

On the q -superlinear convergence of Newton's method for solving nonsmooth equations

by

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Newton's method for solving nonlinear equations of several variables can be extended to the nonsmooth case by using the generalized Jacobian instead of the gradient. This paper presents a sufficient condition for the q -superlinear convergence of this extension for solving the system of nonsmooth equations defined by Lipschitz continuous functions.

Keywords: Newton's method, locally q -superlinear convergence, nonsmooth equations, generalized Jacobian, Lipschitz continuous function.

1. Introduction

Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a continuously differentiable function (i.e. a smooth function). Consider the nonlinear equation

$$F(x) = 0 \quad (1)$$

Classically, we solve (1) by the most popular method which is known as Newton's method:

$$x_{k+1} = x_k - JF(x_k)^{-1}F(x_k) \quad (2)$$

where $JF(x_k)$ denotes the Jacobian matrix of F at x_k . Many other methods for solving (1) are related to (2).

In the recent years, due to the growing interest in nonsmooth problems, there have appeared many papers dealing with new versions of Newton's method for solving problem (1) without the differentiability assumption on F . If F is non-differentiable (i.e. $JF(x_k)$ may not exist), then some generalized derivatives

instead of the Jacobian matrix may be used. The class of equations to which such methods may be applied is considerably larger than the class of differentiable functions. In particular, the Lipschitz continuity of F allows for natural extensions of a number of analytic key results useful in proving convergence theorems. The two related papers by Pang (1990;1991) extend the classical Newton method for solving nonlinear equations to the B -differentiable functions and apply it to nonlinear complementarity and variational inequality problems. Also, another extension was proposed in the papers by Harker and Xiao (1990) and Ip and Kyparisis (1992) under the assumption of B -differentiability of F , and the method was applied to nonlinear complementarity problems.

A new extension of Newton's method to semismooth nonlinear equations was presented by Qi and Sun (1993) and Qi (1991), also, Kummer (1988) presented this method under the assumption that the set $\partial F(x^*)$ is single-valued, where x^* is the zero of F . Another extension of Newton's method to a different class of nonsmooth equations was studied by Robinson (1988).

We establish in this paper the local convergence of a new version of Newton's method which was proposed by Qi and Sun (1993)

$$x_{k+1} = x_k - V_k^{-1} F(x_k) \quad (3)$$

(where $V_k \in \partial F(x_k)$ with $\partial F(x_k)$ being the generalized Jacobian of F at x_k in the sense of Clarke, 1988) for solving the system of semismooth equation defined by Lipschitz continuous functions. The utility of this method is to a large extent dependent on the speed with which $\{x_k\}$ converges to x^* . In order to be practical, a method should be at least q -linear in the sense of Ortega (1970) and if the method is to be competitive, then it should, in fact, converge q -superlinearly in the sense of Dennis (1983). Our aim is to prove the q -superlinear convergence of this new extension of Newton's method without assuming differentiability of F at x^* .

The paper is organized as follows: In the second section we recall some definitions and theorems related to our work as a theoretical background. In the third section we establish the sufficient condition for the locally q -superlinear convergence of algorithm (3).

2. Theoretical background

In this section, we recall some definitions and facts. Throughout this paper, we shall use the assumption that the function $F: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is Lipschitz continuous, where \mathbf{R}^n stands for the n -dimensional real Euclidean space. We denote the Euclidean norm $\|\cdot\|_2$ in \mathbf{R}^n by $\|\cdot\|$, i.e. $\|x\| = \langle x, x \rangle^{1/2}$.

