Control and Cybernetics

vol. **24** (1995) No. 2

A method for calculation of generalized integral criterion $\int_0^\infty \epsilon^{2k}(t)\,dt,\ k = 1, 2, \ldots$ for control systems with time-delay[1]

by

H. Górecki, A. Korytowski, M. Zaczyk

Institute of Automatics
Technical University of Mining and Metallurgy
Cracow, Poland

In the paper a generalized criterion of optimization is proposed. The generalization takes into account even powers higher than 2 of the dynamic error under the integral. The proposed method is also suitable for control systems with time-delay. All the calculations were made using symbolic computation package MAPLE V on computer IBM PC/486.

1. Introduction

In the paper the problem of calculation of the integral

$$\int_0^\infty \epsilon^{2k}(t)\,dt,\ k = 1, 2, \ldots \tag{1}$$

is considered. The dynamic error $\epsilon(t)$ is the solution of a system of linear differential equations with time delay. The proposed approach is the generalization of the method which is given in Górecki, Popek (1984).

2. Solution of the problem

The proposed method is applicable for systems involving a single time-delay in either forward or the feedback path, where the dynamic error transform $E(s)$ is of the form

$$E(s) = \frac{B(s) + D(s)e^{-s\tau}}{A(s) + C(s)e^{-s\tau}} \tag{2}$$

1 This work was supported by the Polish State Committee for Scientific Research (grant 7 0086 91 01).

in which $A(s)$ to $D(s)$ are polynomials in s.

The general assumption is that the system represented by (2) is stable and that the integral (1) exists.

For the sake of simplicity we will demonstrate the method on the following example:

Let us calculate the integral

$$J_4 = \int_0^\infty \epsilon^4 (t)\, dt \tag{3}$$

for the control system which consists of the object represented by the transfer function

$$G_0(s) = \frac{e^{-s\tau}}{s} \tag{4}$$

and the proportional controller

$$G_R(s) = K \tag{5}$$

The transfer function of the closed-loop system is

$$G_0(s) = \frac{Ke^{-s\tau}}{s + Ke^{-s\tau}} \tag{6}$$

The transform of the dynamic error for the unit step input is

$$E(s) = \frac{1}{s}\left[1 - \frac{Ke^{-s\tau}}{s + Ke^{-s\tau}}\right] = \frac{1}{s + Ke^{-s\tau}} \tag{7}$$

It is desirable to calculate the transform $F(s)$ of the square dynamic error by using the transform of the error $E(s)$. We calculate

$$F(S) = \int_0^\infty \epsilon^2(t)e^{-st}dt, \ \mathrm{Re}\ s \geq 0 \tag{8}$$

using the convolution theorem for transforms in the s-domain which corresponds to the product of the original functions in the t-domain.

$$F(S) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} E(\sigma)E(s-\sigma)d\sigma \tag{9}$$

The substitution of (7) into (9) gives

$$F(S) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{1}{\sigma + Ke^{-\sigma\tau}} \cdot \frac{1}{(s-\sigma) + Ke^{-(s-\sigma)\tau}} d\sigma \tag{10}$$

The application of partial fractions leads to the decomposition

$$F(S) = -\frac{1}{4\pi j}\left\{\int_{-j\infty}^{j\infty} \frac{\sigma - Ke^{-\sigma\tau}}{\sigma + Ke^{-\sigma\tau}} \cdot \frac{d\sigma}{\sigma(\sigma - s) + K^2e^{-s\tau}} + \right. \tag{11}$$
$$\left. + \int_{-j\infty}^{j\infty} \frac{(s-\sigma) - Ke^{-(s-\sigma)\tau}}{(s-\sigma) + Ke^{-(s-\sigma)\tau}} \cdot \frac{d\sigma}{\sigma(\sigma - s) + K^2e^{-s\tau}}\right\}$$

Observe that the function

$$g(\sigma) = \frac{\sigma - Ke^{-\sigma\tau}}{\sigma + Ke^{-\sigma\tau}} \tag{12}$$

has all poles in the left half-plane by assumption, and the function $\sigma \longrightarrow g(s-\sigma)$ has all poles in the right half-plane Re $\sigma >$ Re s. We have

$$\sigma(\sigma - s) + K^2e^{-s\tau} = (\sigma - \sigma_1)(\sigma - \sigma_2) \tag{13}$$

where

$$\sigma_{1,2} = \frac{1}{2}\left(s \pm \sqrt{s^2 - 4K^2e^{-s\tau}}\right) \tag{14}$$

