & \text{for} \quad t_{n+1} \le t \le t_{n+2} \end{cases} \qquad (36)$$

and $Y(t_n)$ is the theoretical value of the solution of (5) and Y_n is given by (26). By Jódar, Ponsoda (1993), if $0 = t_0 \le t \le t_N = Nh = \delta$, we have

$$\max_{0\le t\le\delta} \|Y(t) - T_Y(t)\| \le \frac{rh^2}{8} \max_{0\le t\le\delta} \|Y^{(2)}(t)\| \qquad (37)$$

and

$$\|T_Y(t) - S_Y(t)\| \le \max_{0\le n\le N} \|Y(t_n) - Y_n\| \qquad (38)$$

where, from (6), $r = n\sqrt{3}$ because $Y(t)$ are $\mathbf{C}^{3n\times n}$ valued function.

Note that form the expression for $Y^{(2)}(t)$ it follows that

$$\max_{0\le t\le\delta} \|U^{(2)}(t)\| \le \max_{0\le t\le\delta} \|Y^{(2)}(t)\| \le \exp(\delta k_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| \{k_0^2 + k_1\} \qquad (39)$$

Then, from (25), (9), (37) and (38), (39),

$$\begin{aligned} \|U(t) - S_U(t)\| &\le \|U(t) - T_U(t)\| + \|T_U(t) - S_U(t)\| \\ &\le h^2 \exp(\delta k_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\| \\ &\quad \left\{ \frac{r}{8}(k_0^2 + k_1) + \frac{\delta}{6}\exp(2\delta k_0)\{k_0^3 + 3k_1k_0 + k_2\} \right\} = h^2\gamma \end{aligned} \qquad (40)$$

Let M be defined by (9) and let h_0 be the positive number defined by

$$h_0 = \left\{ \frac{1}{2}(1 - M\delta)\gamma^{-1} \right\}^{1/2} \qquad (41)$$

From (40) and (9), and provided $h < h_0$,

$$\|U(t) - S_U(t)\| \le \frac{1}{2}(1 - M\delta) \le \frac{1}{2}\|U^{-1}(t)\|^{-1};\ 0 \le t \le \delta \qquad (42)$$

Thus given an admissible error ϵ, taking $h < \min\left\{h_0, \frac{1}{k_0}, 1\right\}$ such that

$$\ln(h) < \frac{1}{4}\ln\left(\frac{\epsilon}{\alpha_1+\alpha_2}\right) \tag{51}$$

then from (49), we conclude that $\hat{W}(t)$ defined by (43) is an approximate solution of problem (2), whose error is uniformly upper bounded by ϵ in all the interval $[0,\delta]$. Hence, the following main result has been proved:

THEOREM 3.1 *With the above notation, let $\epsilon > 0$ and let δ and M be the constants defined by Lemma 2.1. Let h be a positive number such that $Nh = \delta$, $h < \min\left\{h_0, \frac{1}{k_0}, 1\right\}$ and (51) is satisfied. Let U_n and V_n be the matrix sequences defined by (32), where Y_n is given by (26). Then $\hat{W}(t)$ defined by (43) in the interval $t_n = nh \le t \le t_{n+1} = (n+1)h$, for $0 \le n \le N-1$, is a continuous approximate solution of problem (2), whose error is uniformly upper bounded by ϵ in $[0,\delta]$.*

Example. Let us consider the coupled Riccati system (1) where

$$A(t) = -\frac{1}{2}t^2 I_{2x2};\ Q_1(t) = \frac{1}{2}t^2 I_{2x2}$$

$$Q_2(t) = \frac{1}{2}t^2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix};\ S_1(t) = \frac{1}{2}t^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix};\ S_2(t) = t^2 I_{2x2}$$

$$t \in [0,1];\ r = \sqrt{3n^2} = 2\sqrt{3};\ W_f = 0$$

Then

$$S(t) = \begin{bmatrix} A(t) & S_1(t) & S_2(t) \\ Q_1(t) & -A^T(t) & 0 \\ Q_2(t) & 0 & -A^T(t) \end{bmatrix} = \frac{t^2}{2}\begin{bmatrix} -1 & 0 & 2 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S(t) = \frac{t^2}{2}S;\ S'(t) = tS;\ S^{(2)}(t) = S$$