DEFINITION 2.1 *Dennis, Schnabel (1983). A function F is Lipschitz continuous with a constant L in a set X if, for any $x, y \in X$,*

$$\|F(x) - F(y)\| \leq L \|x - y\|.$$

DEFINITION 2.2 *Dennis, Schnabel (1983).* Let $x \in \mathbf{R}^n, x_k \in \mathbf{R}^n$ and $k = 0, 1, \dots$. The sequence $\{x_k\}$ is said to converge q -superlinearly to x if, for some sequence $\{c_k\}$ that converges to 0, we have

$$\|x_{k+1} - x\| \leq c_k \|x_k - x\| \text{ for all } k.$$

DEFINITION 2.3 *Clarke (1988).* The generalized Jacobian of F at x , denoted by $\partial F(x)$, is the convex hull of all $n \times n$ matrices Z obtained as the limit of a sequence of the form $JF(x_i)$ where $x_i \rightarrow x$ and $x_i \in \Omega_F$ (Ω_F is the set of points at which F fails to be differentiable). In short

$$\partial F(x) = \text{co} \left\{ \lim_{i \rightarrow \infty} JF(x_i) : x_i \rightarrow x, x_i \in \Omega_F \right\}.$$

PROPOSITION 2.1 *Clarke (1988).*

- a) $\partial F(x)$ is a nonempty convex compact subset of $\mathbf{R}^{n \times n}$.
- b) ∂F is closed at x , that is, if $x_i \rightarrow x, Z_i \in \partial F(x_i), Z_i \rightarrow Z$, then $Z \in \partial F(x)$.
- c) ∂F is upper semicontinuous at x : for any $\epsilon > 0$, there exist $\delta > 0$ such that for any y in $x + \delta B$,

$$\partial F(y) \subset \partial F(x) + \epsilon B_{n \times n}$$

where B and $B_{n \times n}$ denote, respectively, the open unit balls in \mathbf{R}^n and $\mathbf{R}^{n \times n}$.

DEFINITION 2.4 *Chaney (1982).* Let $\{x_k\}$ be a sequence in an open set $D \subset \mathbf{R}^n$ which converges in norm to $x \in D$ and suppose that $x_k = x$ for all k , and u is a nonzero vector in \mathbf{R}^n . Then we say that $\{x_k\}$ converges in norm to x in the direction u (denoted by $x_k \xrightarrow{u} x$) in case it is true that the sequence

$$\left\{ \frac{x_k - x}{\|x_k - x\|} \right\} \rightarrow \frac{u}{\|u\|} \text{ as } k \rightarrow \infty.$$

DEFINITION 2.5 Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be locally Lipschitzian on an open convex set $D \subset \mathbf{R}^n$, and let x be in D and u a nonzero vector in \mathbf{R}^n . The set $\partial_u F(x)$ (called the generalized Jacobian of F in the direction u at a point x) is defined to be the convex hull of all V in $\mathbf{R}^{n \times n}$ for each of which there exist sequences $\{x_k\}$ in \mathbf{R}^n and $\{V_k\}$ in $\mathbf{R}^{n \times n}$ such that

- (a) $\{x_k\}$ converges to x in the direction u ,
- (b) $\{V_k\}$ converges to V ,
- (c) V_k belongs to $\partial F(x_k)$ for every k . Thus

$$\partial_u F(x) = \text{co} \left\{ V : V = \lim_{k \rightarrow \infty} V_k, V_k \in \partial F(x_k), x_k \xrightarrow{u} x \right\}.$$

We note that $\partial_u F(x) \subseteq \partial F(x)$, and the set $\partial_u F(x)$ consists of those generalized gradients of F at x which come to x from the direction u .

THEOREM 2.1 (*Mean Value Theorem*) Clarke (1988). Let x and y be points in \mathbf{R}^n , and suppose that F is Lipschitz continuous on an open set $D \subset \mathbf{R}^n$ containing the closed line segment $[x, y]$; then

$$F(y) - F(x) \in \text{co } \partial F([x, y])(y - x).$$

THEOREM 2.2 (*Carathéodory Theorem*) Demyanov, Vasilev (1981). Let D be a nonempty subset of \mathbf{R}^n . Any vector $x \in \text{co } D$ can be represented as a convex combination of $n + 1$, or fewer, vectors from the set D . That is, if $x \in \text{co } D$, then there exist vectors $x_0, \dots, x_n \in D$ such that

$$x = \sum_{i=0}^n \alpha_i x_i, \text{ where } \alpha_i \geq 0, \sum_{i=0}^n \alpha_i = 1.$$