Taking the contours of integration in such a way that σ_1 and σ_2 are on their right sides and closing the contour of the first integral on the right side, and the contour of the second integral on the left side, and applying Cauchy's Residuum Theorem we obtain that

$$F(s) = \frac{1}{2(\sigma_2 - \sigma_1)}[g(\sigma_1) - g(\sigma_2)] \tag{15}$$

The substitution of (12) into (15) gives

$$F(s) = \frac{\sigma_1 Ke^{-\sigma_2\tau} - \sigma_2 Ke^{-\sigma_1\tau}}{(\sigma_2 - \sigma_1)(\sigma_1 + Ke^{-\sigma_1\tau})(\sigma_2 + Ke^{-\sigma_2\tau})} \tag{16}$$

Applying de l'Hospital's rule to (16) for $\sigma_2 \longrightarrow \sigma_1$ we obtain

$$F(s) = -\frac{(1 + \sigma_1\tau)Ke^{-\sigma_1\tau}}{(\sigma_1 + Ke^{-\sigma_1\tau})(\sigma_2 + Ke^{-\sigma_2\tau})} \tag{17}$$

The substitution of (14) into (17) yields

$$F(s) = -\frac{\left[1 + \frac{\tau}{2}\left(s + \sqrt{s^2 - 4K^2e^{-s\tau}}\right)\right] Ke^{-\frac{\tau}{2}\left(s+\sqrt{s^2-4K^2e^{-s\tau}}\right)}}{\left[\frac{1}{2}\left(s + \sqrt{s^2 - 4K^2e^{-s\tau}}\right) + Ke^{\left(s+\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right]} \cdot \tag{18}$$
$$\frac{1}{\left[\frac{1}{2}\left(s - \sqrt{s^2 - 4K^2e^{-s\tau}}\right) + Ke^{-\frac{\tau}{2}\left(s-\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right]}$$

The consideration of the denominator of (18) and its derivative leads to the conclusion that

$$s_1 = s_2 \tag{19}$$

and fulfil the equation

$$\frac{1}{2}s + Ke^{-\frac{1}{2}s\tau} = 0 \tag{20}$$

We therefore claim that the denominator of $F(s)$ and its derivative have common roots $s_1 = s_2 = s$, which fulfil the equation

$$\frac{1}{2}s + Ke^{-\frac{1}{2}s\tau} = 0 \tag{21}$$

Proof. The denominator of $F(s)$ is equal to

$$\begin{aligned} M(s) \quad = \quad & \left[\frac{1}{2}\left(s + \sqrt{s^2 - 4K^2e^{-s\tau}}\right) + Ke^{-\frac{\tau}{2}\left(s+\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right] \cdot \\ \cdot \quad & \left[\frac{1}{2}\left(s - \sqrt{s^2 - 4K^2e^{-s\tau}}\right) + Ke^{-\frac{\tau}{2}\left(s-\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right] \end{aligned} \tag{22}$$

The equation $M(s) = 0$ has the roots which fulfil the following equations

$$Ke^{-\frac{\tau}{2}\left(s_i+\sqrt{s_i^2-4K^2e^{-s_i\tau}}\right)} = -\frac{1}{2}\left(s_i + \sqrt{s_i^2 - 4K^2e^{-s_i\tau}}\right) \quad i = 1, \dots \infty \tag{23}$$

and

$$Ke^{-\frac{\tau}{2}\left(s_j-\sqrt{s_j^2-4K^2e^{-s_j\tau}}\right)} = -\frac{1}{2}\left(s_j - \sqrt{s_j^2 - 4K^2e^{-s_j\tau}}\right) \quad j = 1, \dots \infty \tag{24}$$