Note that

$$\left\|\begin{bmatrix} I \\ W_f \end{bmatrix}\right\| = 1;\ \left\|\begin{bmatrix} -1 & 0 & 2 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 2 \end{bmatrix}\right\| = 3.442978652$$

hence from (42) and the perturbation lemma Ortega, Rheinboldt (1972), the invertibility of $S_u(t)$ is assured for all t in the interval $[0, \delta]$.

Expressions (3) and (32) provide the analytical approximate solution of problem (2) in $[0, \delta]$, defined by

$$\begin{aligned} \hat{W}(t) &= S_V(t)S_U^{-1}(t) = \\ &= \{(t_{n+1}-t)V_n + (t-t_n)V_{n+1}\}\{(t_{n+1}-t)U_n + (t-t_n)U_{n+1}\}^{-1} \quad (43) \\ & t_n \leq t \leq t_{n+1} \end{aligned}$$

where $S_V(t)$ and $S_U(t)$ are defined by (33)-(36). In order to compute the error of $\hat{W}(t)$, write

$$\begin{aligned} W(t) - \hat{W}(t) &= V(t)U^{-1}(t) - S_V(t)S_U^{-1}(t) = \\ &= [V(t) - S_V(t)]U^{-1}(t) + V(t)[U^{-1}(t) - S_U^{-1}(t)] - \quad (44) \\ &- [V_t - S_V t][U^{-1}(t) - S_U^{-1}(t)] \end{aligned}$$

In a way analogous to the comments preceding (39)-(40), it is easy to show that for $h < h_0$ and for $0 \leq t \leq \delta$,

$$\|V(t) - S_V(t)\| \leq \gamma h^2 \quad (45)$$

where γ is defined by (40). From (12) and the relationship $V(t) = \begin{bmatrix} 0 & I \end{bmatrix} Y(t)$, it follows that

$$\|V(t)\| \leq \exp(\delta k_0) \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\|; \; 0 \leq t \leq \delta \quad (46)$$

Let h_0 be defined by (41), if $h < h_0$, then (42) implies the invertibility of $S_U(t)$ in all the interval $[0, \delta]$, and from (7) we can write

$$\begin{aligned} &\|U^{-1}(t) - S_U^{-1}(t)\| \\ &\leq (1 - \|U^{-1}(t)\|\|U(t) - S_U(t)\|)^{-1}\|U^{-1}(t)\|^2\|U(t) - S_U(t)\| \quad (47) \\ &\leq 2\|U^{-1}(t)\|^2\|U(t) - S_U(t)\| \end{aligned}$$

Now using (9), (40) and (47), it follows that

$$\|U^{-1}(t) - S_U^{-1}(t)\| \leq 2(1 - M\delta)^{-2}\gamma h^2 \quad (48)$$

where γ is defined by (40). Taking into account (9) and (44)-(48), if $0 \leq t \leq \delta$ and $h < \min(h_0, 1/k_0)$,

$$\|W(t) - \hat{W}(t)\| \leq \alpha_1 h^2 + \alpha_2 h^4, \text{ for } 0 \leq t \leq \delta \quad (49)$$

where

$$\begin{aligned} \alpha_1 &= \gamma(1 - M\delta)^{-1}\left\{1 + 2(1 - M\delta)^{-1}\exp(\delta k_0)\left\|\begin{bmatrix} I \\ W_f \end{bmatrix}\right\|\right\} \quad (50) \\ \alpha_2 &= 2(1 - M\delta)^{-2}\gamma^2 \end{aligned}$$

DIECI L., (1992) Numerical Integration of The Differential Riccati Equations and Some Related Issues. *SIAM J. Num. Anal.* 29, vol. 3(1992), 781-815.

FLETT T.M., (1980) *Differential Analysis.* Cambridge Univ. Press. New York, 1980.