COROLLARY 2.1 Marek, Žitný (1983). Let X be the set of all nonsingular matrices in $\mathbf{R}^{n \times n}$, then:

- (a) X is open in $\mathbf{R}^{n \times n}$,
- (b) The mapping $x \rightarrow x^{-1}$ of the set X into $\mathbf{R}^{n \times n}$ is continuous.

LEMMA 2.1 Let f and g be two functions from \mathbf{R}^n to \mathbf{R} defined on an arbitrary set X . Then the following inequality holds:

$$\inf_{x \in X} (f(x) + g(x)) \leq \sup_{x \in X} f(x) + \inf_{x \in X} g(x)$$

Proof. We assume that the inequality does not hold; then we have

$$\inf_{x \in X} (f(x) + g(x)) > \sup_{x \in X} f(x) + \inf_{x \in X} g(x).$$

Thus there exists $\alpha \in \mathbf{R}$ such that, for any $y \in X$,

$$f(y) + g(y) > \alpha > \sup_{x \in X} f(x) + \inf_{x \in X} g(x) \geq f(y) + \inf_{x \in X} g(x)$$

and so,

$$g(y) > \alpha > \inf_{x \in X} g(x) \text{ for any } y \in X.$$

Hence, it follows from the definition of the infimum that there exists $z \in X$ such that, for any $y \in X$,

$$g(y) > \alpha > g(z)$$

In particular, taking $z = y$, we get a contradiction. ■

3. Local convergence of Newton's method

In this section we prove the sufficient condition for the locally q -superlinear convergence of the new version of Newton's method, presented by Qi and Sun (1993). We begin by proving some facts that will be useful for our main theorem.

PROPOSITION 3.1 *Suppose that $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is (locally) Lipschitzian and $\partial F(x)$ is the generalized Jacobian of F at x . If all $V \in \partial F(x)$ are nonsingular, then there exist a neighbourhood $N(x)$ of x and a constant c such that, for any $V \in \text{co} \bigcup_{y \in N(x)} \partial F(y)$, V is nonsingular and $\|V^{-1}\| \leq c$.*

Proof. We assume that the conclusion of the proposition is not true. Then, for any $\delta > 0$ and for any $c \in \mathbf{R}$, there exists

$$V \in \text{co} \bigcup_{y \in B(x, \delta)} \partial F(y)$$

(where $B(x, \delta)$ is the open ball with centre x and radius δ) such that either V is singular or $\|V^{-1}\| > c$.

Now, let $\delta_k = \frac{1}{k}$, $c_k = k$; then, for every k , there exists

$$V_k \in \text{co} \bigcup_{y \in B(x, \delta_k)} \partial F(y)$$

such that either V_k is singular or $\|V_k^{-1}\| > k$. Then, by Carathéodory's Theorem 2.2, there exist $\alpha_{ki} \geq 0$, $\sum_{i=0}^n \alpha_{ki} = 1$, and

$$V_{ki} \in \partial F(y_{ki}), \quad y_{ki} \in B(x, \delta_k), \quad i = 0, 1, \dots, n, \quad (4)$$

such that

$$V_k = \sum_{i=0}^n \alpha_{ki} V_{ki}. \quad (5)$$

Since the sets $\partial F(y)$ are uniformly bounded for y in some neighbourhood of x (see Proposition 2.1(a), (c)), we may assume by choosing a subsequence of $\{V_k\}$, if necessary, that

$$\lim_{k \rightarrow \infty} V_{ki} = W_i, \quad \text{for some } W_i \in \mathbf{R}^{n \times n}, \quad i = 0, 1, \dots, n \quad (6)$$

and

$$\lim_{k \rightarrow \infty} \alpha_{ki} = \alpha_i \quad \text{for some } \alpha_i \geq 0, \quad \sum_{i=0}^n \alpha_i = 1$$