The derivate of (22) with respect to s is equal to

$$\begin{aligned} &\frac{dM(s)}{ds} = \\ &\left[\frac{1}{2}\left(1 + \frac{s + 2K^2\tau e^{-s\tau}}{\sqrt{s^2 - 4K^2e^{-s\tau}}}\right) - \right. \\ &\left. -K\frac{\tau}{2}\left(1 + \frac{s + 2K^2\tau e^{-s\tau}}{\sqrt{s^2 - 4K^2e^{-s\tau}}}\right) e^{-\frac{\tau}{2}\left(s+\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right] \cdot \\ &\cdot \left[\frac{1}{2}\left(s - \sqrt{s^2 - 4K^2e^{-s\tau}}\right) + Ke^{-\frac{\tau}{2}\left(s-\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right] + \\ &+ \left[\frac{1}{2}\left(1 - \frac{s + 2K^2\tau e^{-s\tau}}{\sqrt{s^2 - 4K^2e^{-s\tau}}}\right) - \right. \end{aligned} \tag{25}$$

$$-K\frac{\tau}{2}\left(1-\frac{s+2K^2\tau e^{-s\tau}}{\sqrt{s^2-4K^2e^{-s\tau}}}\right)e^{-\frac{\tau}{2}\left(s-\sqrt{s^2-4K^2e^{-s\tau}}\right)}\Bigg]\cdot$$
$$\cdot\left[\frac{1}{2}\left(s+\sqrt{s^2-4K^2e^{-s\tau}}\right)+Ke^{-\frac{\tau}{2}\left(s+\sqrt{s^2-4K^2e^{-s\tau}}\right)}\right]$$

The substitution of (21) into (22), (23), (24) gives

$$M(s)=0 \tag{26}$$

The substitution of (21) into (25) gives

$$\frac{dM(s)}{ds}=0 \tag{27}$$

We conclude that

$$M(s)=\left[\frac{1}{2}s+Ke^{-\frac{1}{2}s\tau}\right]^2 \tag{28}$$

Returning to (18) and taking (21) and (28) into account gives

$$F(s)=-\frac{\left(1+\frac{1}{2}\tau s\right)Ke^{-\frac{1}{2}s\tau}}{\left[\frac{1}{2}s+Ke^{-\frac{1}{2}s\tau}\right]^2} \tag{29}$$

It is possible now to calculate the integral (3) using Parseval's theorem

$$J_4=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}F(s)F(-s)ds \tag{30}$$

$$J_4=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{K^2\left[1-\left(\frac{1}{2}\tau s\right)^2\right]}{\left[\frac{1}{2}s+Ke^{-\frac{1}{2}s\tau}\right]^2\left[-\frac{1}{2}s+Ke^{\frac{1}{2}s\tau}\right]^2}ds \tag{31}$$

It is desirable to decompose the integral (31) into the form

$$\begin{aligned}J_4 &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{c_0(s)+c_1(s)e^{-\frac{1}{2}s\tau}+c_2(s)e^{-s\tau}}{\left[\frac{1}{2}s+Ke^{-\frac{1}{2}s\tau}\right]^2}ds \\ &+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{c_0(-s)+c_1(-s)e^{\frac{1}{2}s\tau}+c_2(-s)e^{s\tau}}{\left[-\frac{1}{2}s+Ke^{\frac{1}{2}s\tau}\right]^2}ds\end{aligned} \tag{32}$$

The application of partial fractions gives

$$c_0=\frac{1}{2}\frac{s^2K^2[\tau^2s^2-4][4K^2-s^2]}{[4K^2+s^2]^3} \tag{33}$$

$$c_1 = 2\frac{sK^3[\tau^2 s^2 - 4]}{[4K^2 + s^2]^2} \tag{34}$$

$$c_2 = -2\frac{K^4[\tau^2 s^2 - 4][4K^2 - s^2]}{[4K^2 + s^2]^3} \tag{35}$$

It is evident that

$$\left.\begin{array}{l} c_0(s) = c_0(-s) \\ c_2(s) = c_2(-s) \\ c_1(s) = -c_1(-s) \end{array}\right\} \tag{36}$$

The substitution on (33)-(36) into the integral (32) and Cauchy's Residuum Theorem give finally

$$\begin{aligned} J_4 \;=\; & \frac{[K\tau(3K^2\tau^2 - 1) + (1 + K^2\tau^2)\cos K\tau](1 - \sin K\tau)}{2K[1 - \sin K\tau]^2} + \\ & + \frac{(1 + K^2\tau^2)^2 \cos K\tau}{2K[1 - \sin K\tau]^2} \end{aligned} \tag{37}$$

For the system considered above the integral

$$J_2 = \int_0^\infty \epsilon^2(t)dt$$

is given by the relation in Marshall, Górecki, Walton, Korytowski (1992)

$$J_2 = \frac{\cos \tau K}{2K[1 - \sin \tau K]}$$

In Figure 1, dependence of J_2/τ and J_4/τ on the τK, is shown.