FREEDMAN H.I., (1969) An Explicit Estimate of the Norm of a Matrix. *SIAM Review*, 11 No. 2 (1969), 254-256.

GOLUB G.H., VAN LOAN C.F., (1983) *Matrix Computations.* Johns Hopkins Univ. Press Baltimore, Maryland, 1983.

JÓDAR L., (1992) Rational Approximate Solutions and Error Bounds for the Nonsymmetric Riccati Matrix Differential Equations. *Appl. Maths. and Computations*, 50 (1992), 23-31.

JÓDAR L., ABOU-KANDIL H., (1988) A Resolution Method for Riccati Differential Systems Coupled in their Quadratic Terms. *SIAM J. Math. Anal.* 19 (1988), 1225-1230.

JÓDAR L., PONSODA E., (1993) Continuous Numerical Solutions and Error Bounds for Matrix Differential Equations. *Proc. First Int. Coll. on Num. Anal.* VSP Utrecht, The Netherlands (1993), 73-88.

KENNEY S.S., LEIPNIK R.B., (1985) Numerical Integration of the Differential Matrix Riccati Equation. *IEEE Trans. Autom. Control*, AC-30 (1985), 962-970.

KUNKEL P., MEHRMANN V., (1990) Numerical Solution of Differential-Algebraic Riccati Equations. *Linear Algebra Appl.* 137/138 (1990), 39-66.

ORTEGA J.M., RHEINBOLDT W.C., (1972) *Iterative Solution of Nonlinear Equations in Several Variables.* Academic Press, New York, 1972.

OSHMAN Y., BAR-ITZHACK I.Y., (1984) Eigenfactor Solution of the Matrix Riccati Equation. A Continuous Square Root Algorithm. *Proc. 23rd. Conf. on Decision and Control.* Las Vegas, 1984, 503-508.

REID W.T., (1972) *Riccati Differential Equations.* Academic Press, New York, 1972.

STARR A.W., HO Y.C., (1969) Non-Zero Sum Differential Games. *J. Optim. Theory and Appl.* 3 (1969), 179-197.

$$\|S\| = 3.487108559$$

$$k_0 = \frac{1}{2}\|S\| = 1.74355428; \; K_1 = K_2 = \|S\| = 3.487108559$$

$$q_0 = \frac{1}{2}\left\| \begin{bmatrix} -1 & 0 & 2 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 2 \end{bmatrix} \right\| = 1.721489326$$

We choose δ such that

$$0 < \delta \leq 1 \text{ and } \delta k_0 + \ln(\delta) < -\ln\left(q_0 \left\| \begin{bmatrix} I \\ W_f \end{bmatrix} \right\|\right)$$

$$1.74355428\delta + \ln(\delta) < -0.5431898034$$

We can take $\delta = 0.325$, and then

$$M = q_0 \exp(\delta k_0) \left\| \begin{bmatrix} I \\ 0 \end{bmatrix} \right\| = 3.0033887936$$

$$\gamma = 12.99447389; \; h_0 = 0.1862361999; \; h < \min(1, h_0, 1/k_0) < h_0$$

and α_i, $i = 1, 2$ given by (50), takes the values

$$\alpha_2 = 132854.2695; \; \alpha_1 = 930.8402212$$

N (Number of knots)	h (stepsize)	ϵ (Error)
10	0.0325	1.131420718
20	0.01625	0.25550637918
50	0.0065	0.03956515252
100	0.00325	0.00984682191
500	0.00065	0.0003933037
1000	0.000325	0.00009832148

Acknowledgements

This work has been partially supported by D.G.I.C.Y.T. grant PB93-0381 and the Generalitat of Valencia grant GV1118/93.

References

Abou-Kandil H., Bertrand P., (1986) Analytic Solution for a Class of Linear Quadratic Open-Loop Nash Games. *Internat. J. Control.* 43(1986), 997-1002.

Cruz J.B., Chen C.I., (1971) Series Nash Solution of Two-Person Non-Zero Sum Linear Quadratic Games. *j. Optim. Theory Appl.* 7(1991), 240-257.