Hence the limit

$$W = \lim_{k \rightarrow \infty} V_k \quad (7)$$

exists and is equal to

$$W = \sum_{i=0}^n \lim_{k \rightarrow \infty} \alpha_{ki} V_{ki} = \sum_{i=0}^n \alpha_i W_i. \quad (8)$$

Moreover, for any i , we have

$$\lim_{k \rightarrow \infty} y_{ki} = x. \quad (9)$$

Since $\partial F : y \rightarrow \partial F(y)$ is upper semicontinuous (Proposition 2.1(c)), conditions (4), (6) and (9) imply

$$W_i \in \partial F(x), i = 0, 1, \dots, n.$$

Then, by (8) and the convexity of $\partial F(x)$ (Proposition 2.1(a)) we have

$$W \in \partial F(x).$$

Choosing again a subsequence of $\{V_k\}$ if necessary, we may consider two cases:

- (i) V_k is singular for infinitely many k ; then, by (7), W is also singular, which contradicts the assumption that all $V \in \partial F(x)$ are nonsingular.
- (ii) V_k is nonsingular and $\|V_k^{-1}\| > k$, for every k ; then

$$\|V_k^{-1}\| \rightarrow \infty \text{ as } k \rightarrow \infty. \quad (10)$$

Since $W \in \partial F(x)$ and all the elements of $\partial F(x)$ are nonsingular, we deduce that W^{-1} exists and, by (7) and the continuity of the map $V \rightarrow \|V^{-1}\|$ (Corollary 2.1), we have

$$\|V_k^{-1}\| \rightarrow \|W^{-1}\| \text{ as } k \rightarrow \infty,$$

a contradiction with (10). This completes the proof. ■

LEMMA 3.1 *Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be locally Lipschitzian on an open convex set $D \subset \mathbf{R}^n$ and let $\partial F(z)$ be the generalized Jacobian of F at $z \in D$. If all $V \in \partial F(z)$ are nonsingular, then there exists a neighbourhood $N(z)$ of z and a constant $\alpha > 0$, such that, for all $x, y \in N(z)$, one has*

$$\|F(x) - F(y)\| \geq \alpha \|x - y\|. \quad (11)$$

Proof. Since all $V \in \partial F(z)$ are nonsingular, it follows from Proposition 3.1 that there exists a neighbourhood $N(z)$ of z and a constant c , such that for any $V \in \text{co} \bigcup_{u \in N(z)} \partial F(u)$, V is nonsingular and $\|V^{-1}\| \leq c$. We may assume that $N(z)$ is convex; then, for any $x, y \in N(z)$, we have the closed line segment $[y, x] \subset N(z)$ and so,

$$\text{co } \partial F([y, x]) \subset \text{co} \bigcup_{u \in N(z)} \partial F(u).$$

Therefore, any element

$$V_{x,y} \in \text{co } \partial F([y, x])$$

is nonsingular and

$$\|V_{x,y}^{-1}\| \leq c.$$

Then, by The Mean-Value Theorem 2.1,

$$F(x) - F(y) \in \text{co } \partial F([y, x])(x - y),$$

which means that, for some

$$V_{x,y} \in \text{co } \partial F([y, x]),$$

we have

$$\|F(x) - F(y)\| = \|V_{x,y}(x - y)\|. \quad (12)$$

Therefore

$$\|x - y\| = \|V_{x,y}^{-1} V_{x,y}(x - y)\| \leq \|V_{x,y}^{-1}\| \|V_{x,y}(x - y)\| \leq c \|V_{x,y}(x - y)\|.$$

Then, by (12)

$$\|x - y\| \leq c \|F(x) - F(y)\|$$

Now, take $\alpha = \frac{1}{c}$; then

$$\|F(x) - F(y)\| \geq \frac{1}{c} \|x - y\| = \alpha \|x - y\|,$$

for all $x, y \in N(z)$. ■

The following theorem is the main result of this paper.