Figure 2 shows step responses of the system for the optimal K, for integrals J_2 and J_4, (for a fixed $\tau = 0.5$).

It is evident that this method can be applied for the calculation of the general integral J_{2K}.

The results of integral evaluation presented above were obtained using the symbolic computation package MAPLE V, refer to Char, Geddes, Gonnet, Leong Monagan, Watt (1993).

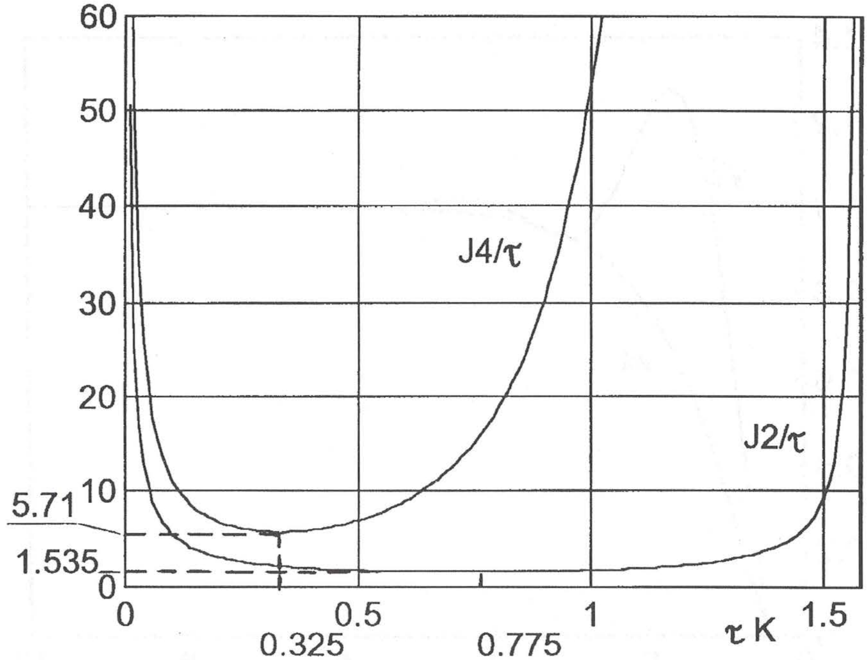
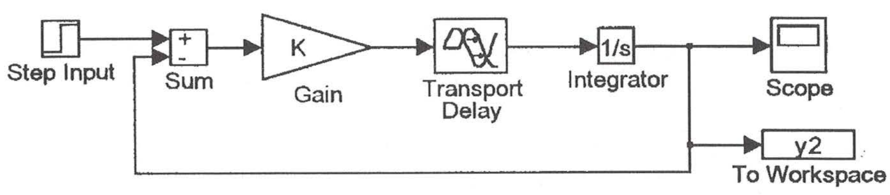


Figure 1. Dependence of J_2/τ and J_4/τ on τK.



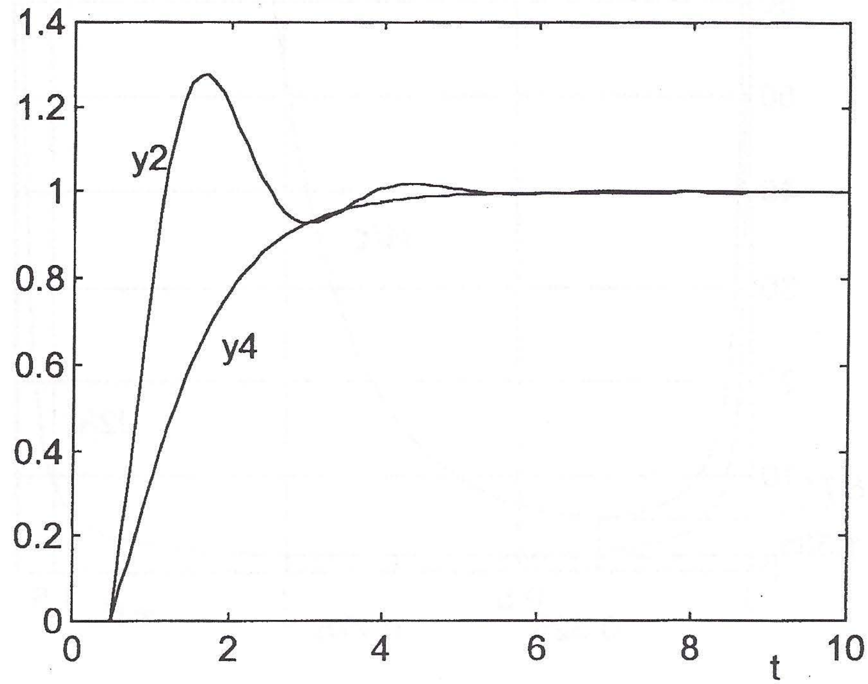


Figure 2. Step responses of the system for the optimal K (for $\tau = 0.5$).