THEOREM 3.1 Suppose that $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is Lipschitz continuous on an open convex set $D \subset \mathbf{R}^n$. Let $\{V_k\}$ be a sequence of nonsingular matrices in $\mathbf{R}^{n \times n}$, and suppose that, for some $x_0 \in D$, the sequence $\{x_k\}$ generated by the algorithm

$$x_{k+1} = x_k - V_k^{-1}F(x_k) \text{ for some } k = 0, 1, \dots \quad (13)$$

remains in D and converges to x^* in the direction u where u is a nonzero vector in \mathbf{R}^n , and $x_k = x^*$ for all but finitely many k . Suppose, in addition, that all elements $V^* \in \partial_u F(x^*)$ are nonsingular, and

$$\lim_{k \rightarrow \infty} \sup_{V^* \in \partial_u F(x^*)} \frac{\|(V_k - V^*)s_k\|}{\|s_k\|} = 0, \quad (14)$$

where $s_k = x_{k+1} - x_k$, and $\partial_u F(x^*)$ is the set defined in Definition 2.5. Then the sequence $\{x_k\}$ converges q -superlinearly to x^* , and $F(x^*) = 0$.

Before proving the theorem, we need the following

LEMMA 3.2 Let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be Lipschitz continuous on an open convex set $D \subset \mathbf{R}^n$ and let $\partial F(x^*)$ be the generalized Jacobian of F at $x^* \in D$. Then the sequence $\{x_k\}$ generated by algorithm (13) satisfies the condition

$$\lim_{k \rightarrow \infty} \inf_{V^* \in \partial_u F(x^*)} \frac{\|F(x_{k+1}) - F(x_k) - V^*s_k\|}{\|s_k\|} = 0, \quad (15)$$

where $s_k = x_{k+1} - x_k$, and $\partial_u F(x^*)$ is the set defined in Definition 2.5.

Proof. By The Mean-Value Theorem 2.1, we have

$$F(x_{k+1}) - F(x_k) \in \text{co } \partial F([x_k, x_{k+1}])s_k$$

and by Carathéodory's Theorem 2.2, there exist $\alpha_{ki} \geq 0$, $\sum_{i=0}^n \alpha_{ki} = 1$, and

$$V_{ki}^* \in \partial F(x_{ki}^*), \quad (16)$$

$x_{ki}^* \in [x_k, x_{k+1}]$, $i = 0, 1, \dots, n$, such that

$$F(x_{k+1}) - F(x_k) = \sum_{i=0}^n \alpha_{ki} V_{ki}^* s_k. \quad (17)$$

We may assume that $\|u\| = 1$ (if not use $\frac{u}{\|u\|}$ instead of u).

For $i = 0, 1, \dots, n$, x_{ki}^* lies on the close line segment $[x_k, x_{k+1}]$, put

$$x'_k = \frac{x_k - x^*}{\|x_k - x^*\|}$$

then x'_k converges to u (Definition 2.4). Let $x'_{ki} = t(x_{ki}^* - x^*)$ for some $t > 0$ such that x'_{ki} lies on the segments $[x'_k, x'_{k+1}]$

$$\frac{x'_{ki}}{\|x'_{ki}\|} = \frac{x_{ki}^* - x^*}{\|x_{ki}^* - x^*\|},$$

so that the convergence of x'_k to 0 in direction u is equivalent to the convergence of x_{ki}^* to x^* in the direction u . We have x'_{ki} in the closed line segment $[x'_k, x'_{k+1}]$, so that we can represent x'_{ki} as a convex combination

$$x'_{ki} = \beta_{ki}x'_k + (1 - \beta_{ki})x'_{k+1} \text{ for some } \beta_{ki} \in [0, 1].$$

Since

$$x'_{ki} = \beta_{ki}(x'_k - u) + (1 - \beta_{ki})(x'_{k+1} - u) + u,$$

and

$$x'_k \rightarrow u, x'_{k+1} \rightarrow u \text{ and } |\beta_{ki}| \leq 1, |1 - \beta_{ki}| \leq 1,$$

we get

$$x'_{ki} \rightarrow u \text{ and } \|x'_{ki}\| \rightarrow \|u\| = 1.$$

Thus

$$x_{ki}^* \rightarrow x^* \text{ as } k \rightarrow \infty \text{ for each } i = 0, 1, \dots, n. \quad (18)$$

u

We may assume by choosing appropriate subsequences that

$$\lim_{k \rightarrow \infty} V_{ki}^* = V_i^* \text{ for all } i \quad (19)$$

and

$$\lim_{k \rightarrow \infty} \alpha_{ki} = \alpha_i \text{ for some } V_{ki}^* \in \mathbf{R}^{n \times n} \text{ and } \alpha_i \geq 0, \sum_{i=0}^n \alpha_i = 1$$

From (16), (18), and (19) we have

$$V_i^* \in \partial_u F(x^*),$$

and, by the convexity of $\partial_u F(x^*)$ (Definition 2.5)

$$\sum_{i=0}^n \alpha_i V_i^* \in \partial_u F(x^*). \quad (20)$$

Then, using (17) and (20), we obtain

$$\begin{aligned}
& \inf_{V^* \in \partial_u F(x^*)} \frac{\|F(x_{k+1}) - F(x_k) - V^* s_k\|}{\|s_k\|} \leq \\
& \leq \frac{\|F(x_{k+1}) - F(x_k) - \sum_{i=0}^n \alpha_i V_i^* s_k\|}{\|s_k\|} = \\
& = \frac{\|\sum_{i=0}^n \alpha_i V_{ki}^* s_k - \sum_{i=0}^n \alpha_i V_i^* s_k\|}{\|s_k\|} \leq \\
& \leq \frac{\|\sum_{i=0}^n (\alpha_i V_{ki}^* - \alpha_i V_i^*)\| \|s_k\|}{\|s_k\|} = \\
& = \sum_{i=0}^n \|\alpha_i V_{ki}^* - \alpha_i V_i^*\| \rightarrow 0 \text{ as } k \rightarrow \infty,
\end{aligned}$$

which completes the proof. ■

Proof (of Theorem 3.1). Let $e_k = x_k - x^*$. From (13) we have

$$\begin{aligned}
x_{k+1} &= x_k - V_k^{-1} F(x_k), \\
0 &= V_k(x_{k+1} - x_k) + F(x_k), \\
0 &= V_k s_k + F(x_k).
\end{aligned}$$

Then, for any $V^* \in \partial_u F(x^*)$,

$$0 = (V_k - V^*)s_k + F(x_k) + V^* s_k$$

so that

$$-F(x_{k+1}) = (V_k - V^*)s_k - F(x_{k+1}) + F(x_k) + V^* s_k;$$

then

$$\frac{\|F(x_{k+1})\|}{\|s_k\|} \leq \frac{\|(V_k - V^*)s_k\|}{\|s_k\|} + \frac{\|-F(x_{k+1}) + F(x_k) + V^* s_k\|}{\|s_k\|}.$$

Hence, by Lemma 2.1, we have

$$\begin{aligned}
& \inf_{V^* \in \partial_u F(x^*)} \frac{\|F(x_{k+1})\|}{\|s_k\|} \leq \sup_{V^* \in \partial_u F(x^*)} \frac{\|(V_k - V^*)s_k\|}{\|s_k\|} + \\
& + \inf_{V^* \in \partial_u F(x^*)} \frac{\|F(x_{k+1}) - F(x_k) - V^* s_k\|}{\|s_k\|},
\end{aligned}$$

and so,

$$\frac{\|F(x_{k+1})\|}{\|s_k\|} \leq \sup_{V^* \in \partial_u F(x^*)} \frac{\|(V_k - V^*)s_k\|}{\|s_k\|} +$$

$$+ \inf_{V^* \in \partial_u F(x^*)} \frac{\|F(x_{k+1}) - F(x_k) - V^* s_k\|}{\|s_k\|}.$$