References

CHAR B.W., GEDDES K.O., GONNET G.H., LEONG B.L., MONAGAN M.B., WATT S.W. (1993) Maple V Library Reference Manual; Maple V Language Reference Manual, Springer-Verlag, 1993.

GÓRECKI H., KORYTOWSKI A., MARSHALL J.E., WALTON K. (1992) Time - delay systems stability and performance criteria with applications. Ellis Horwood Ltd.

GÓRECKI H., POPEK, L. (1984) Parametric optimization problem for control systems with time-delay, Proc. 9th IFAC Word Congress, Budapest.

LEVIN, B. (1964) Distribution of zeros of entire functions A.M.S. Trans. of Math., monograph No.5.

CONFERENCE ANNOUNCEMENT AND CALL FOR PAPERS

IEA $\frac{96}{}$ AIE

The Ninth International Conference on Industrial & Engineering Applications of Artificial Inteligence & Expert Systems

June 4-7, 1996, ARCOS Fukuoka, Tenjin Chuo-ku Fukuoka, Japan

Topics of interest include, but are not limited to:

Automated Problem Solving
CAD/CAM
Case-based Reasoning
Computer Vision
Connectionist Models
Dependability of AE/ES
Distributed AI Architectures
Expert Systems
Fuzzy Logic & Soft Computing
Genetic Algorithms
Heuristic Searching
Intelligent Computer Network
Intelligent Databases
Intelligent Interface
Intelligent Tutoring
KBS Methodologies
Knowledge Acquisition
Knowledge Representation
Machine Learning
Model-Based Reasoning
Natural Language Processing
Neural Networks
Planning & Scheduling
Practical Applications
Reasoning Under Uncertainty
Robotics
Sensor Fusion
Spatial & Temporal Reasoning
Speech Recognition
System Integration
Tools
Verification & Validation of KBS

Authors are invited to submit five copies of paperss, written in English, of up to 10 single-spaced pages, presenting the results of original research or innovative practical applications relevant to one or more of the listed areas of interest. Practical experiences with state-of-art AI methodologies are also acceptable when they reflect lessons of unique value to the conference attendees. Shorter works, up to 6 pagess, to be presented in 10 minutes, may be submitted as SHORT PAPERS representing work in progress or suggesting possible research directions. (Please indicate "short papers" in the submission letter in this case.) Submissions should be received by the Program Co-Chair Takushi Tanaka of the IEA/AIE-96 conference office by November 8, 1995. All papers should include a key word list. Notification of the review process will be made by January 22, 1996, and final copies of papers will be due for inclusion in the conference proceedings by February 20, 1996. References will be asked to nominate papers for a Best Paper Prize to be announced at the conference. All papers, but particularly those nominated for the Best Paper competition, will be automatically considered for a place in the Journal of Applied Intelligence.

Prof. Moonis Ali
General Chair of IEA/AIE-96
Department of Computer Science
Southwest Texas State University
San Marcos TX 78666-4616 USA
Phone: (+1) 512-245-3409
Fax: (+1) 512-245-8750
Email: ma04@swt.edu

Prof. Takushi Tanaka
IEA/AIE-96 Conference Office
Fukuoka Institute of Technology
Wajiro-Higashi, Higashi-ku
Fukuoka 811-02, Japan
Phone: (+81) 92-606-3131
Fax: (+81) 92-606-1342
Email: tanaka@fit.ac.jp

World Wide Web – http://www.fit.ac.jp/ieaaie.html
The procedings will be published and will be avaliable at the conference. Copies of the procedings of earlier conferences are avaliable from: *Gordon and Breach Science Publishers, Customer Service*, P.O. Box 786, Cooper Station, New York, NY 10276; Fax: (+!) 212-645-2459