Taking the limits of both sides as $k \rightarrow \infty$ and using (14) as well as (15), we infer that

$$\frac{\|F(x_{k+1})\|}{\|s_k\|} \rightarrow 0, \text{ as } k \rightarrow \infty, \quad (21)$$

which implies $F(x^*) = \lim_{k \rightarrow \infty} F(x_k) = 0$.

By Lemma 3.1, there exist a neighbourhood $N(x^*)$ of x^* and $\alpha > 0$, such that (11) holds for all $x, y \in N(x^*)$. Then, for all sufficiently large k , we have $x_{k+1} \in N(x^*)$, and so,

$$\|F(x_{k+1})\| = \|F(x_{k+1}) - F(x^*)\| \geq \alpha \|x_{k+1} - x^*\| = \alpha \|e_{k+1}\|. \quad (22)$$

Combining (21) and (22), we obtain

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \frac{\|F(x_{k+1})\|}{\|s_k\|} \geq \lim_{k \rightarrow \infty} \alpha \frac{\|e_{k+1}\|}{\|s_k\|} \geq \\ &\geq \lim_{k \rightarrow \infty} \alpha \frac{\|e_{k+1}\|}{\|e_k\| + \|e_{k+1}\|} = \lim_{k \rightarrow \infty} \alpha \frac{P_k}{1 + P_k} \end{aligned}$$

where $P_k = \frac{\|e_{k+1}\|}{\|e_k\|}$, and $\frac{P_k}{1+P_k} \rightarrow 0$ as $k \rightarrow \infty$.

Hence $\{P_k\}$ converges to zero as $k \rightarrow \infty$, so that $\{x_k\}$ converges q -superlinearly to x^* (Definition 2.2). ■

In the following example we verify analytically condition (14).

Example.

The objective function $F : \mathbf{R} \rightarrow \mathbf{R}$ is given by

$$F(x) = \begin{cases} x^2 + 2x & x \geq 0 \\ x/2 & x \leq 0 \end{cases}$$

The equation $F(x^*) = 0$ has a solution $x^* = 0$.

The set $\partial F(x^*) = \text{co } \{1/2, 2\}$

$$\text{The set } \partial_u F(x^*) = \begin{cases} \{2\} & \text{for } u = 1 \\ \{1/2\} & \text{for } u = -1 \end{cases}$$

We choose sequence $\{x_k\}$ convergent to x^* , and verify analytically that condition (14) is satisfied (for the sequence). Taking the direction $u = 1$, we have $\partial_u F(x^*) = \{2\}$, thus

$$\begin{aligned} \sup_{V^* \in \partial_u F(0)} \frac{\|(2x_k + 2 - V^*)s_k\|}{\|s_k\|} &\leq \\ &\leq \sup_{V^* \in \partial_u F(0)} \|2x_k + 2 - V^*\| = \|2x_k + 2 - 2\|; \end{aligned}$$

obviously, $\lim_{k \rightarrow \infty} \|2x_k\| = 0$.

Taking the direction $u = -1$, we have $\partial_u F(x^*) = \{1/2\}$, thus

$$\sup_{V^* \in \partial_u F(0)} \frac{\|(1/2 - V^*)s_k\|}{\|s_k\|} \leq \sup_{V^* \in \partial_u F(0)} \|1/2 - V^*\|;$$

obviously, $\lim_{k \rightarrow \infty} \|1/2 - 1/2\| = 0$. This implies that condition (14) holds and algorithm (13) generates the sequence $\{x_k\}$ which converges to x^* q -superlinearly.

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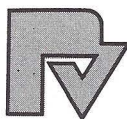
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